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A nonlinear parabolic problem on a Riemannian manifold without boundary arising in climatology

J.I. Díaz

Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid

E-mail: jidiaz@sunma1.mat.ucm.es

L. Tello

Departamento de Análisis Económico, Economía Cuantitativa, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid

E-mail: lourdes.tello@uam.es

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Abstract

We present some results on the mathematical treatment of a global two-dimensional diffusive climate model. The model is based on a long time averaged energy balance and leads to a nonlinear parabolic equation for the averaged surface temperature. The spatial domain is a compact two-dimensional Riemannian manifold without boundary simulating the Earth. We prove the existence of bounded weak solutions via a fixed point argument. Although, the uniqueness of solutions may fail, in general, we give a uniqueness criterion in terms of the behaviour of the solution near its "ice caps".

1. Introduction

This work is concerned with the nonlinear parabolic problem of the form

$$(P)\begin{cases} u_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \;\in\; QS(x)\beta(u) - \mathcal{G}(u) + f & \text{in } (0,T)\times\mathcal{M}, \\ u(x,0) = u_0(x) & \text{in } \mathcal{M}, \end{cases}$$

where $\mathcal{M}$ is a C^∞ two-dimensional compact connected oriented Riemannian manifold without boundary (and so, no boundary conditions are needed in (P)). We assume a fixed time $T > 0$ as well as $Q > 0$, $S \in L^\infty(\mathcal{M})$ and $p \geq 2$. The function $\mathcal{G}$ is increasing and β represents a bounded maximal monotone graph in $\mathbb{R}^2$ (e.g. of the Heaviside type).

The problem (P) arises in the modeling of some problems in Climatology. The so-called *Energy Balance Models* were introduced in 1969 by M.I. Budyko and W.D. Sellers, independently. They are diagnostic models intended to understand the evolution of the global climate on a long time scale. Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters. This kind of models has been used in the study of the Milankovitch theory of the ice-ages.

The model is obtained from an energy balance equation for the Earth's surface:

$$\text{heat variation} \;=\; R_a - R_e + D, \tag{1}$$

where R_a and R_e represent the absorbed solar and the emitted terrestrial energy flux, respectively. D represents the heat diffusion, given by a second order diffusion operator.

Another characteristic is the spatial domain, it is the whole Earth's surface and the time scale is considered relatively large. In seasonal models a smaller scale of time is introduced, in order to analyze the effect of the seasonal cycles in the climate and in particular in the ice caps formation.

Let us express each component of the above balance in mathematical terms. The distribution of temperature $u(t,x)$ is expressed pointwise after a standard average process, where the spatial variable x is in the Earth's surface which may be identified with a compact Riemannian manifold without boundary $\mathcal{M}$ (for instance, the two-sphere S^2), and t is the time variable. The *heat variation* is the product of the heat capacity c and the partial derivative of the temperature u with respect to the time. The *absorbed energy* R_a depends on the planetary *coalbedo* β. The coalbedo function represents the fraction of the incoming radiation flux which is absorbed by the surface. In ice-covered zones, reflexion is greater than over oceans, therefore, the coalbedo is smaller. One observes that there is a sharp transition between zones of high and low coalbedo. In the energy balance climate models, a main change of the coalbedo occurs in a neighbourhood of a temperature $u = -10^0 C$. This variation is modelled by a discontinuous function in the *Budyko model* and it will be treated as a maximal monotone graph in $\mathbb{R}^2$

$$\beta(u) = \begin{cases} \beta_i & u < -10 \\ [\beta_i, \beta_w] & u = -10 \\ \beta_w & u > -10\,, \end{cases}$$

where β_i and β_w represent the coalbedo in the ice-covered zone and the free-ice zone, respectively and $0 < \beta_i < \beta_w < 1$ (the value of these constants has been estimated by observation from satellites).

In the *Sellers model*, β is a function more regular (at least, Lipschitz), as for instance

$$\beta(u) = \begin{cases} \beta_i & u < u_i, \\ \beta_i - \left(\frac{u-u_i}{u_w-u_i}\right)(\beta_i - \beta_w) & u_i \leq u \leq u_w, \\ \beta_w & u > u_w, \end{cases}$$

where u_i and u_w are fixed temperatures closed to $-10^0 C$.

In both models, the absorbed energy is given (formally) by

$$R_a = QS(x)\beta(x, u)$$

where $S(x)$ is the *insolation function* and Q is the so-called *solar constant.*

The Earth's surface and atmosphere, warmed by the Sun, reemit part of the absorbed solar flux as an infrared long-wave radiation. This energy R_e is represented, in the Budyko model according to the Newton cooling law, that is,

$$R_e = Bu + C\,. \tag{2}$$

Here, B and C are positive parameters, which are obtained by observation, and can depend on the *greenhouse effect.* However, in the Sellers model, R_e is expressed according to the Stefan-Boltzman law

$$R_e \;=\; \sigma u^4, \tag{3}$$

where σ is called *emissivity constant* and u is in Kelvin degrees.

The *heat diffusion* D is given by the divergence, with negative sign, of the conduction heat flux F_c and the advection heat flux F_a. Fourier's law expresses

$$F_c \;=\; -k_c \nabla u$$

where k_c is the *conduction coefficient.* The advection heat flux is given by

$$F_a \;=\; -v\nabla u$$

and it is known (see e.g. (Childress-Ghil, 1987)) that the speed of the atmospheric flux v on the planetary scale can be incorporated by a diffusion coefficient k_a. So, $D \;=\; \mathrm{div}(k\nabla u)$ with $k = k_c + k_a$. In the pioneering models, the diffusion coefficient

k was considered as a positive constant. Later, P.H. Stone, (Stone, 1972) proposed a coefficient $k = |\nabla u|$, in order to consider the negative feedback in the eddy fluxes. Therefore, the heat diffusion is represented by the operator $D = \text{div}(|\nabla u|\nabla u)$. The formulation proposed in (P) is more general and it includes as particular case $p = 3$ (the case $p = 2$ gives a linear diffusion, that is, k constant). These physical laws, incorporated in (1), lead to problem (P) where $R_e(u) = \mathcal{G}(u) - f$ and where c is a constant given (which can be assumed equal to one by rescaling). Of course, this hypothesis simplifies the Earth's geography. E.g. c is greater over oceans than over land, but such a formulation adds more mathematical difficulties than the presence of the (possibly) discontinuous "source" term $QS(x)\beta(u)$. The general case $c = c(x)$ requires a different study, which is the subject of the work (Bermejo, Díaz and Tello, 1998) where the numerical analysis is also studied.

The goal of the present paper is to carry over previous results (North, 1979), (Hetzer, 1990), (Xu, 1991), (Díaz, 1993) and others, for a one-dimensional simplified problem. Such simplification considers the averaged temperature over each parallel as the unknown. So, the two-dimensional model (P) is reduced in a one-dimensional model when $\mathcal{M}$ is the two-sphere and considering the spherical coordinates. Therefore, the obtained model is

$$(P_1)\begin{cases} u_t - \big((1-x^2)^{p/2}\big)|u_x|^{p-2}u_x)_x \in QS(x)\beta(u) - R_e(u) & \text{in } (0,T)\times(-1,1), \\ \big(1-x^2\big)|u_x|^{p-2}u_x = 0 & \text{in } x=-1,\ x=1, \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases}$$

where $x = \sin\theta$ and θ is the latitude. Artificial boundary conditions has been introduced in (P_1), justified by the fact that meridional heat flux in the poles must be zero. The existence of solutions and the free boundary (the curve separating the regions $\{x : u(x,t) < -10\}$ and $\{x : u(x,t) > -10\}$) in problem (P_1) with linear diffusion ($p = 2$) was studied in (Xu, 1991) and, later, for the nonlinear case ($p \geq 2$) by (Díaz, 1993), where the uniqueness and nonuniqueness of solutions was studied. We are concerned with similar existence and uniqueness results for more general two-dimensional model whose spatial domain is a Riemannian manifold without boundary $\mathcal{M}$.

2. Preliminaries: analysis on manifolds

The physical origin of the problems of concern suggests to consider spatial domains which are not open subsets of $\mathbb{R}^n$. This is the case for the Earth's surface, which is typically treated as a two-sphere. A suitable general framework for our investigations is class of C^∞ two-dimensional connected compact oriented Riemannian manifolds without boundary. Let $\mathcal{M}$ belong to that class. It is well known that the differentiable structure of $\mathcal{M}$ allows us to extend certain notions from Differential Calculus in $\mathbb{R}^2$ to the two-dimensional manifolds. This approach uses local charts $(W_\lambda, \mathbf{w}_\lambda)$ where $\mathbf{w}_\lambda : W_\lambda \to \mathbb{R}^2$ is a C^∞ diffeomorphism and $\mathbf{p} \in W_\lambda \subset \mathcal{M}$. The pair $(w^1_\lambda(\mathbf{p}), w^2_\lambda(\mathbf{p}))$ may be considered as the *coordinates* of $\mathbf{p}$, which we denote by $(\theta_\lambda, \varphi_\lambda)$.

A function $f : \mathcal{M} \to \mathbb{R}$ is differentiable of class $C^r(\mathcal{M})$ if for each $\mathbf{p} \in \mathcal{M}$ there exits a chart $(W_\lambda, \mathbf{w}_\lambda)$ in $\mathbf{p}$ such that $f \circ \mathbf{w}_\lambda^{-1}$ is defined on a neighbourhood $E \subset \mathbb{R}^2$ of $\mathbf{w}_\lambda(\mathbf{p})$ and it is of class $C^r(E, \mathbb{R})$. Every function $f : \mathcal{M} \to \mathbb{R}$ has a *local representation* $\tilde{f}(\theta, \varphi) = f \circ \mathbf{w}_\lambda^{-1}(\theta, \varphi)$. Sometimes, we denote the local representation of f by f. We also recall that the *tangent space* in $\mathbf{p}$ is denoted by $T_\mathbf{p}\mathcal{M}$. The *tangent bundle* $T\mathcal{M}$ is $\cup_{\mathbf{p}\in\mathcal{M}} T_\mathbf{p}\mathcal{M}$. Let $(W_\lambda, \mathbf{w}_\lambda)$ be a chart in $\mathbf{p} \in \mathcal{M}$ and let $\{\theta_\lambda, \varphi_\lambda\}$ be the associated coordinates system, then the set $\{\frac{\partial}{\partial\theta_\lambda}|_\mathbf{p}, \frac{\partial}{\partial\varphi_\lambda}|_\mathbf{p}\}$ is a basis of $T_\mathbf{p}\mathcal{M}$ and so, dim $T_\mathbf{p}\mathcal{M} = 2 =$ dim $\mathcal{M}$. Notice that if $f : \mathcal{M} \to \mathbb{R}$ then

$$\frac{\partial}{\partial\theta}(f)|_\mathbf{p} = \frac{\partial(f \circ w_\lambda^{-1})}{\partial\theta}|_{\mathbf{w}_\lambda(\mathbf{p})} \qquad \frac{\partial}{\partial\varphi}(f)|_\mathbf{p} = \frac{\partial(f \circ w_\lambda^{-1})}{\partial\varphi}|_{\mathbf{w}_\lambda(\mathbf{p})}.$$

The global definition is obtained by using a partition of unity subordinated to a covering of $\mathcal{M}$ which is given by the domains of the charts. Such partition allow us to extend local properties (on each chart) to global properties (on the whole manifold).

We consider a *Riemannian metric* $\mathbf{g}$ on the manifold $\mathcal{M}$, that is, for each $\mathbf{p} \in \mathcal{M}$, $T_\mathbf{p}\mathcal{M}$ has a inner product $g_\mathbf{p} : T_\mathbf{p}\mathcal{M} \times T_\mathbf{p}\mathcal{M} \to \mathbb{R}$ which is a differentiable function of $\mathbf{p}$. Let g_{ij} denote the coefficients of a matrix associated to $g_\mathbf{p}$. The coefficients of $\mathbf{g}$ for points of the domain W_λ are expressed by $g_{ij}(\theta, \varphi)$ depending of the coordinates θ, φ of $\mathbf{q} = \mathbf{x}^{-1}(\mathbf{p})$ and so, a metric g^λ on W_λ is obtained. The definition of $g_\mathbf{p}$ on $T_\mathbf{p}\mathcal{M} \times T_\mathbf{p}\mathcal{M}$ is determined by

$$g_\mathbf{p}\Big(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\Big) = g_{11}, \qquad g_\mathbf{p}\Big(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\Big) = g_{22}, \qquad g_\mathbf{p}\Big(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\Big) = g_{12} = g_{21},$$

and so, if $v = v_1 \frac{\partial}{\partial\theta} + v_2 \frac{\partial}{\partial\varphi}$ we have that

$$g_{\mathbf{p}}(\mathbf{v},\mathbf{v}) = (v_1,\ v_2) \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Finally, the partition of unity $(\alpha_\lambda)_{\lambda\in\Lambda}$ subordinated to $\{W_\lambda\}_{\lambda\in\Lambda}$ allows us to define a Riemannian metric $\mathbf{g}$ from $\mathbf{g}^\lambda$, $\mathbf{g} = \sum_{\lambda\in\Lambda} \alpha_\lambda \mathbf{g}^\lambda$. When the considered manifold is the sphere of radius R ($\mathcal{M} = S_R^2$) with the atlas of the spherical coordinates, we have $g_{11} = R^2 \sin^2\varphi$, $g_{22} = R^2$, $g_{12} = g_{21} = 0$.

