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Strictly and uniformly monotone sequential Musielak-Orlicz spaces

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Abstract

Characterizations of various monotonicity properties (UM, LUM, STM) in the sequential Orlicz spaces $l_\phi\, h_\phi$, for the Luxemburg norm $\|\cdot\|_\phi$, are considered (cf. [23]). It follows that the strong monotonicity (STM) and the uniform monotonicity (UM) do not coincide in general for the counting measure case. Some applications to (dominated) best approximation are also given.

1. Preliminaries

Let $l_\phi = l_\phi(\mu)$ be a sequential Orlicz space for the counting measure μ ([16], [17], [15], [27], [31], [11]). In [23] (Theorem 2.7), for μ nonatomic, it was proved that the strict monotonicity (STM) and the uniform monotonicity (UM) (and hence all the intermediate monotonicity properties) coincide for the Luxemburg norm (the case of the Orlicz norm is considered in [14]). For the counting measure such equivalence is not longer true. It is proved below for l_ϕ that the monotonicity properties UM, LUM, CWLUM, H_+STM and STM (cf. [23]) fall into two groups and that within these two groups they are equivalent each to other. We proceed by characterizations of these properties. The methods we apply in this paper differ essentially from that for the nonatomic case ([23]).

We begin with brief recalling of some basic definitions. A Banach lattice X, with the positive cone X_+, is called uniformly monotone (UM) (cf. [3], Chap. XV,

[23], [14], [1], [4], [5], [6], [26]), if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$, such that $\|f - g\| \leq 1 - \delta(\epsilon)$, whenever $f \geq g \geq 0, \|f\| = 1$ and $\|g\| \geq \epsilon$. Equivalently, for all sequences (f_n), (g_n) in X_+ with $\|f_n\| = 1$ and $f_n \geq g_n$ there holds $\|g_n\| \longrightarrow 0$ whenever $\|f_n - g_n\| \longrightarrow 1$. X is said to be a strictly monotone space (STM), if for all $f, g \in X_+$ such that $f \geq g$ there holds $\|f - g\| < \|f\|$ whenever $\|g\| > 0$. The STM and UM properties can be viewed as the rotundity (R) and the uniform rotundity (UR) restricted to compatible elements in X_+ ([23], [14]). If in the definition of the UM property $f_n = f$ for all n, or we look for $l(g_n) \longrightarrow 0$ as $n \longrightarrow +\infty$ instead of $\|g_n\| \longrightarrow 0$, for some positive functional l, we arrive to the concept of the local uniform monotonicity (LUM), or to the the weak uniform monotonicity (WUM), respectively. If both these properties take place simultaneously then we get the weak local uniform property (WLUM). Further, if for all positive functionals $l \in S(X^*)$ (S - the unit sphere), $f_n \in S(X_+)$ and $g_n \in X_+$ with $f_n \geq g_n$, there holds $\|g_n\| \longrightarrow 0$ whenever $l(f_n - g_n) \longrightarrow 1$, then X is said to be a CWUM space (i.e. weakly uniformly monotone in different sense; cf. [19] for an analogy with the CWUR property studied originally under a different name in [8], [30]). If in this definition $f = f_n$ for all n then X is said to be CWLUM space (localization of CWUM). Finally, X is said to have a H_+ property ([23]), if $f \geq g_n \geq 0, \|f\| = 1$ and the weak-$*$ convergence $g_n \longrightarrow f$ imply the norm convergence $\|f - g_n\|_\phi \longrightarrow 0$. From the definitions it follows that UM $\Rightarrow$ LUM $\Rightarrow$ WLUM, LUM $\Rightarrow$ CWLUM $\Rightarrow$ H_+ and STM, UM $\Rightarrow$ WUM $\Rightarrow$ WLUM $\Rightarrow$ STM, UM $\Rightarrow$ CWUM $\Rightarrow$ CWLUM. Let us point out that H_+ always implies the order continuity in Banach lattice X and that CWLUM is equivalent to H_+STM (i.e. H_+ and STM) (cf. [23], [14]). In the paper, applying these implications, all the mentioned properties (except the property WUM) are characterized directly in terms of the function ϕ for Musielak-Orlicz spaces l_ϕ.