An easy modification of a well known result (see e.g. (Díaz, 1985) Chap. 4) gives the following.

Lemma 1

Let $\{\mathbf{e}_\theta, \mathbf{e}_\varphi\}$ be a basis of $T_{\mathbf{p}}\mathcal{M}$ and let ξ and η belong to $T_{\mathbf{p}}\mathcal{M}$,

$$\xi = \xi_1 \mathbf{e}_\theta + \xi_2 \mathbf{e}_\varphi\,, \qquad \eta = \eta_1 \mathbf{e}_\theta + \eta_2 \mathbf{e}_\varphi\,.$$

Then, if $p \geq 2$ and $|\xi| = \sqrt{g_{\mathbf{p}}(\xi,\xi)}$, we have that

$$g_{\mathbf{p}}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \geq C|\xi - \eta|^p.$$

We introduce, as in (Aubin, 1982), (Boothby, 1975), (Chavel, 1984) and (Gallot-Hullin-Lafontaine, 1987) some operators arising in partial differential equations on manifolds. The gradient of a differentiable function $f : \mathcal{M} \to \mathbb{R}$ is given by the vector field $\mathrm{grad}_{\mathcal{M}} f : \mathcal{M} \to T\mathcal{M}$ such that each point $\mathbf{p} \in \mathcal{M}$ maps into the vector $\mathrm{grad}_{\mathcal{M}} f(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$ defined by

$$\mathrm{grad}_{\mathcal{M}} f_{|\mathbf{p}} = \sum_{i,j} g^{ij} \frac{\partial \tilde{f}}{\partial y_j} \frac{\partial}{\partial y_i}\,,$$

where g^{ij} are the coefficients of the inverse matrix $g_{\mathbf{p}}$ and $\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$ is the basis of $T_{\mathbf{p}}\mathcal{M}$ associated to the coordinates system $\{y_1, y_2\}$ in $\mathbf{p}$. We also use the notation ∇f for the gradient of f. Let $X = h_1 \frac{\partial}{\partial\theta} + h_2 \frac{\partial}{\partial\varphi}$ be a vector field of $T_{\mathbf{p}}\mathcal{M}$, it is defined the *divergence* of X as the scalar field

$$\mathrm{div}\ X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial\theta}\big(h_1\sqrt{g}\big) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial\varphi}\big(h_2\sqrt{g}\big)\,,$$

where $\sqrt{g} = \sqrt{g_{11}g_{22} - g_{12}^2}$. Moreover, if $\mathbf{e}_\theta = \frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial\theta}$, $\mathbf{e}_\varphi = \frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial\varphi}$ form an orthogonal basis of $T_{\mathbf{p}}\mathcal{M}$ then div$X$ may be expressed as

$$\text{div } X = g_{\mathbf{p}}(D_{\mathbf{e}_\theta}(X), \mathbf{e}_\theta) + g_{\mathbf{p}}(D_{\mathbf{e}_\varphi}(X), \mathbf{e}_\varphi).$$

Since the vectors $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\varphi}$ can be nonconstant, we introduce the *covariant derivatives*

$$D_{\frac{\partial}{\partial y_i}}\Big(\frac{\partial}{\partial y_j}\Big) = \Gamma_{ij}^k \frac{\partial}{\partial y_k} \qquad \text{where } \ y_1 = \theta,\ y_2 = \varphi, \tag{1}$$

where Γ_{ij}^k are the *Christoffel symbols*, defined by

$$\Gamma_{ij}^k = \sum_{l=1,2} \frac{1}{2}\big(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\big) g^{kl}. \tag{2}$$

For the case $X = |\text{grad}_{\mathcal{M}} u|^{p-2}\text{grad}_{\mathcal{M}} u$ and the above orthonormal basis it is not difficult to see that

$$\begin{aligned}
&\text{div}\big(|\nabla u|^{p-2}\nabla u\big) = \\
&= \frac{u_\theta}{g_{11}}\frac{p-2}{2}\left(\frac{u_\theta^2}{g_{11}} + \frac{u_\varphi^2}{g_{22}}\right)^{(p-4)/2}\left(\frac{2u_\theta u_{\theta\theta}}{g_{11}} + u_\theta^2\frac{\partial}{\partial\theta}\Big(\frac{1}{g_{11}}\Big) + \frac{2u_\varphi u_{\varphi\theta}}{g_{22}} + u_\varphi^2\frac{\partial}{\partial\theta}\Big(\frac{1}{g_{22}}\Big)\right) \\
&\ + \frac{u_\varphi}{g_{22}}\frac{p-2}{2}\left(\frac{u_\theta^2}{g_{11}} + \frac{u_\varphi^2}{g_{22}}\right)^{(p-4)/2}\left(\frac{2u_\theta u_{\theta\varphi}}{g_{11}} + u_\theta^2\frac{\partial}{\partial\varphi}\Big(\frac{1}{g_{11}}\Big) + \frac{2u_\varphi u_{\varphi\varphi}}{g_{22}} + u_\varphi^2\frac{\partial}{\partial\varphi}\Big(\frac{1}{g_{22}}\Big)\right) \\
&\ + \frac{1}{\sqrt{g_{11}}}\left(\frac{u_\theta^2}{g_{11}} + \frac{u_\varphi^2}{g_{22}}\right)^{(p-2)/2}\left(\frac{u_{\theta\theta}}{\sqrt{g_{11}}} + 2u_\theta\frac{\partial}{\partial\theta}\Big(\frac{1}{\sqrt{g_{11}}}\Big) + \frac{u_\theta}{\sqrt{g_{11}}}\Gamma_{11}^1 + \frac{u_\varphi\sqrt{g_{11}}}{g_{22}}\Gamma_{12}^1\right) \\
&\ + \frac{1}{\sqrt{g_{22}}}\left(\frac{u_\theta^2}{g_{11}} + \frac{u_\varphi^2}{g_{22}}\right)^{(p-2)/2}\left(\frac{u_{\varphi\varphi}}{\sqrt{g_{22}}} + 2u_\varphi\frac{\partial}{\partial\varphi}\Big(\frac{1}{\sqrt{g_{22}}}\Big) + \frac{u_\varphi}{\sqrt{g_{22}}}\Gamma_{22}^2 + \frac{u_\theta\sqrt{g_{22}}}{g_{11}}\Gamma_{21}^2\right).
\end{aligned}$$

So, if $p = 2$ we get the Laplace-Beltrami operator on $\mathcal{M}$,

$$\begin{aligned}
\Delta u = \frac{u_{\theta\theta}}{g_{11}} + 2u_\theta\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sqrt{g_{11}}}\Big) + \frac{u_\theta}{g_{11}}\Gamma_{11}^1 + \frac{u_\varphi}{g_{22}}\Gamma_{12}^1 \\
+ \frac{u_{\varphi\varphi}}{g_{22}} + 2u_\varphi\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial\varphi}\Big(\frac{1}{\sqrt{g_{22}}}\Big) + \frac{u_\varphi}{g_{22}}\Gamma_{22}^2 + \frac{u_\theta}{g_{11}}\Gamma_{21}^2\,.
\end{aligned}$$

In the particular case $\mathcal{M} = S^2_R$, we arrive at

$$\begin{aligned}
&\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \\
&= \frac{u_\theta}{R^2\sin^2\varphi}\frac{p-2}{2}\left(\frac{u_\theta^2}{R^2\sin^2\varphi}+\frac{u_\varphi^2}{R^2}\right)^{(p-4)/2}\left(\frac{2u_\theta u_{\theta\theta}}{R^2\sin^2\varphi}+\frac{2u_\varphi u_{\varphi\theta}}{R^2\sin^2\varphi}\right)\\
&\quad+\frac{u_\varphi}{R^2}\frac{p-2}{2}\left(\frac{u_\theta^2}{R^2\sin^2\varphi}+\frac{u_\varphi^2}{R^2}\right)^{(p-4)/2}\left(\frac{2u_\theta u_{\theta\varphi}}{R^2\sin^2\varphi}-\frac{2u_\theta^2\cot\varphi}{R^2\sin^2\varphi}+\frac{2u_\varphi u_{\varphi\varphi}}{R^2}\right)\\
&\quad+\frac{1}{R\sin\varphi}\left(\frac{u_\theta^2}{R^2\sin^2\varphi}+\frac{u_\varphi^2}{R^2}\right)^{(p-2)/2}\left(\frac{u_{\theta\theta}}{R\sin\varphi}+\frac{u_\varphi\cos\varphi}{R}\right)\\
&\quad+\frac{1}{R}\left(\frac{u_\theta^2}{R^2\sin^2\varphi}+\frac{u_\varphi^2}{R^2}\right)^{(p-2)/2}\left(\frac{u_{\varphi\varphi}}{R}\right).
\end{aligned}$$

We use the special notation

$$\Delta_p u := \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)\,. \tag{6}$$

We recall some function spaces on manifolds. Denote by $\mathcal{D}(\mathcal{M})$ the space of functions $C^\infty(\mathcal{M})$ with compact support on $\mathcal{M}$. In this case, since $\mathcal{M}$ is compact, $\mathcal{D}(\mathcal{M}) = C^\infty(\mathcal{M})$. $L^2(\mathcal{M})$ represents the space of functions $u : \mathcal{M} \to \mathbb{R}$ measurable on $\mathcal{M}$ such that $\int_{\mathcal{M}} |u|^2 dA < +\infty$, i.e.

$$\sum_{\lambda\in\Lambda}\int_{\mathbf{w}_\lambda(W_\lambda)} \alpha_\lambda\big|u(\mathbf{w}_\lambda^{-1}(\theta_\lambda,\varphi_\lambda))\big|^2\sqrt{\det\mathbf{g}^\lambda}\,d\theta_\lambda d\varphi_\lambda < +\infty\,.$$

$L^2(\mathcal{M})$ is endowed with the usual inner product and the norm

$$(f,g)_{L^2(\mathcal{M})} := \int_{\mathcal{M}} fg dA, \qquad \|f\|_{L^2(\mathcal{M})} := (f,f)^{1/2}_{L^2(\mathcal{M})}.$$

$L^p(\mathcal{M})$ is defined from the Riemannian density dA as the set of measurable functions on $\mathcal{M}$ such that

$$\int_{\mathcal{M}} |u|^p dA \;<\; +\infty \quad \text{if} \quad 1 \le p < \infty, \text{ and } \operatorname{ess\,sup}|u| \;<\; +\infty \quad \text{if} \quad p = \infty\,.$$

So, $\big(L^2(\mathcal{M}), (,)\big)$ is a Hilbert space and $(L^p(\mathcal{M}), \|\cdot\|_p)$ is a Banach space, where $1 \le p \le \infty$. $L^2(T\mathcal{M})$ represents the Hilbert space of the vector fields $X : \mathcal{M} \to T\mathcal{M}$

endowed with the inner product in L^2 induced by $\mathbf{g}$ in $T_{\mathbf{p}}\mathcal{M}$, i.e. let $X = h_1\mathbf{e}_\theta + h_2\mathbf{e}_\varphi$, $X \in L^2(T\mathcal{M})$ if

$$\int_{\mathcal{M}} < X, X > dA < +\infty \,.$$

Since we assume $\mathcal{M}$ is compact, $H^1(\mathcal{M})$ is defined as the closure of $C^\infty(\mathcal{M})$ for the inner product

$$\begin{aligned}(f,h)_{H^1(\mathcal{M})} &:= (f,h)_{L^2(\mathcal{M})} + (\mathrm{grad}_{\mathcal{M}} f, \mathrm{grad}_{\mathcal{M}} h)_{L^2(T\mathcal{M})} \\ &= \sum_{\lambda\in\Lambda} \Big\{ \int_{\mathbf{w}_\lambda(W_\lambda)} \alpha_\lambda f\big(\mathbf{w}_\lambda^{-1}(\theta_\lambda,\varphi_\lambda)\big) h\big(\mathbf{w}_\lambda^{-1}(\theta_\lambda,\varphi_\lambda)\big)\sqrt{\det \mathbf{g}^\lambda}\, d\theta_\lambda d\varphi_\lambda \\ &\quad + \int_{\mathbf{w}_\lambda(W_\lambda)} \alpha_\lambda g^\lambda(\mathrm{grad}_{\mathcal{M}} f, \mathrm{grad}_{\mathcal{M}} h)\sqrt{\det \mathbf{g}^\lambda}\, d\theta_\lambda d\varphi_\lambda \Big\}.\end{aligned}$$

If $s \in \mathbb{N}$, we denote by $H^s(\mathcal{M})$ the closure of $C^\infty(\mathcal{M})$ for the norm

$$\|\cdot\|_{H^s(\mathcal{M})} = \left(\int_{\mathcal{M}} \Big(\sum_{1\le k\le s} \sum_{i_j=1,2,\ \ j=1,..k} |D_{i_1}D_{i_2}...D_{i_k}u|^2 + |u|^2 \Big) dA \right)^{1/2},$$

where $D_1 = D_{e_\theta}$, $D_2 = D_{e_\varphi}$ and $|D_{i_1}D_{i_2}...D_{i_k}u|^2 = g(D_{i_1}D_{i_2}...D_{i_k}u, D_{i_1}D_{i_2}...D_{i_k}u)$. If $m \in \mathbb{N}$, we denote by $W^{m,p}(\mathcal{M})$ the set of measurable functions u on $\mathcal{M}$ such that

$$\|u\|_{m,p} := \left(\int_{\mathcal{M}} \Big(\sum_{1\le k\le m} \sum_{i_j=1,2,\ j=1,..k} |D_{i_1}D_{i_2}...D_{i_k}u|^p + |u|^p \Big) dA \right)^{1/p} < \infty \,.$$

As in the case where $\mathcal{M}$ is an open set of $\mathbb{R}^n$, it follows that $H^s(\mathcal{M})$ is a Hilbert space and $W^{m,p}(\mathcal{M})$ is a Banach space. Moreover, $u \in W^{1,p}(\mathcal{M})$ if and only if $u \in L^p(\mathcal{M})$ and $\mathrm{grad}_{\mathcal{M}} u \in L^p(T\mathcal{M})$ (in the weak sense).

As we will see in the following section, the mathematical treatment of (P) leads us to introduce the following "energy space"

$$V := \left\{ u : \mathcal{M} \to \mathbb{R},\ u \in L^2(\mathcal{M}),\ \nabla_{\mathcal{M}} u \in L^p(T\mathcal{M}) \right\},$$

which is a reflexive Banach space if $1 < p < \infty$. A useful technical result is the following

Theorem 1

Let $\mathcal{M}$ be a two-dimensional compact Riemannian manifold. Then

$$\textit{if} \quad p = 2, \ \ V \hookrightarrow L^q(\mathcal{M}),\ \forall q \in [2,\infty), \qquad\qquad \textit{if} \quad p > 2, \ \ V \hookrightarrow L^\infty(\mathcal{M})\,.$$

with continuous embedding.

By using the compactness of the manifold $\mathcal{M}$, it is not difficult to extend known results about compact embeddings for open sets.

Theorem 2

Let $2 \leq p < \infty$ then the embedding $V \subset L^2(\mathcal{M})$ is compact.

Although the result is well known for standard Sobolev spaces (see (Aubin, 1982), pp. 44), we have estimated the constants of these embeddings which will be useful in some later computations. More precisely, the constants

$$C_{1,2,q} = 2\mu^{2/q} k(2,q)^2 \nu^{-1} \max\{1,\mu\} \left(1 + \sup|\nabla\alpha_\lambda|\right)^2, \tag{7}$$

$$\begin{aligned} C_{1,p,\infty} = \; & 2^{p-1} k(p,\infty)^p \max\left\{\nu^{-p/2}, \nu^{-1}\mu^{p/2}\right\} \\ & \times \left(1 + C_{1,2,p} \sup|\nabla\alpha_\lambda|\right)^p \max\left\{1, |\mathcal{M}|^{(p-2)/2}\right\} \end{aligned} \tag{8}$$

satisfy

$$\|f\|^2_{L^q(\mathcal{M})} \leq C_{1,2,q}\left(\|\nabla f\|^2_{L^2(T\mathcal{M})} + \|f\|^2_{L^2(\mathcal{M})}\right),$$
$$\|f\|^p_{L^\infty(\mathcal{M})} \leq C_{1,p,\infty}\left(\|\nabla f\|^p_{L^p(T\mathcal{M})} + \|f\|^p_{L^2(\mathcal{M})}\right),$$

for all $f \in V$. Here, the constants $k(2,p)$ and $k(p,\infty)$ are independent of $\mathcal{M}$ and the constants ν and μ are given by

$$\nu \parallel X \parallel^2 \leq \; \mathbf{g}(X,X) \; \leq \; \mu \parallel X \parallel^2 \quad \forall X \in T\mathcal{M},$$

(such constants exist thank to $\mathcal{M}$ is a compact manifold).

3. Existence of solution to the problem (P)

Motivated by model background described in Section 1, we introduce the following structure hypotheses: $p \geq 2$, $Q > 0$,

- $(H_\mathcal{M})$ $\mathcal{M}$ is a C^∞ two-dimensional compact connected oriented Riemannian manifold of $\mathbb{R}^3$ without boundary,
- (H_β) β is a bounded maximal monotone graph in $\mathbb{R}^2$, i.e. $|z| \leq M$ $\forall z \in \beta(s)$, $\forall s \in \mathbb{R}$.
- $(H_\mathcal{G})$ $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\mathcal{G}(0) = 0$, and $|\mathcal{G}(\sigma)| \geq C|\sigma|^r$ for some $r \geq 1$,
- (H_s) $S : \mathcal{M} \to \mathbb{R}$, $S \in L^\infty(\mathcal{M})$, $s_1 \geq S(x) \geq s_0 > 0$ a.e.$x \in \mathcal{M}$,
- (H_f^T) $f \in L^\infty((0,T) \times \mathcal{M})$, (resp. (H_f^∞) $f \in L^\infty((0,\infty) \times \mathcal{M})$),
- (H_0) $u_0 \in L^\infty(\mathcal{M})$.

The possible discontinuity in the coalbedo function causes that (P) does not have classical solutions in general, even if the data u_0 and f are smooth. Therefore, we introduce the notion of weak solution with respect to the "energy space" V.

Definition 1. We say that $u : \mathcal{M} \to \mathbb{R}$ is a *bounded weak solution of* (P) if

i) $u \in C([0,T]; L^2(\mathcal{M})) \cap L^\infty((0,T) \times \mathcal{M}) \cap L^p(0,T;V)$

ii) there exists $z \in L^\infty((0,T) \times \mathcal{M})$ with $z(t,x) \in \beta(u(t,x))$ a.e. $(t,x) \in (0,T) \times \mathcal{M}$ such that

$$\begin{aligned}
&\int_{\mathcal{M}} u(T,x)v(T,x)dA - \int_0^T < v_t(t,x), u(t,x) >_{V' \times V} dt \\
&\quad + \int_0^T \int_{\mathcal{M}} < |\nabla u|^{p-2}\nabla u,\ \nabla v > dAdt + \int_0^T \int_{\mathcal{M}} \mathcal{G}(u) v dAdt \\
&= \int_0^T \int_{\mathcal{M}} QS(x)z(t,x)vdAdt + \int_0^T \int_{\mathcal{M}} fvdAdt + \int_{\mathcal{M}} u_0(x)v(0,x)dA \\
&\forall v \in L^p(0,T;V) \cap L^\infty((0,T) \times \mathcal{M}) \text{ such that } v_t \in L^{p'}(0,T;V'),
\end{aligned}$$

where $<,>_{V' \times V}$ denotes the duality product in $V' \times V$.

In this section we prove the following result

Theorem 3

Let $u_0 \in L^\infty(\mathcal{M})$. *There exists at least a bounded weak solution of* (P). *Moreover, if* $T = +\infty$ *and* f *verifies* (H_f^∞)*, the solution* u *of* (P) *can be extended to* $[0,\infty) \times \mathcal{M}$ *such that* $u \in C([0,\infty), L^2(\mathcal{M})) \cap L^\infty((0,\infty) \times \mathcal{M}) \cap L^p_{loc}((0,\infty); V)$.

As in the one-dimensional model (Díaz, 1993), we use the techniques of (Díaz-Vrabie, 1987), based in fixed point arguments which are useful for nonmonotone equations, possibly multivalued.

3.1. The operator A. Properties. A comparison principle

We define the operator A as follows

$$\begin{array}{rcl}
A : D(A) \subset L^2(\mathcal{M}) & \longrightarrow & L^2(\mathcal{M}) \\
u & \longrightarrow & -\Delta_p u + \mathcal{G}(u)\,,
\end{array} \tag{9}$$

where $D(A) = \{u \in L^2(\mathcal{M}) : -\Delta_p u + \mathcal{G}(u) \in L^2(\mathcal{M})\}$ (recall (6)).

We have

Proposition 1

Let us define $\phi : D(\phi) \subset L^2(\mathcal{M}) \to \mathbb{R}$ *by*

$$\phi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathcal{M}} |\nabla u|^p dA + \int_{\mathcal{M}} G(u) dA & u \in D(\phi) \\ +\infty & u \notin D(\phi) \end{cases} \tag{10}$$

where $G(u) = \displaystyle\int_0^u \mathcal{G}(\sigma) d\sigma$ *and*

$$D(\phi) := \Big\{ u \in L^2(\mathcal{M}),\ \nabla u \in L^p(T\mathcal{M}) \text{ and } \int_{\mathcal{M}} G(u) dA < +\infty \Big\}.$$

Then

i) ϕ *is proper, convex and lower semicontinuous in* $L^2(\mathcal{M})$.
ii) $A = \partial\phi$ *where* A *is given by (9) and* $\overline{D(A)} = L^2(\mathcal{M})$.
iii) A *generates a compact semigroup of contractions* $S(t)$ *on* $L^2(\mathcal{M})$ *for* $t \in (0, T)$.

The proof is an easy adaptation of some well known results (see e.g. (Brezis, 1973) or (Barbu, 1976)) for bounded open sets (a detailed proof can be found in (Tello, 1996)). We also have a comparison principle for the operator A defined in (9). Now, we consider the operator A from $W^{1,p}(\mathcal{M})$ into $L^{p'}(\mathcal{M})$ and so,

Proposition 2 (Weak comparison principle)

Let $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ *be a continuous and strictly increasing function. Let* $f, \tilde{f} \in L^2(\mathcal{M})$ *such that* $f \leq \tilde{f}$ *and let* $u, \tilde{u} \in D(A)$ *with* $\mathcal{G}(u), \mathcal{G}(\tilde{u}) \in L^2(\mathcal{M})$ *weak solutions of the equations*

$$(P_f) \qquad -\Delta_p u + \mathcal{G}(u) = f$$
$$(P_{\tilde{f}}) \qquad -\Delta_p \tilde{u} + \mathcal{G}(\tilde{u}) = \tilde{f}\,.$$

then $u \leq \tilde{u}$ *en* $\mathcal{M}$. *Actually, if* f *and* $\tilde{f}$ *are arbitrary in* $L^2(\mathcal{M})$ *and we replace* $\mathcal{G}(u)$ *with* $\mathcal{G}(u) + u$ *we have*

$$\|(u - \tilde{u})^+\|_{L^2(\mathcal{M})} \ \leq\ \|(f - \tilde{f})^+\|_{L^2(\mathcal{M})}\,,$$

i.e. the operator $\Delta_p u + \mathcal{G}(u)$ *is T-accretive in* $L^2(\mathcal{M})$.

Once more, the proof is an easy adaptation of the known results (see (Tello, 1996) for details).

3.2. Multivalued functions. A fixed point theorem

Here we recall several concepts in multivalued functions that we need for our later purpose.

DEFINITION 2. Let X be a Banach space and let Ω be a measurable set of $\mathbb{R}^n$. A mapping $F : \Omega \mapsto 2^X$ is called measurable if for each closed subset $C \subset X$, the set $F^{-1}(C) := \{y \in \Omega : \; F(y) \cap C \neq \emptyset\}$ is Lebesgue measurable in $\mathbb{R}^n$.

DEFINITION 3. Let U be a topological space. A mapping $F : U \mapsto 2^X$ is called continuous (resp. weakly continuous) at $u \in U$, if (i) $F(u)$ is nonempty bounded closed and convex set, (ii) for every open (resp. weakly open) subset $D \subset X$ satisfying $F(u) \subset D$ there exists a neighbourhood V of u, such that $F(v) \subset D$ $\forall v \in V$.

We are interested in applying a version of the Schauder-Tychonoff Theorem which we state as in (Vrabie, 1987), in the form

Theorem 4

Let X be a Banach space. Assume that (i) $K \subset X$ is a nonempty, convex and weakly compact set, (ii) $\mathcal{L} : K \to 2^X$ with nonempty convex and closed values, such that $\mathcal{L}(u) \subset K$ $\forall u \in K$. If graph($\mathcal{L}$) is weakly $\times$ weakly sequentially closed, then $\mathcal{L}$ has at least one fixed point in K, i.e. $\exists u \in K$ such that $u \in \mathcal{L}(u)$.