Sequential Musielak-Orlicz spaces l_ϕ consist of all (real) sequences $f = (x_n)$ satisfying $I_\phi(\alpha f) = \Sigma_{n=1}^{+\infty} \phi_n(\alpha |x_n|) < +\infty$ for some $\alpha > 0$ depending on f. By ϕ we mean a function $\phi(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{N} \longrightarrow \overline{\mathbb{R}}_+$, or a sequence $\phi = (\phi_n(\cdot))$ of functions $\phi_n(\cdot) = \phi(\cdot, n)$ $(n \in \mathbb{N})$, such that $\phi_n(0) = 0$, $\phi_n(\cdot)$ are nontrivial, convex, lsc and continuous at zero. We will write $\phi < +\infty$ (resp. $0 < \phi$), if $\phi_n(r) < +\infty$ (resp. $0 < \phi_n(r)$) for all $n \in \mathbb{N}$) and for all $r > 0$.

The space h_ϕ consists of all sequences $f = (x_n)$ such that $I_\phi(\alpha f) < +\infty$ for all $\alpha > 0$ (cf. [11] for different approach concerning the definition of h_ϕ). Further, let l_ϕ^a be a subspace of l_ϕ on which the Luxemburg norm $\|f\|_\phi = \inf\{\alpha > 0 : I_\phi(f/\alpha) \leq 1\}$ is absolutely continuous, i.e.

$$l_\phi^a = \{f \in l_\phi : f \geq f_m \geq 0, f_m \searrow 0 \;\;\Rightarrow\;\; \|f_m\|_\phi \to 0\}.$$

From the definitions it follows that $h_\phi \subset l^a_\phi \subset l_\phi$ ([31], [16], [2]).

Following [16] ϕ is said to satisfy a δ^0_2-condition, if there exist $\gamma, \delta > 0$, a nonnegative sequence (α_n) and a natural number k, for which $\Sigma^{+\infty}_{n=k}\alpha_n < +\infty$, such that for all $n \in \mathbb{N}$ and $r > 0$ satisfying $\phi_n(r) \leq \delta$ there holds $\phi_n(2r) \leq \gamma\, \phi_n(r) + \alpha_n$ $(n \in \mathbb{N})$. We will then write $\phi \in \delta^0_2$ or $\phi \in \delta_2$ whenever in the above definition we have $k = 1$. If $\phi < +\infty$, or ϕ does not depend on n, the δ^0_2-condition and the δ_2-condition coincide. The role of the δ^0_2-condition explain the following results: (a) $l^a_\phi = l_\phi$ if and only if $\phi \in \delta^0_2$(see Theorem 2.6 below), (b) $h_\phi = l_\phi$ if and only if $\phi \in\delta^0_2$ and $\phi < +\infty$, [17]). Let us point out that: (c) $h_\phi = l^a_\phi$ if and only if $\phi < +\infty$ ([31]). In [11] it was proved that $l^a_\phi = cl_{\|\cdot\|_\phi}\{e_n\}$ (- the closure in l_ϕ). Thus the spaces h_ϕ and l^a_ϕ are different in general except the case when $\phi < +\infty$.

Remark. It can be verified that $\mathrm{supp}(h_\phi) = \mathbb{N}$ if and only if $\phi < +\infty$. If $\mathrm{supp}(h_\phi) \neq \mathbb{N}$, we can confine to all these functions ϕ_n with n in some subset $\mathbb{N}_0$ of $\mathbb{N}$ which are finite. Therefore, without loss of generality, one can assume that $\phi < +\infty$ whenever the space h_ϕ is considered.

Following [16] ϕ is said to satisfy a $(\star)$-condition, if for each $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that for all $x > 0$ and $n \in \mathbb{N}$ there holds $\phi_n(x) > 1 - \epsilon$ whenever $\phi_n((1 + \delta)x) > 1$. Let us introduce a weaker condition then the $(\star)$-condition. Let (r_n) be a positive sequence satisfying $\phi_n(r_n) = 1$ $(n \in \mathbb{N})$. If such a sequence exists, we will write $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$. We say that ϕ satisfies a $(\star\star)$-condition, if any of the following (equivalent) conditions is satisfied:

1. For each sequence (θ_n) in $(0, 1]$ converging to 1 there holds
$$\phi_n(\theta_n r_n) \longrightarrow 1 \text{ whenever } n \longrightarrow +\infty.$$
2. For each $\epsilon \in (0, 1)$ there holds $\eta_\phi(\epsilon) > 0$, where
$$\eta_\phi(\epsilon) = \inf_n \inf_{r:\phi_n(r)\leq 1-\epsilon} \left(1 - \frac{r}{r_n}\right).$$

We leave a simple proof of this equivalence. From 2 it follows that the condition $(\star\star)$ means that all the values $r > 0$ such that $y = \phi_n(r)$ remain below the horizontal line $y = 1 - \epsilon$ must be uniformly (with respect to n) far from the vertical lines at $x = r_n$ in the sense that $r/r_n \leq \eta_\phi(\epsilon)$. If $\phi_n(s) = s$ for all n then $\eta_\phi(\epsilon) = \epsilon$.