3.3. Proof of the existence of solution

Let us consider the Cauchy problem associated to the operator A defined by (9),

$$(P_h) \begin{cases} \dfrac{du}{dt}(t) + Au(t) \ni h(t) & t \in (0,T), \text{ in } X = L^2(\mathcal{M}) \\ u(0) = u_0, & u_0 \in L^2(\mathcal{M})\,. \end{cases}$$

The properties of A given by Proposition 1 and the abstract results of (Brezis, 1973) guarantee that (P_h) has a unique solution (in the sense of semigroups) in $C([0,T]; L^2(\mathcal{M}))$ for every $h \in L^2((0,T); L^2(\mathcal{M}))$. Furthermore u is a weak solution in the sense of distributions and verifies $u \in L^p((0,T); V)$, $\sqrt{t}u_t \in L^2((0,T); L^2(\mathcal{M}))$, $u \in W^{1,2}((\delta,T); L^2(\mathcal{M}))$, $0 < \delta < T$. Since β is bounded, it is clear that there exists $h \in L^2((0,T); L^2(\mathcal{M}))$ such that $h \in QS\beta(u) + f$

a.e.$(t,x) \in (0,T) \times \mathcal{M}$. Let us see that the study of the existence of solutions for the problem (P) can be reduced to prove that an operator $\mathcal{L}$ has at least a fixed point. Let $Y = L^p((0,T);L^2(\mathcal{M}))$. Let us define $\mathcal{L} : K \to 2^{L^p((0,T);L^2(\mathcal{M}))}$ by the following process. First of all we define

$$K = \left\{ z \in L^p((0,T);L^\infty(\mathcal{M})) \ : \ \|z(t)\|_{L^\infty(\mathcal{M})} \leq C_0 \text{ a.e.} t \in (0,T) \right\}$$

with $C_0 = Qs_1M + \|f\|_{L^\infty((0,T);L^\infty(\mathcal{M}))}$. Let us see that K verifies the hypotheses of Theorem 4.

i) Obviously, K is nonempty and convex (due to the triangle inequality for $L^\infty(\mathcal{M})$ norm). Moreover, K is weakly compact in $L^p((0,T);L^2(\mathcal{M}))$. Indeed, since $L^p((0,T);\, L^2(\mathcal{M}))$ is a reflexive Banach space, it suffices to show that K is bounded in $L^p((0,T);L^2(\mathcal{M}))$ and weakly closed. Clearly

$$\|z(t)\|_{L^2(\mathcal{M})} \leq C\|z(t)\|_{L^\infty(\mathcal{M})} \leq CC_0 = C_1 \ \text{ a.e.} t \in (0,T).$$

Taking the essential supremum of both sides of this inequality, we get

$$\|z(t)\|_{L^\infty((0,T);L^2(\mathcal{M}))} \leq C\|z(t)\|_{L^\infty((0,T);L^\infty(\mathcal{M}))} \leq C_1\,,$$

and therefore

$$\|z(t)\|_{L^p((0,T);L^2(\mathcal{M}))} \leq C_2 \ \ \forall z \in K.$$

Let us see that K is weakly closed: if z is in the closure of K then there exists a sequence $\{z_n\} \subset K$ such that $z_n \rightharpoonup z$ in $L^p((0,T);L^2(\mathcal{M}))$. Thus,

$$z_{nk}(t) \rightharpoonup z(t) \text{ in } L^q(\mathcal{M}) \ \forall q \in (1,\infty), \text{ a.e. } t \in (0,T)$$

and from $\|z\|_{L^\infty(\mathcal{M})} = \lim_{q\to\infty} \|z\|_{L^q(\mathcal{M})}$ and

$$\|z\|_{L^q(\mathcal{M})} \leq \lim_{n\to\infty} \sup \|z_n\|_{L^q(\mathcal{M})} \leq C_0, \ \forall q,$$

we have that $z \in K$.

Now, we fix $u_0 \in L^2(\mathcal{M})$ and define the *solution operator* (or generalized Green operator)

$$\begin{array}{rccl} I_0 : & K & \to & C([0,T];L^2(\mathcal{M})) \\ & z & \to & v \end{array}$$

where v is the solution of (P_h) associated to $h \equiv z$. Since A is a m-accretive operator, $\forall z \in L^p((0,T);L^2(\mathcal{M}))$ there exists a unique solution in $C([0,T];L^2(\mathcal{M}))$ (we recall that $K \subset L^p((0,T); L^2(\mathcal{M})))$. So, the operator I_0 is well defined. Given $f \in L^2(\mathcal{M})$, we also define the *superposition operator* associated with $QS(x)\beta + f(x)$,

$$\begin{aligned} \mathcal{F}: \ L^2(\mathcal{M}) &\rightarrow \ 2^{L^2(\mathcal{M})} \\ v \quad &\rightarrow \ \{h \in L^2(\mathcal{M}): \ h(x) \in QS(x)\beta(v(x)) + f(x) \ \ \text{a.e.} x \in \mathcal{M}\}, \end{aligned} \tag{11}$$

i.e. $\frac{h-f}{QS} \in \beta(v)$ a.e. $x \in \mathcal{M}$. In the general case $f \in L^p((0,T); L^2(\mathcal{M}))$ this operator is defined $\mathcal{F}: \ L^p((0,T); L^2(\mathcal{M})) \ \rightarrow \ 2^{L^p((0,T);L^2(\mathcal{M}))}$. Finally, we define $\mathcal{L}$ by

$$\mathcal{L}(z) = \left\{h \in L^p((0,T); L^2(\mathcal{M})): \ h(t) \in \mathcal{F}(I_0(z)(t)) \text{ in } L^2(\mathcal{M}) \text{ a.e.} t \in (0,T)\right\}.$$

Let us see that $\mathcal{L}$ verifies the hypotheses of Theorem 4. In the following proposition we are going to state some properties of the previously defined operators which are proved easily (see for instance (Tello, 1996)).

Proposition 3

Let $\beta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ be a bounded maximal monotone graph and let $\mathcal{F} : L^2(\mathcal{M}) \rightarrow 2^{L^2(\mathcal{M})}$ be a superposition operator associated with the graph $QS\beta + f$ defined by (11). Then

i) $\mathcal{F}$ has nonempty closed convex values;
ii) $\mathcal{F}$ is bounded in $L^2(\mathcal{M})$;
iii) $\mathcal{F}(w) \subset L^\infty(\mathcal{M})$ $\forall w \in L^2(\mathcal{M})$, in the case $f \in L^\infty(\mathcal{M})$ (or $f \in L^\infty((0,T) \times \mathcal{M})$;
iv) $\mathcal{F} : L^2(\mathcal{M}) \rightarrow 2^{L^\infty(\mathcal{M})}$ is bounded;
v) the graph of $\mathcal{F}$ is strongly $\times$ weakly sequentially closed in $L^2(\mathcal{M}) \times L^2(\mathcal{M})$.

Also we need the following

Proposition 4 ((Vrabie, 1987), Corollary 2.3.2).

Let $A : D(A) \subset L^2(\mathcal{M}) \rightarrow 2^{L^2(\mathcal{M})}$ be defined by $Ax = \partial\phi(x)$ for each $x \in D(A) = D(\partial\phi)$, where $\phi : L^2(\mathcal{M}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, l.s.c., convex function of compact type. Then, for each $u_0 \in \overline{D(A)}$, the mapping

$$\begin{aligned} I_0 : \quad L^p((0,T); L^2(\mathcal{M})) \quad &\rightarrow \quad C([0,T]; L^2(\mathcal{M})) \\ z \qquad\qquad &\rightarrow \quad v \end{aligned}$$

where v is the solution of (P_h) associated with $h \equiv z$, is sequentially continuous from $L^p((0,T); L^2(\mathcal{M}))-$weak into $C([0,T]; L^2(\mathcal{M}))-$strong.

From structure hypotheses, it is obvious that $\mathcal{L}(u) \subset K\ \forall u \in K$. Finally, let us prove that graph$(\mathcal{L})$ is weakly $\times$ weakly sequentially closed in $L^p((0,T);\, L^2(\mathcal{M}))$ $\times\ L^p((0,T); L^2(\mathcal{M}))$. Indeed, let $(z,h) \in \overline{\text{graph}(\mathcal{L})}^{\text{weak}\times\text{weak}}$ then there exists $(z_n, h_n) \in$ graph$(\mathcal{L})$ such that

$$\begin{aligned} z_n &\rightharpoonup z \text{ weakly in } L^p((0,T); L^2(\mathcal{M}))\,, \\ h_n &\rightharpoonup h \text{ weakly in } L^p((0,T); L^2(\mathcal{M}))\,, \end{aligned}$$

$h_n \in \mathcal{L}(z_n)\ \forall n \in \mathbb{N}$ (i.e. $h_n(t) \in \mathcal{F}(I_0(z_n)(t))$ a.e.$t \in (0,T)\ \forall n$). Let us see that ($h \in \mathcal{L}(z)$ i.e. $h(t) \in \mathcal{F}(I_0(z)(t))$ a.e.$t \in (0,T)$). From Proposition 4,

$$I_0(z_n) \to I_0(z) \in C\big([0,T], L^2(\mathcal{M})\big)$$

and therefore

$$I_0(z_n)(t) \to I_0(z)(t) \text{ in } L^2(\mathcal{M}) \ \text{ a.e. } \ t \in (0,T)\,.$$

Using that graph$(\mathcal{F})$ is strongly $\times$ weakly sequentially closed, we get the following

$$\left.\begin{aligned} &(I_0(z_n)(t), h_n(t)) \in \text{ graph}\mathcal{F} \subset L^2(\mathcal{M}) \times L^2(\mathcal{M}) \text{ a.e. } \ t \in (0,T)\,, \\ &I_0(z_n)(t) \to I_0(z)(t) \text{ in } L^2(\mathcal{M}) \text{ a.e.} t \in (0,T)\,, \\ &h_n(t) \to h(t) \text{ in } L^2(\mathcal{M}) \text{ a.e.} t \in (0,T)\,, \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{aligned} &(I_0(z)(t), h(t)) \in \text{ graph}\mathcal{F} \text{ a.e.} t \in (0,T) \\ &\ \text{i.e. } h(t) \in \mathcal{F}(I_0(z)(t)) \text{ a.e.} t \in (0,T)\,. \end{aligned}\right.$$

Thus, $h \in \mathcal{L}(z)$ and graph$(\mathcal{L})$ is weakly $\times$ weakly sequentially closed.

So, we have verified the assumptions of Theorem 4 and it proves the first assertion of Theorem 3.

3.4. Proof of the global existence

In order to complete the proof of Theorem 3, we are going to show that the solution u can be continued up to $t = \infty$, when (H_f^∞) is fulfilled. Let us see that $u \in C([0,\infty), L^2(\mathcal{M}))$. In fact, multiplying the equation by u, and integrating on $\mathcal{M}$, we get

$$\int_{\mathcal{M}} u_t u dA + \int_{\mathcal{M}} |\nabla u|^p dA + \int_{\mathcal{M}} \mathcal{G}(u) u dA = \int_{\mathcal{M}} QSzudA + \int_{\mathcal{M}} fudA, \quad z \in \beta(u). \tag{12}$$

We recall that for all $z \in \beta(u)$ we have that $m \le z \le M$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}} |u|^2 dA + \int_{\mathcal{M}} |\nabla u|^p dA + c\int_{\mathcal{M}} |u|^2 dA \le C + \epsilon \|u\|^2_{L^2(\mathcal{M})}$$

where $C = C(\epsilon, \|S\|_\infty, M, \|f\|_{L^\infty((0,\infty);L^\infty(\mathcal{M}))})$. Then

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathcal{M})} \le -C_1\|u(t)\|^2_{L^2(\mathcal{M})} + C_2, \quad C_1, C_2 > 0 \tag{13}$$

and by Gronwall's inequality,

$$\|u(t)\|^2_{L^2(\mathcal{M})} \le e^{-C_1 t}\|u_0\|^2_{L^2(\mathcal{M})} + \frac{C_2}{C_1}(1 - e^{-C_1 t}).$$

The above expression tends to $\frac{C_2}{C_1}$ as $t \to +\infty$ and therefore

$$\|u(t)\|_{L^2(\mathcal{M})} \le k \quad \forall t > 0 \quad \text{(with } k \text{ independent of } t\text{)}. \tag{14}$$

By a well known result (see e.g. (Cazenave - Haraux, 1990) Theorem 4.3.4) u can be extended to $(0,\infty)$, hence, $u \in C([0,\infty); L^2(\mathcal{M}))$.

We also have that $u \in L^p_{loc}((0,\infty); V)$. In fact, from the estimate (14) we deduce that $u \in L^\infty((0,\infty); L^2(\mathcal{M}))$ and then $u \in L^p_{loc}((0,\infty); L^2(\mathcal{M}))$. Let us estimate the norm $\|\nabla u\|_{L^p(T\mathcal{M})}$. Integrating (12) on $(0,T)$, we arrive at

$$\frac{1}{2}\int_{\mathcal{M}} (|u(T)|^2 - |u_0|^2) dA + \int_0^T \int_{\mathcal{M}} |\nabla u|^p dA dt + \int_0^T \int_{\mathcal{M}} |u|^2 dA dt$$
$$\le Q\|S\|_\infty \int_0^T \int_{\mathcal{M}} |u| dA dt + \int_0^T \|f\|_{L^2(\mathcal{M})}\|u\|_{L^2(\mathcal{M})} dt.$$

By the continuous embedding $L^\infty(0,T) \subset L^1(0,T)$ and Young's inequality, this is majorised by

$$C_0 Q\|S\|_\infty M\|u\|_{L^\infty((0,\infty);L^1(\mathcal{M}))} + \frac{\epsilon}{2}\int_0^T \int_{\mathcal{M}} |u|^2 dA dt + C_\epsilon \|f\|_{L^\infty((0,\infty);L^2(\mathcal{M}))},$$

where $C_0 = C_0(T)$. So, we arrive at

$$\int_{\mathcal{M}} |u(T)|^2 dA + \int_0^T \int_{\mathcal{M}} |\nabla u|^p dA dt + \left(C - \frac{\epsilon}{2}\right)\int_0^T \int_{\mathcal{M}} |u|^2 dA dt \le k_2$$

where $k_2 = k_2(\int_{\mathcal{M}} |u_0|^2, Q, \|S\|_\infty, M, \|u\|_{L^\infty((0,\infty);L^1(\mathcal{M}))}, \|f\|_{L^\infty((0,\infty);L^2(\mathcal{M}))}, C_0)$. In particular,

$$\int_{\mathcal{M}} |u(T)|^2 \le k_2, \quad \int_0^T \int_{\mathcal{M}} |\nabla u|^p \le k_2 \qquad \forall T.$$

Thus $u \in L^p_{loc}((0,\infty); V)$. In order to finish the proof of Theorem 3 we establish that $u \in L^\infty((0,\infty) \times \mathcal{M})$ as follows:

Lemma 2

Let $u_0 \in L^\infty(\mathcal{M})$ and $f \in L^\infty((0,\infty) \times \mathcal{M})$ then $u \in L^\infty((0,\infty) \times \mathcal{M})$.

Proof. Let $\overline{u}(x,t)$ be the unique solution of the problem

$$\begin{cases} \overline{u}_t - \Delta_p \overline{u} + \mathcal{G}(\overline{u}) = MQS(x) + f^+(t,x) & \text{on} \quad (0,\infty)\times\mathcal{M} \\ \overline{u}(0,x) = u_0^+(x) = \ \max\{0, u_0(x)\} & \text{on} \quad \mathcal{M}. \end{cases}$$

We notice that $-\Delta_p u + \mathcal{G}(u)$ is a maximal monotone operator in $L^2(\mathcal{M})$ which guarantees the existence of $\overline{u}$. Since $u_0^+ \geq 0$, $MQS(x)+f^+(t,x) \geq 0$ and the operator under consideration is T-accretive in $L^2(\mathcal{M})$, it is clear that $\overline{u} \geq 0$. Moreover

$$\|\overline{u}\|_{L^\infty((0,\infty)\times\mathcal{M})} \ \leq \ L := \max\left\{\|u_0^+\|_\infty \ , \ \mathcal{G}^{-1}(\|MQS\|_\infty + \|f^+\|_\infty)\right\}.$$

Observe that L is an upper solution and the operator A is T-accretive in $L^2(\mathcal{M})$, which guarantees the above relation. Analogously, if $\underline{u}(x,t)$ is the unique solution of

$$\begin{cases} \underline{u}_t - \Delta_p \underline{u} + \mathcal{G}(\underline{u}) = mQS(x) + f^-(x) \\ \underline{u}(0,x) = u_0^-(x) = \ \min\ \{0, u_0(x)\} \end{cases}$$

we have

$$\|\underline{u}\|_{L^\infty((0,\infty)\times\mathcal{M})} \leq \max\left\{\|u_0^-\|_\infty \ , \ \mathcal{G}^{-1}(\|mQS\|_\infty + \|f^-\|_\infty)\right\}.$$

Finally, since

$$\begin{array}{ccc} \underline{u}_t - \Delta_p \underline{u} + \mathcal{G}(\underline{u}) \leq & u_t - \Delta_p u + \mathcal{G}(u) & \leq \overline{u}_t - \Delta_p \overline{u} + \mathcal{G}(\overline{u}) \\ u_0^- \leq & u_0 & \leq u_0^+ \end{array}$$

and A is T-accretive in $L^2(\mathcal{M})$, $\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t)$. It follows that

$$\|u\|_{L^\infty((0,\infty)\times\mathcal{M})} \leq \max\left\{\|\overline{u}\|_{L^\infty((0,\infty)\times\mathcal{M})}, \|\underline{u}\|_{L^\infty((0,\infty)\times\mathcal{M})}\right\}, \tag{15}$$

$\forall t > 0$. $\square$

4. On the uniqueness of solutions

The question of uniqueness has different answers for the different coalbedo functions under consideration, depending on whether the coalbedo is supposed to be discontinuous or not. For the Sellers model (β locally Lipschitz), the uniqueness is obtained by standard methods (see (Díaz, 1993)). Although (P) is a pde of parabolic type, in the Budyko model (β multivalued), there are cases of nonuniqueness. E.g. one finds (Díaz, 1993) infinitely many for a one-dimensional model with constant f and initial condition u_0 satisfying

$$\left.\begin{array}{l} u_0 \in C^\infty(I),\ u_0(x) = u_0(-x)\ \forall x \in [0,1], \\ u_0^{(k)}(0) = 0 \text{ where } k = 1,2,\ u_0(0) = -10 \\ u_0'(x) < 0,\ x \in (0,1),\ u_0'(1) = 0 \end{array}\right\} \tag{16}$$

Notice that these initial data u_0 are very "flat" at the level -10. This nonuniqueness result for the Budyko model with a suitable initial datum carries over to the two-dimensional model when $\mathcal{M} = S^2$. Each solution $u_1(t,x)$ of an 1D model generates a solution $u_2(t,x,y)$ of 2D model by rotation about the axis through the poles (notice that the initial datum $u_2(0,x,y)$ is independent of the longitude), i.e. $u_2(t,x,y) = u_1(t,\sin\theta)$ where $(x,y) \in S^2$ with latitude θ. It is not difficult to prove that u_2 is a solution of (P) for the initial datum $u_1(0,\mathrm{sen}\theta)$.

4.1. Existence of a maximal solution and a minimal solution

We start by proving the following

Lemma 3

The problem (P) has a maximal solution u^* and a minimal solution u_*, i.e. u^* and u_* are solutions of (P) such that every solution u of (P) verifies that $u_* \leq u \leq u^*$ in $(0,T) \times \mathcal{M}$.

Proof. Let $\overline{u}$ be the solution of the problem

$$(\overline{P}) \begin{cases} u_t - \Delta_p u + \mathcal{G}(u) = QS(x)M + f(x,t) & \text{in } (0,T)\times\mathcal{M}, \\ u(x,0) = u_0(x) & \text{in } \mathcal{M}, \end{cases}$$

and let $\underline{u}$ be the solution of the problem

$$(\underline{P}) \begin{cases} u_t - \Delta_p u + \mathcal{G}(u) = QS(x)m + f(x,t) & \text{in } (0,T)\times\mathcal{M}, \\ u(x,0) = u_0(x) & \text{in } \mathcal{M}. \end{cases}$$

Clearly $\overline{u}$ and $\underline{u}$ are upper solution and lower solution of problem (P), respectively. In fact, every $z \in L^2((0,T);L^2(\mathcal{M}))$ verifying $z \in \beta(\overline{u})$ fulfills $z \leq M$ a.e. on $(0,T)\times\mathcal{M}$. Analogously, if $z \in \beta(\underline{u})$ then $m \leq z$.

We construct sequences $\{u^k\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ by the following iteration process

$$(P^k)\begin{cases} u_t^k - \Delta_p u^k + \mathcal{G}(u^k) \;=\; QS(x)\overline{\beta}(u^{k-1}) + f(x,t) & \text{in} \quad (0,T)\times\mathcal{M}, \\ u^k(x,0) = u_0(x) & \text{in} \quad \mathcal{M}. \end{cases}$$

$$(P_k)\begin{cases} u_{kt} - \Delta_p u_k + \mathcal{G}(u_k) \;=\; QS(x)\underline{\beta}(u_{k-1}) + f(x,t) & \text{in} \quad (0,T)\times\mathcal{M}, \\ u_k(x,0) = u_0(x) & \text{in} \quad \mathcal{M}, \end{cases}$$

where $u^1 = \overline{u}$ and $u_1 = \underline{u}$ and where $\overline{\beta}$ and $\underline{\beta}$ are real monotone functions verifying

$$\begin{array}{lll} \overline{\beta}(s)\in\beta(s) & \text{such that} & \text{if } z\in\beta(s) \text{ then } z\leq\overline{\beta}(s) \\ \underline{\beta}(s)\in\beta(s) & \text{such that} & \text{if } z\in\beta(s) \text{ then } \underline{\beta}(s)\leq z\,. \end{array}$$

In other words, $\overline{\beta}$ and $\underline{\beta}$ coincide with β in the set where β is not multivalued and are equal to $\max\{\beta(s)\}$ and $\min\{\beta(s)\}$ for s in the set where β is multivalued, respectively.

Step 1. The sequences $\{u^k\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ are monotone.

Let us see that $u^2 \leq u^1 = \overline{u}$. Take the test function $(u^2-\overline{u})^+$ in (P^2) and $(\overline{P})$. Considering the difference we obtain

$$\int_0^T\int_{\mathcal{M}}(u_t^2-\overline{u}_t)(u^2-\overline{u})^+\int_0^T\int_{\mathcal{M}} < |\nabla u^2|^{p-2}\nabla u^2 - |\nabla\overline{u}|^{p-2}\nabla\overline{u}, \nabla(u^2-\overline{u})^+ >$$

$$+\int_0^T\int_{\mathcal{M}}(\mathcal{G}(u^2)-\mathcal{G}(\overline{u}))(u^2-\overline{u})^+ \;=\; \int_0^T\int_{\mathcal{M}}QS(x)(\overline{z}-M)(u^2-\overline{u})^+$$

where $\overline{z} = \overline{\beta}(\overline{u})$. Since $QS(x)\geq 0$ a.e. $x\in\mathcal{M}$ and $(\overline{z}(x,t)-M)\leq 0$ a.e. $(t,x)\in(0,T)\times\mathcal{M}$, $(u^2(x,t)-\overline{u}(x,t))^+\geq 0$ a.e. $(t,x)\in(0,T)\times\mathcal{M}$, we have

$$0\leq\frac{1}{2}\int_{\mathcal{M}}|(u^2(T)-\overline{u}(T))^+|^2\leq\int_0^T\int_{\mathcal{M}}QS(x)(\overline{z}-M)(u^2-\overline{u})^+\leq 0\,.$$

Hence, $u^2\leq\overline{u}$. Let us assume, by induction, $u^{k-1}\leq ...\leq u^2\leq\overline{u}$ and let us see that $u^k\leq u^{k-1}$. Arguing as before, we arrive at

$$0\leq\frac{1}{2}\int_{\mathcal{M}}|(u^k(T)-u^{k-1}(T))^+|^2\leq\int_0^T\int_{\mathcal{M}}QS(x)(z^{k-1}-z^{k-2})(u^k-u^{k-1})^+.$$

From the monotonicity of $\overline{\beta}$ we deduce that

$$\frac{1}{2}\int_{\mathcal{M}} |(u^k(T) - u^{k-1}(T))^+|^2 \;=\; 0$$

and so, $u^k \leq u^{k-1}$. Analogously, $\{u_k\}$ is a monotone nondecreasing sequence. In this way we have constructed two monotone sequences, that is, $\{u^k\}_{k\in\mathbb{N}}$ verifies $\overline{u} \geq u^2 \geq ... \geq u^{k-1} \geq u^k \geq ...$ and $\{u_k\}_{k\in\mathbb{N}}$ verifies $... \geq u_k \geq u_{k-1} \geq ... \geq u_2 \geq \underline{u}$. Moreover, since the upper and the lower solutions $\overline{u}$ and $\underline{u}$ are ordered, reasoning by induction we arrive at $u^k \geq u_k\ \forall k$. From $\overline{u} \geq \underline{u}$ and assuming $u^{k-1} \geq u_{k-1}$ we have

$$0 \leq \frac{1}{2}\int_{\mathcal{M}} |(u_k(T) - u^k(T))^+|^2 \leq \int_0^T \int_{\mathcal{M}} QS(x)(z_{k-1} - z^{k-1})(u_k - u^k)^+ \leq 0\,.$$

This follows immediately from β is a monotone graph, $u^{k-1} \geq u_{k-1}$ and so $z_{k-1} - z^{k-1} \leq 0$ a.e. $(x,t) \in (0,T) \times \mathcal{M}$.

Step 2. The sequences $\{u^k\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ are convergents in $L^p((0,T); L^2(\mathcal{M}))$.

The monotone sequence $\{\overline{u} - u^k\}_{k\in\mathbb{N}}$ verifies that $\overline{u} - u^k \geq 0$ and $\displaystyle\int_0^T \int_{\mathcal{M}} \overline{u} - u^k \leq \int_0^T \int_{\mathcal{M}} \overline{u} - \underline{u} = C$ then, $\sup \int_0^T \int_{\mathcal{M}} \overline{u} - u^k \leq C$. So, by the monotone convergence theorem we can conclude that there exists $\phi \in L^1((0,T) \times \mathcal{M})$ such that $\overline{u} - u^k \;\to\; \phi$ a.e. $(t,x) \in (0,T) \times \mathcal{M}$ and $\overline{u} - u^k \;\to\; \phi$ in $L^1((0,T) \times \mathcal{M})$. Now, calling $u^* = \overline{u} - \phi$ we have that $u^k \to u^*$ in $L^1((0,T) \times \mathcal{M})$. In order to obtain the convergence in $L^p((0,T); L^2(\mathcal{M}))$ $(p > 1)$ we consider the sequence $\{|\overline{u} - u^k|^p\}$, which also verifies the hypotheses of the monotone convergence theorem. Similar reasoning yields that $\{u_k\}$ converges to u_*.

Step 3. u^* and u_* are solutions of (P). Consider the weak formulations of (P^k) and (P_k) and study what happen when $k \to \infty$. We have the following a priori estimates:

$$\|u^k(T)\|_{L^2(\mathcal{M})} \leq C_1 \qquad \|u_k(T)\|_{L^2(\mathcal{M})} \leq C_1$$
$$\|u^k\|_{L^2(0,T;V)} \leq C_1 \qquad \|u_k\|_{L^2(0,T;V)} \leq C_1\,.$$

If $u_0 \in V$, we also have that $\|u_t^k\|_{L^2((0,T);L^2(\mathcal{M}))} \leq C_2$, $\|u_{kt}\|_{L^2((0,T);L^2(\mathcal{M}))} \leq C_2$. Thus, we obtain the convergences: $u^k \rightharpoonup u^*$ weakly in $L^2(0,T;V)$, $u^k \to u^*$ strongly in $L^2((0,T); L^2(\mathcal{M}))$, $u_k \rightharpoonup u_*$ weakly in $L^2(0,T;V)$ and $u^k \to u_*$ strongly in $L^2((0,T); L^2(\mathcal{M}))$. Since β is a maximal monotone graph then $z^k \rightharpoonup z^* \in \beta(u^*)$ weakly in $L^2((0,T); L^2(\mathcal{M}))$ and $z_k \rightharpoonup z_* \in \beta(u_*)$ weakly in $L^2((0,T); L^2(\mathcal{M}))$.

From the above estimates we also deduce that

$$|\nabla u^k|^{p-2}\nabla u^k \rightharpoonup Y^* \quad \text{weakly in} \quad L^2(0,T;L^{p'}(T\mathcal{M})),$$
$$|\nabla u_k|^{p-2}\nabla u_k \rightharpoonup Y_* \quad \text{weakly in} \quad L^2(0,T;L^{p'}(T\mathcal{M})).$$

In order to obtain the weak formulation of (P) as limit of the weak formulations of (P^k) and (P_k) as $k \to \infty$, we need establish that $Y^* = |\nabla u^*|^{p-2}\nabla u^*$ and $Y_* = |\nabla u_*|^{p-2}\nabla u_*$. This is a consequence of the fact

$$\lim_{k\to\infty} \int_0^T \int_{\mathcal{M}} < |\nabla u^k|^{p-2}\nabla u^k - |\nabla \chi|^{p-2}\nabla\chi \ , \ \nabla u^* - \nabla\chi > dAdt \ \geq \ 0\,, \qquad (17)$$

for any $\chi \in V$. Taking $\chi = u^* + \lambda\xi$ in (17), with $\lambda < 0$ and letting $\lambda \to \infty$, and again with $\lambda > 0$ we obtain

$$\int_0^T \int_{\mathcal{M}} < Y^* - |\nabla u^*|^{p-2}\nabla u^* \ , \ \nabla\xi > dAdt \ = \ 0\,.$$

The proof of (17) has been detailed in (Tello, 1996). Analogously, we get $Y_* = |\nabla u_*|^{p-2}\nabla u_*$. Finally, taking $k \to \infty$ in the weak formulation of (P_k) and (P^k) we have the conclusion of this step.

Final step. Every solution u of (P) verifies that $u_* \leq u \leq u^*$. Let u be a solution of the problem (P). Let us see that $u_k \leq u \leq u^k$. We have that $u_1 \leq u \leq u^1$. Indeed, take the test function $(u_1 - u)^+$ in the weak formulation of $(\underline{P})$ and (P). Now, taking the difference of the obtained expressions, we arrive at

$$0 \leq \frac{1}{2}\int_{\mathcal{M}} |(u_1 - u)^+|^2 \leq \int_0^T \int_{\mathcal{M}} QS(x)(m-z)(u_1-u)^+ \leq 0\,.$$

This implies that $(u_1 - u)^+ = 0$ a.e. t, that is, $u_1 \leq u$. Similarly, $u \leq u^1$.

Again, by induction, assume $u_{k-1} \leq u \leq u^{k-1}$ and take $(u_k - u)^+$ in the weak formulation of (P_k) and (P). Arguing as in step 1, we have $u_k \leq u$.

Finally, since $u_k \to u_*$ and $u^k \to u^*$ in $L^\infty(\mathcal{M})$ $\forall t \in (0,T)$, we conclude that $u_* \leq u \leq u^*$. Hence, u_* is the minimal solution of (P) and u_* is the maximal solution. □

4.2. Uniqueness of nondegenerated functions

A criterion of uniqueness for one-dimensional latitude dependent models of Budyko type was given in (Díaz, 1993) under so-called nondegeneracy hypotheses. The goal of this section is to extend these arguments to the two-dimensional models on a manifold. First, we introduce the notion of nondegeneracy for functions defined on a manifold $\mathcal{M}$.

Definition 4. Let $w \in L^\infty(\mathcal{M})$. We say that w satisfies the strong nondegeneracy property (resp. weak) if there exist $C > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$

$$|\{x \in \mathcal{M} : |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$$

(resp. $|\{x \in \mathcal{M} : 0 < |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$), where $|E|$ denotes the Lebesgue measure on the manifold $\mathcal{M}$ for all $E \subset \mathcal{M}$.

Theorem 5

(i) Assume that there exists a solution u of (P) such that $u(t)$ verifies the strong nondegeneracy property for any $t \in [0, T]$ then u is the unique bounded weak solution of (P).
(ii) There exists at most one solution of (P) verifying the weak nondegeneracy property.

The proof is based on the fact that β generates a continuous operator from $L^\infty(\mathcal{M})$ to $L^q(\mathcal{M})$ $\forall q \in [1, \infty)$, although β is discontinuous, when the domain of such operator is the set of functions verifying the strong nondegeneracy property. More precisely, we have

Lemma 4

(i) Let $w, \hat{w} \in L^\infty(\mathcal{M})$ and assume that w satisfies the strong nondegeneracy property. Then for any $q \in [1, \infty)$ there exists $\tilde{C} > 0$ such that for any $z, \hat{z} \in L^\infty(\mathcal{M})$ with $z(x) \in \beta(w(x))$ and $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in \mathcal{M}$, we have that

$$\| z - \hat{z} \|_{L^q(\mathcal{M})} \leq (b_w - b_i) \min \left\{ \tilde{C} \| w - \hat{w} \|^{(p-1)/q}_{L^\infty(\mathcal{M})} , \ |\mathcal{M}|^{1/q} \right\}. \tag{18}$$

(ii) If $w, \hat{w} \in L^\infty(\mathcal{M})$ and satisfy the weak nondegeneracy property then

$$\int_{\mathcal{M}} \big(z(x) - \hat{z}(x)\big)\big(w(x) - \hat{w}(x)\big) dA \leq (b_w - b_i) C \| w - \hat{w} \|^p_{L^\infty(\mathcal{M})} . \tag{19}$$

Proof of Lemma 4

Let ϵ_0 be given as in the Definition 4 for w. If $\| w - \hat{w} \|_{L^\infty(\mathcal{M})} > \epsilon_0$ then

$$\| z - \hat{z} \|_{L^q(\mathcal{M})} \leq (b_w - b_i)|\mathcal{M}|^{1/q} \leq (b_w - b_i) \frac{|\mathcal{M}|^{1/q}}{(\epsilon_0)^{(p-1)/q}} \| w - \hat{w} \|^{(p-1)/q}_{L^\infty(\mathcal{M})} .$$

Assume now that $\| w - \hat{w} \|_{L^\infty(\mathcal{M})} \leq \epsilon_0$. Define the coincidence sets

$$A := \left\{ x \in \mathcal{M} : w(x) = -10 \right\}, \quad \hat{A} := \left\{ x \in \mathcal{M} : \hat{w}(x) = -10 \right\},$$

and consider the decomposition $\mathcal{M} = A \cup \mathcal{M}_+ \cup \mathcal{M}_-$, $\quad \mathcal{M} = \hat{A} \cup \hat{\mathcal{M}}_+ \cup \hat{\mathcal{M}}_-$, where $\mathcal{M}_+ := \{x \in \mathcal{M} : w(x) > -10\}$, $\quad \mathcal{M}_- := \{x \in \mathcal{M} : w(x) < -10\}$ and where $\hat{\mathcal{M}}_+, \hat{\mathcal{M}}_-$ are defined in the some way but with $\hat{w}$ replacing w. Let $z, \hat{z}$ be defined as in the statement. Then

$$|z(x) - \hat{z}(x)| \leq (b_w - b_i) \quad \text{if} \quad x \in A \cup \hat{A} \cup (\mathcal{M}_+ \cap \hat{\mathcal{M}}_-) \cup (\mathcal{M}_- \cap \hat{\mathcal{M}}_+)$$

$$z(x) = \hat{z}(x) \quad \text{if} \quad x \in (\mathcal{M}_+ \cap \hat{\mathcal{M}}_+) \cup (\mathcal{M}_- \cap \hat{\mathcal{M}}_-)$$

$$\| z - \hat{z} \|_{L^q(\mathcal{M})} \leq (b_w - b_i) \min \left\{ |A \cup \hat{A} \cup (\mathcal{M}_+ \cap \hat{\mathcal{M}}_-) \cup (\mathcal{M}_- \cap \hat{\mathcal{M}}_+)|^{1/q}, |\mathcal{M}|^{1/q} \right\}. \quad (20)$$

On the other hand, if $\epsilon < \epsilon_0$ then $\left(A \cup \hat{A} \cup (\mathcal{M}_+ \cap \hat{\mathcal{M}}_-) \cup (\mathcal{M}_- \cap \hat{\mathcal{M}}_+) \right) \subset B_\epsilon$, where $B_\epsilon := \{x \in \mathcal{M} : -10 - \epsilon \leq w(x) \leq -10 + \epsilon\}$. Indeed, it is clear that $A \subset B_\epsilon$. Furthermore, $\hat{w}(x) - \| w - \hat{w} \|_{L^\infty(\mathcal{M})} \leq w(x) \leq \| w - \hat{w} \|_{L^\infty(\mathcal{M})} + \hat{w}(x)$ a.e.$x \in \mathcal{M}$. So, it is obvious that $\hat{A} \subset B_\epsilon$. If $x \in \mathcal{M}_+ \cap \hat{\mathcal{M}}_-$ then $-10 < w(x) \leq \epsilon + \hat{w}(x) < -10 + \epsilon$ and so, $x \in B_\epsilon$. Finally, if $x \in \mathcal{M}_- \cap \hat{\mathcal{M}}_+$, we have that $-10 - \epsilon \leq -10 - |w(x) - \hat{w}(x)| \leq \hat{w}(x) + w(x) - \hat{w}(x) \leq w(x) < -10$ and thus $x \in B_\epsilon$. Consequently, the inequality (18) is obtained from the strong nondegeneracy property of w. In order to prove (ii), we assume that w and $\hat{w}$ satisfy the weak nondegeneracy property. Arguing as in (i) we can suppose that $\| w - \hat{w} \|_{L^\infty(\mathcal{M})} \leq \epsilon_0$. Observe that if $x \in A \cap \hat{A}$ then $(z(x) - \hat{z}(x))(w(x) - \hat{w}(x)) = 0$ and if $w(x) \neq -10$ (resp. $\hat{w}(x) \neq -10$) and $x \in \hat{A}$ (resp. $x \in A$) we have that $x \in \{x \in \mathcal{M} : 0 < |w(x) + 10| \leq \epsilon\}$ (resp. $\{x \in \mathcal{M} : 0 < |\hat{w}(x) + 10| \leq \epsilon\}$). So, we arrive at (19). □

Proof of Theorem 5

Step 1. *Estimates.* Assume that there exist two bounded weak solutions u and $\hat{u}$ of (P), where u verifies the strong nondegeneracy property, i.e.

$$u_t - \Delta_p u + \mathcal{G}(u) = QSz + f \quad \text{in} \quad (0,T) \times \mathcal{M}$$

$$\hat{u}_t - \Delta_p \hat{u} + \mathcal{G}(\hat{u}) = QS\hat{z} + f \quad \text{in} \quad (0,T) \times \mathcal{M}$$

$$u(0) = \hat{u}(0) = u_0 \,,$$

for some $z \in \beta(u)$ and $\hat{z} \in \beta(\hat{u})$. Taking the test function $(u - \hat{u})$ in the weak formulation of these problems and considering the difference, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathcal{M}} |u(t) - \hat{u}(t)|^2 dA + \int_{\mathcal{M}} (\mathcal{G}(u) - \mathcal{G}(\hat{u}))(u - \hat{u}) dA \\ &\quad + \int_{\mathcal{M}} < |\nabla u(t)|^{p-2} \nabla u(t) - |\nabla \hat{u}(t)|^{p-2} \nabla \hat{u}(t),\ \nabla u(t) - \nabla \hat{u}(t) > dA \\ &= Q \int_{\mathcal{M}} S(x)(z(x,t) - \hat{z}(x,t))(u(x,t) - \hat{u}(x,t)) dA \,. \end{aligned} \quad (21)$$

Let us estimate these integrals. Since $p \geq 2$, the expression

$$\int_{\mathcal{M}} < |\nabla u(t)|^{p-2}\nabla u(t) - |\nabla \hat{u}(t)|^{p-2}\nabla \hat{u}(t),\ \nabla u(t) - \nabla \hat{u}(t) > dA$$

is majorised by

$$C_0 \int_{\mathcal{M}} |\nabla u(t) - \nabla \hat{u}(t)|^p dA$$

(see Lemma 1), in particular if $p = 2$ then $C_0 = 1$. Moreover, using the embeddings of Theorem 1 we have that if $p > 2$

$$\tilde{C}_0 \int_{\mathcal{M}} |\nabla u(t) - \nabla \hat{u}(t)|^p dA \geq \frac{C_0}{C_{1,p,\infty}} \|u - \hat{u}\|^p_{L^\infty(\mathcal{M})} - \tilde{C}_0 \parallel u - \tilde{u} \parallel^2_{L^2(\mathcal{M})}, \qquad (22)$$

where $\tilde{C}_0 = C_0 \parallel u - \tilde{u} \parallel_{L^\infty((0,T);L^2(\mathcal{M}))}$ and where $C_{1,p,\infty}$ is given in (8). If $p = 2$, we have that $V = H^1(\mathcal{M})$ and the continuous embedding $V \subset L^\sigma(\mathcal{M})$ for all $\sigma \in [1, \infty)$. So, for any $\sigma \in [1, \infty)$, we have

$$\int_{\mathcal{M}} |\nabla u(t) - \nabla \hat{u}(t)|^2 dA \geq \frac{1}{C_{1,2,\sigma}} \|u - \hat{u}\|^2_{L^\sigma(\mathcal{M})} - \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})}, \qquad (23)$$

where $C_{1,2,\sigma}$ is defined in (7). Since u and $\hat{u}$ are bounded weak solutions of (P), it is obvious that $u - \hat{u} \in L^\infty(\mathcal{M})$. This fact allows us to use the property

$$\parallel \cdot \parallel_{L^\infty(\mathcal{M})} = \lim_{\sigma \to \infty} \frac{\parallel \cdot \parallel_{L^\sigma(\mathcal{M})}}{|\mathcal{M}|^{1/\sigma}}\,.$$

That is, $\forall \epsilon > 0\ \exists \sigma_0 > 1$ such that $\forall \sigma > \sigma_0$, we have that

$$\parallel u - \hat{u} \parallel^2_{L^\infty(\mathcal{M})} \leq \frac{\parallel u - \hat{u} \parallel^2_{L^\sigma(\mathcal{M})}}{|\mathcal{M}|^{2/\sigma}} + \epsilon\,.$$

From the monotonicity of $\mathcal{G}$ we have

$$\int_{\mathcal{M}} \big(\mathcal{G}(u) - \mathcal{G}(\hat{u})\big)(u - \hat{u}) dA \ \geq\ 0\,.$$

Since u verifies the strong nondegeneracy property, we can apply Lemma 4 for $q = 1$ to get the estimate

$$\int_{\mathcal{M}} QS(z - \hat{z})(u - \hat{u}) dA \leq C_l Q \parallel S \parallel_{L^\infty(\mathcal{M})} \parallel u - \hat{u} \parallel^p_{L^\infty(\mathcal{M})}$$

where $C_l = (b_w - b_i)\hat{C}$. Inserting the above estimates into (21), we arrive at

$$\frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} \leq \Big(C_l Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \frac{C_0}{C_{1,p,\infty}} \Big) \parallel u - \hat{u} \parallel^p_{L^\infty(\mathcal{M})} + \tilde{C}_0 \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})}, \tag{24}$$

in the case $p > 2$, and obtain for the case $p = 2$,

$$\frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} \leq \Big(C_l Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \frac{|\mathcal{M}|^{2/\sigma}}{C_{1,2,\sigma}} \Big) \parallel u - \hat{u} \parallel^2_{L^\infty(\mathcal{M})} + \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} + \frac{\epsilon}{C_{1,2,\sigma}}, \tag{25}$$

Step 2. Now, we distinguish two cases,

CASE 1: if $C_l Q \parallel S \parallel_\infty - \dfrac{C_0}{C_{1,p,\infty}} \leq 0$ and $p > 2$, then

$$\frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} \leq \tilde{C}_0 \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})}.$$

Now, by Gronwall's Lemma, we deduce that

$$\parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} \leq e^{2\tilde{C}_0 t} \parallel u_0 - \hat{u}_0 \parallel^2_{L^2(\mathcal{M})} = 0.$$

This proves part $i)$ of Theorem 5 for this case. Now, for $p = 2$, if $C_l Q \parallel S \parallel_\infty - \dfrac{1}{C_{1,2,\sigma}} \leq 0$, arguing as before, we obtain that

$$\begin{aligned} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} &\leq e^{2t} \parallel u_0 - \hat{u}_0 \parallel^2_{L^2(\mathcal{M})} - \frac{2\epsilon}{C_{1,2,\sigma}}(e^{2t} - 1) \\ &\leq -2\epsilon C_l Q \parallel S \parallel_\infty (e^{2t} - 1). \end{aligned}$$

Finally, since the above inequality is true for all ϵ, we conclude the uniqueness.

CASE 2: if $C_l Q \parallel S \parallel_\infty - \frac{1}{C_{1,p,\infty}} > 0$, we consider a suitable rescaling ($\mathcal{M} \mapsto \mathcal{M}_\delta$). That is, a dilatation D of magnitude $\delta > 0$ on the manifold $(\mathcal{M}, \mathbf{g})$, $D : \mathcal{M} \subset \mathbb{R}^3 \to \mathbb{R}^3$, $D(x) = \tilde{x} = \delta x$. This dilatation allows us to define an atlas $\{(\tilde{W}_\lambda, \tilde{\mathbf{w}}_\lambda)\}$ on $\mathcal{M}_\delta$ as follows $\tilde{W}_\lambda := D(W_\lambda)$, $\tilde{w}_\lambda : \tilde{W}_\lambda \to \mathbb{R}^2$, $\tilde{w}_\lambda(\tilde{x}) = \mathbf{w}_\lambda(\frac{\tilde{x}}{\delta})$. We can consider a partition of unity $\{\tilde{\alpha}_\lambda\}$ subordinated to the covering $\{\tilde{W}_\lambda\}$ of $\mathcal{M}_\delta$ defined by $\tilde{\alpha}_\lambda(\tilde{x}) = \alpha_\lambda(\tilde{x}/\delta)$, where $\{\alpha_\lambda\}$ is a partition of unity subordinated to the covering

$\{W_\lambda\}$ of $\mathcal{M}$. In particular, we have that $\sup|\tilde{\alpha}_\lambda| = \sup|\alpha_\lambda|$. Moreover, D defines a metric $\tilde{\mathbf{g}}$ on $\mathcal{M}_\delta$, and its coefficients $\tilde{g}_{ij}$ depend on $\mathbf{g}$ in the following way: $\tilde{g}_{ij} = \delta^2 g_{ij}$. It follows that

$$\begin{array}{ccccc} \nu \parallel \psi \parallel^2 & \leq & \mathbf{g}(\psi,\psi) & \leq & \mu \parallel \psi \parallel^2 \quad \forall \psi \in T\mathcal{M} \\ \tilde{\nu} \parallel \tilde{\psi} \parallel^2 & \leq & \tilde{\mathbf{g}}(\tilde{\psi},\tilde{\psi}) & \leq & \tilde{\mu} \parallel \tilde{\psi} \parallel^2 \quad \forall \tilde{\psi} \in T\mathcal{M}_\delta \end{array}$$

where $\tilde{\nu} = \delta^2 \nu$ and $\tilde{\mu} = \delta^2 \mu$. Obviously $(\mathcal{M}_\delta, \tilde{\mathbf{g}})$ is also a Riemannian manifold of dimension 2.

Let u a real function defined on $\mathcal{M}$. Its local representation in the new coordinates is given by $\tilde{u} : \mathcal{M}_\delta \to \mathbb{R}$, $\tilde{u}(\tilde{x}) = u(\frac{\tilde{x}}{\delta})$. Moreover, the derivatives verify the following relation:

$$\frac{\partial \tilde{u}}{\partial \tilde{x}_i}(\tilde{x}) = \frac{1}{\delta}\frac{\partial u}{\partial x_i}\Big(\frac{\tilde{x}}{\delta}\Big) \qquad i = 1,2,3\,.$$

Now, the local representation of p-Laplacian operator, detailed in the above section, is given by

$$\delta^p \mathrm{div}_{\mathcal{M}_\delta}\left(|\nabla_{\mathcal{M}_\delta}\tilde{u}|^{p-2}\nabla_{\mathcal{M}_\delta}\tilde{u}\right) = \mathrm{div}_{\mathcal{M}}\left(|\nabla_{\mathcal{M}}u|^{p-2}\nabla_{\mathcal{M}}u\right).$$

So, the equation in the new coordinates is

$$(P_\delta)\begin{cases} \tilde{u}_t - \delta^p \mathrm{div}_{\mathcal{M}_\delta}\left(|\nabla_{\mathcal{M}_\delta}\tilde{u}|^{p-2}\nabla_{\mathcal{M}_\delta}\tilde{u}\right) + \mathcal{G}(\tilde{u}) \in QS\beta(\tilde{u}) + f \quad \text{in} \quad (0,T)\times\mathcal{M}_\delta \\ \tilde{u}(0,\tilde{x}) = u_0\Big(\frac{\tilde{x}}{\delta}\Big). \end{cases}$$

Clearly, if $\tilde{u}$ is a solution of (P_δ) then $u : \mathcal{M} \to \mathbb{R}$ defined by $u(x) = \tilde{u}(\delta x)$ is a solution of (P). Moreover, the uniqueness of (P_δ) implies the uniqueness of (P), and conversely. *Let us see that there exists $\delta > 0$ such that the solution of (P_δ) is unique.* Let u_δ and $\tilde{u}_\delta$ two solutions of (P_δ) such that u_δ verifies the strong nondegeneracy property. Arguing as in step 1, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}_\delta}\left|u_\delta(t) - \hat{u}_\delta(t)\right|^2 dA_\delta \\ &\quad + \delta^p \int_{\mathcal{M}_\delta} < |\nabla u_\delta|^{p-2}\nabla u_\delta - |\nabla \hat{u}_\delta|^{p-2}\nabla \hat{u}_\delta, \nabla u_\delta - \nabla \hat{u}_\delta > dA_\delta \\ &\quad + \int_{\mathcal{M}_\delta}\big(\mathcal{G}(u_\delta) - \mathcal{G}(\hat{u}_\delta)\big)(u_\delta - \hat{u}_\delta) dA_\delta = Q\int_{\mathcal{M}_\delta} S_\delta(z_\delta - \hat{z}_\delta)(u_\delta - \hat{u}_\delta) dA_\delta \end{aligned}$$

for some $z_\delta \in \beta(u_\delta)$ and $\hat{z}_\delta \in \beta(\hat{u}_\delta)$. Here, S_δ is defined by $S_\delta : \mathcal{M}_\delta \to \mathbb{R}$, $S_\delta(\tilde{x}) = S(\frac{\tilde{x}}{\delta})$. (24) and (25) allow us to estimate $u_\delta - \hat{u}_\delta$ for u_δ and $\hat{u}_\delta$ solutions of (P_δ). So, if $p > 2$,

$$\frac{1}{2}\frac{d}{dt} \parallel u_\delta - \hat{u}_\delta \parallel^2_{L^2(\mathcal{M}_\delta)} \\ \leq \Big(C_{l,\delta} Q \parallel S_\delta \parallel_{L^\infty(\mathcal{M}_\delta)} - \frac{C_0 \delta^p}{C_{1,p,\infty,\delta}}\Big) \parallel u_\delta - \hat{u}_\delta \parallel^p_{L^\infty(\mathcal{M}_\delta)} + \tilde{C}_0 \parallel u_\delta - \hat{u}_\delta \parallel^2_{L^2(\mathcal{M}_\delta)}, \quad (26)$$

and for $p = 2$,

$$\frac{1}{2}\frac{d}{dt} \parallel u_\delta - \hat{u}_\delta \parallel^2_{L^2(\mathcal{M}_\delta)} \leq \Big(C_{l,\delta} Q \parallel S_\delta \parallel_{L^\infty(\mathcal{M}_\delta)} - \frac{\delta^2 |\mathcal{M}_\delta|^{1/\sigma}}{C_{1,2,\sigma,\delta}}\Big) \parallel u_\delta - \hat{u}_\delta \parallel^2_{L^\infty(\mathcal{M}_\delta)} \\ + \parallel u_\delta - \hat{u}_\delta \parallel^2_{L^2(\mathcal{M}_\delta)} + \frac{\epsilon}{C_{1,2,\sigma,\delta}}. \quad (27)$$

Now, we are concerned with the dependence of the constants $C_{l,\delta}$, $C_{1,p,\infty,\delta}$ and $C_{1,2,\sigma,\delta}$ on δ. Let us consider the Banach space $V_\delta = \{u \in L^2(\mathcal{M}_\delta) : \nabla u \in L^p(T\mathcal{M}_\delta)\}$.

In order to estimate $C_{l,\delta}$, we study how the estimates of Lemma 4 for $q = 1$ change when we replace $\mathcal{M}$ with $\mathcal{M}_\delta$. We have that

$$\parallel z_\delta - \hat{z}_\delta \parallel^2_{L^1(\mathcal{M}_\delta)} \leq (b_w - b_i)\tilde{C}_\delta \parallel u_\delta - \hat{u}_\delta \parallel^{p-1}_{L^\infty(\mathcal{M}_\delta)}$$

where $\tilde{C}_\delta = \max\{C_\delta, \frac{|\mathcal{M}_\delta|}{\epsilon_0^{p-1}}\} = \delta^2 \max\{C, \frac{|\mathcal{M}|}{\epsilon_0^{p-1}}\} = \delta^2 \tilde{C}$ and C, C_δ are the constants of nondegeneracy for $\mathcal{M}$ and $\mathcal{M}_\delta$, respectively. So, $C_{l,\delta} = \delta^2 C_l$.

Now, we recall that $C_{1,2,\sigma,\delta}$ verifies

$$\parallel f \parallel^2_{L^\sigma(\mathcal{M}_\delta)} \leq C_{1,2,\sigma,\delta} \big(\parallel \nabla f \parallel^2_{L^2(T\mathcal{M}_\delta)} + \parallel f \parallel^2_{L^2(\mathcal{M}_\delta)} \big).$$

Since $\tilde{\nu} = \delta^2 \nu$, $\tilde{\mu} = \delta^2 \mu$ and $|\tilde{\alpha}_\lambda| = \frac{1}{\delta}|\alpha_\lambda|$, we get

$$C_{1,2,\sigma,\delta} = 2\delta^{4/\sigma - 2} \mu^{2/\sigma} k(r,p,\sigma)^2 \nu^{-1} \max\{1, \delta^2 \mu\} \Big(1 + \sup \frac{1}{\delta} |\nabla \alpha_\lambda|\Big)^2,$$

where k is a positive constant independent of δ.

The constant $C_{1,p,\infty,\delta}$, depends on the continuity constant for the embedding $V_\delta \subset L^\infty(\mathcal{M}_\delta)$. More precisely, if $\delta = 1$ and $p > 2$ we obtain the constant given in (8). Therefore,

$$C_{1,p,\infty,\delta} = 2^{p-1} k(p,r)^p \max\{\delta^{-p} \nu^{-p/2}, \delta^{p-2} \nu^{-1} \mu^{p/2}\} \Big(1 + \delta^{2/p - 1} \mu^{1/p} k(r,2,p) \nu^{-1/2} \\ \times \max\{1, \delta \mu^{1/2}\} \big(1 + \sup \frac{1}{\delta} |\nabla \alpha_\lambda|\big) \sup \frac{1}{\delta} |\nabla \alpha_\lambda|\Big)^p \max\{1, \delta^{p-2} |\mathcal{M}|^{(p-2)/2}\}.$$

Finally, we define the constant $\mathbf{K_{p,\delta}}$,

$$K_{p,\delta} \;:= \begin{cases} C_{l,\delta} Q \parallel S_\delta \parallel_{L^\infty(\mathcal{M}_\delta)} - \dfrac{\delta^2 |\mathcal{M}|^{2/\sigma}}{C_{1,2,\sigma,\delta}} & \text{if} \quad p = 2\,, \\ C_{l,\delta} Q \parallel S_\delta \parallel_{L^\infty(\mathcal{M}_\delta)} - \dfrac{\delta^p C_0}{C_{1,p,\infty,\delta}} & \text{if} \quad p > 2\,. \end{cases}$$

Clearly $\parallel S_\delta \parallel_{L^\infty(\mathcal{M}_\delta)} = \parallel S \parallel_{L^\infty(\mathcal{M})}$. So, observing the dependence of the above constants on δ, we have that if $p \geq 2$ then $\lim\limits_{\delta \to 0} K_{p,\delta} = 0$. This fact allows us to reduce the proof to case 1.

In order to prove part (ii) of Theorem 5, we assume that there exist two bounded weak solutions u and $\hat{u}$ of (P) which verify the weak nondegeneracy property. Arguing as in (i), we have that the term

$$\frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} + \frac{C_0}{C_{1,p,q}} \parallel u - \hat{u} \parallel^p_{L^\infty(\mathcal{M})}$$

is majorised by

$$\int_{\mathcal{M}} QS(z - \hat{z})(u - \hat{u})dA \;+\; \tilde{C}_0 \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})}$$

where $C_{1,p,q} = C_{1,p,\infty}$ if $p > 2$, $C_{1,2,\sigma}$ if $p = 2$ and where $\tilde{C}_0$ is defined in (22). Thus, using (ii) of Lemma 4, we have

$$\frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_2 \leq \left(C_d Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \frac{C_0}{C_{1,p,q}} \right) \parallel u - \hat{u} \parallel^p_\infty + \tilde{C}_0 \parallel u - \hat{u} \parallel^2_2$$

where C_d is the constant of the weak nondegeneracy property (Lemma 4). It follows the uniqueness as in (i), by studying the sign of the constant $C_d Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \dfrac{1}{C_{1,p,q}}$ and by rescaling when it is negative. □

4.3. A criterion of existence of nondegenerated solutions for the one-dimensional model

We are concerned with conditions for u_0 and f under which a nondegenerated solution exists. This questions allows different answers and can be formulated in a setting which is more general than that of problem (P). For $p = 2$ we have

Proposition 5

Let $w \in C^1((-1,1))$ such that there exists $\epsilon_0 > 0$ satisfying

(i) the set $\{x \in (-1,1) : \; |w(x) + 10| \leq \epsilon_0\}$ has a finite number of connected components I_j with $j = 1, .., N$ and for any j there exists $x_j \in I_j$ such that $w(x_j) = -10$,

(ii) there exists $\delta_0 > 0$ such that if $x \in I_j$ then $|w_x(x)| \geq \delta_0$.

Then w satisfies the strong nondegeneracy property.

Proof. Let $\epsilon \in (0, \epsilon_0)$ and $x \in \{y \in (-1,1) : |w(y)+10| \le \epsilon\}$, $x \in I_j$. We can apply the mean value theorem on the connected component I_j. There exist $x' \in I_j$ between x and x_j such that $w(x) - w(x_j) = w_x(x')(x - x_j)$. So, $|x - x_j| = \frac{|w(x)-w(x_j)|}{|w_x(x')|}$, and since $x' \in I_j$, $|w_x(x')| \ge \delta_0$, we conclude that

$$|x - x_j| \le \frac{|w(x)+10|}{\delta_0} \le k\epsilon \,,$$

for some $k > 0$. This means that $|I_j| \le 2k\epsilon$ and so $|\{x \in (-1,1) : |w(x)+10| \le \epsilon\}| \le 2Nk\epsilon$. □

Now, we study the case where the function w is a solution of (P_1). First, we approximate u by the solution u_ϵ of the problem

$$\begin{cases} u_t - ((1-x^2)|u_x|^{p-2}u_x)_x = QS(x)\beta_\epsilon(u) - \mathcal{G}_\epsilon(u) + f(x) & \text{in } (0,T)\times(-1,1) \\ (1-x^2)u_x = 0 & \text{in } x = -1 y x = 1 \\ u(x,0) = u_0(x) & \text{in } (-1,1)\,, \end{cases}$$

where β_ϵ and $\mathcal{G}_\epsilon$ are monotone approximations of β and $\mathcal{G}$ of class C^1. We assume, by simplicity, $f, u_0 \in C^1(-1,1)$. Let $|u_\epsilon| \le K$ on $(0,T)\times(-1,1)$. The function $v = u_x$ verifies

$$\begin{cases} v_t - ((1-x^2)|v|^{p-2}v)_{xx} \;=\; v(QS(x)\beta'_\epsilon(u) - \mathcal{G}_\epsilon(u)) + QS_x(x)\beta_\epsilon(u) + f_x(x) \\ (1-x^2)v = 0 \qquad\qquad \text{in } \; x = \pm 1 \\ v(x,0) = v_0(x) \qquad\qquad \text{in } (-1,1)\,. \end{cases}$$

Denote $a(t,x) := QS(x)\beta'_\epsilon(u) - \mathcal{G}_\epsilon(u)$ and $b(t,x) = QS_x(x)\beta_\epsilon(u) + f_x(x)$. Assume there exist $\underline{x}$ and $\overline{x}$ with $-1 < \underline{x} < \overline{x} < 1$ such that

$$(H_1)\begin{cases} S_x(x) \ge 0 \quad \forall x \in (-1,\underline{x})\,, \quad S_x(x) \le 0 \quad \forall x \in (\overline{x},1) \\ f_x(x) \ge 0 \quad \forall x \in (-1,\underline{x}), \quad f_x(x) \le 0 \quad \forall x \in (\overline{x},1) \\ u_{0x}(x) \ge 0 \quad \forall x \in (-1,\underline{x}), \quad u_{0x}(x) \le 0 \quad \forall x \in (\overline{x},1)\,. \end{cases}$$

We also assume

$$(H_2)\begin{cases} u_{\epsilon,x}(t,\underline{x}) \ge 0 \quad \forall t \in [0,T], \quad u_{\epsilon,x}(t,\overline{x}) \le 0 \quad \forall t \in [0,T]\,, \\ u_\epsilon(t,\underline{x}) > -10 \quad \forall t \in [0,T], \quad u_\epsilon(t,\overline{x}) < -10 \quad \forall t \in [0,T]\,. \end{cases}$$

Notice that the first condition of (H_2) is fulfilled if for instance $S(x)$, $f(x)$ and $u_0(x)$ are even functions and $\underline{x} = \overline{x} = 0$. Now, we call $w(t,x) := e^{-\lambda t} v(t,x)$. Then $v = we^{\lambda t}$ and

$$\begin{aligned} w_t =& \; e^{-\lambda t}\left(-\lambda v + ((1-x^2)|v|^{p-2}v)_{xx} - va(t,x) - b(t,x)\right) \\ =& \; -\lambda w + \left((1-x^2)e^{-\lambda t/p-1}|w|^{p-2}w\right)_{xx} - aw - be^{-\lambda t} \\ \geq& \; \left((1-x^2)e^{-\lambda t/p-1}|w|^{p-2}w\right)_{xx} + be^{-\lambda t}\,, \end{aligned}$$

where λ satisfies

$$-\lambda - a(t,x) \;\geq\; 0 \qquad \text{i.e. } \lambda \leq -\sup\, a(t,x).$$

We recall that since $|u_\epsilon| \leq K$, there exists such λ.

Lemma 5

Let $\underline{h}(t,x)$ *verify*

$$\begin{cases} h_t - \left((1-x^2)e^{-\lambda t/p-1}|h|^{p-2}h\right)_{xx} = be^{-\lambda t} & \text{in } (0,T)\times(-1,\underline{x})\,, \\ h(t,\underline{x}) = -10 & \forall t \in (0,T)\,, \\ h(x,0) = u_{0x}(x) & \text{if } x \in (-1,\underline{x})\,. \end{cases}$$

Then

$$u_{\epsilon,x}(t,x) \;\geq\; e^{\lambda t}\underline{h}(t,x) \geq 0 \qquad \forall (t,x) \in (0,T)\times(-1,\underline{x}). \;\square$$

Proof. It suffices to apply the comparison principle to h and $w = e^{-\lambda t}u_{\epsilon,x}$ (which is an upper solution). Notice that the elliptic second order operator associated is T-accretive in $L^1(-1,\underline{x})$ (hence this is a little variation of the results of (Benilan, 1972)). □

Similarly, we can obtain that

$$u_x(t,x) \;\leq\; -e^{\lambda t}\overline{h}(t,x) \leq 0 \qquad \forall (t,x) \in (0,T)\times(\overline{x},1),$$

for suitable λ and $\overline{h}$. These two results imply the condition (ii) of Proposition 5. Now, we assume that $u(t,x)$ satisfies condition (i) and we also assume

$$(H_3) \qquad\qquad u(t,x) \text{ satisfies condition (i) } \forall t \in [0,T]\,.$$

In view of the known results in the literature, it seems not difficult to prove that hypothesis (H_3) is fulfilled if the data $u_0(x)$ and $f(x)$ pass only finitely many times through -10 and $-10 < u(t,x)\ \forall (t,x) \in (0,T)\times(\underline{x},\overline{x})$. □

The conclusion of section 3.4 is that if (H_1), (H_2) and (H_3) are fulfilled, then the solution u, limit of u_ϵ is nondegenerated. Finally, we have the result for two-dimensional solutions by rotation about the axis passing through the poles.

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