Lemma 1.1

For the function ϕ the following statements are equivalent.
(a) ϕ satisfies the condition $(\star)$.
(b) (i) $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$,
(ii) ϕ satisfies the condition $(\star\star)$.

Proof. (a) $\Rightarrow$ (b)(i). If $1 \notin \phi_n[\mathbb{R}_+]$ for some $n \in \mathbb{N}$, then $\phi_n(r_0) < 1$ where $r_0 = \sup\{r > 0 : \phi_n(r) \leq 1\}$ and $r_0 > 0$. Hence, for some $\epsilon > 0$, we have $\phi_n(r_0) \leq 1 - \epsilon$. From (a) it follows that there exists a $\delta > 0$ such that $\phi_n((1+\delta)r_0) \leq 1$, a contradiction with the choice of r_0.

(a) $\Rightarrow$ (b)(ii). If this implication is not true then there exists a sequence (θ_n) such that $0 < \theta_n \leq 1$ and $\theta_n \longrightarrow 1$ with $\phi_n(\theta_n r_n) \leq 1 - \epsilon$ for some $\epsilon \in (0, 1)$. Let n be large enough such that $\theta_n > 1/(1+\delta)$, where $\delta \in (0, 1)$ is from the assumption (a). Then, $1 < (1+\delta)\theta_n \leq \phi_n((1+\delta)\theta_n r_n) \leq 1 - \epsilon$, a contradiction.

(b) $\Rightarrow$ (a). If not, there exist $\epsilon > 0$, a subsequence (n_m) and $0 < x_m$ such that $\phi_{n_m}((1 + 1/m)x_m) > 1$ and $\phi_{n_m}(x_m) \leq 1 - \epsilon$. We can confine ourselves to x_m satisfying $0 < x_m < r_{n_m}$ where $\phi_{n_m}(r_{n_m}) = 1$. Then, $x_m < r_{n_m} < (1 + 1/m)x_m$. Hence, $\delta_m = x_m/r_{n_m} \longrightarrow 1$ with $\delta_m < 1$, for $m \longrightarrow +\infty$. From the assumption (b) it follows that $\phi_{n_m}(\delta_m r_{n_m}) \longrightarrow 1$. On the other hand $\phi_{n_m}(\delta_m r_{n_m}) = \phi_{n_m}(x_m) \leq 1-\epsilon$, a contradiction. □

For the sake of completeness we recall some basic results concerning the sequential Musielak-Orlicz spaces.

Lemma 1.2 ([15])

The following statements are equivalent.
(a) $\|f\|_\phi = 1$ *implies that* $I_\phi(f) = 1$, *for all* $f \in l_\phi$.
(b) (i) The function ϕ *satisfies the* δ_2^0*-condition,*
(ii) $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$.

Let us point out that (b)(ii) is automatically satisfied whenever $\phi < +\infty$, i.e. $\phi_n(r)$ is finite for all $r > 0$ $(n \in \mathbb{N})$.

We will write $\xi_n \uparrow 1$ whenever $\xi_n \leq 1$ and $\xi_n \longrightarrow 1$ for $m \longrightarrow +\infty$. The following lemma is essentially due to Kamińska ([16]) under the assumption that $\phi < +\infty$ and $\phi \in \delta_2$.

Lemma 1.3

The following statements are equivalent.
(a) $\|f_m\|_\phi \uparrow 1$ *implies that* $I_\phi(f_m) \uparrow 1$, *for sequences* $(f_m) \in l_\phi$.
(b) For each $\epsilon \in (0, 1)$ *there exists* $\delta \in (0, 1)$ *such that for all* $f \in l_\phi$ *satisfying* $\|f\|_\phi \geq 1 - \delta$ *there holds* $I_\phi(f) \geq 1 - \epsilon$.
(c) (i) ϕ *satisfies the* δ_2^0*-condition,*
(ii) $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$,
(iii) ϕ *satisfies the condition* $(\star\star)$.

Remarks. The equivalence (a) $\Leftrightarrow$ (b) is clear. The implications (a) $\Rightarrow$ (c)(i), (a) $\Rightarrow$ (c)(ii) follow from Lemma 1.2. By virtue of Lemma 1.1 the conditions (c)(ii), (c)(iii) are equivalent to the condition ($\star$). Applying Lemma 9 from [16] we obtain that $(b) \iff (\star)$ under the δ_2-condition. It can be verified that this lemma is still true for $\phi \in \delta_2^0$. Thus the δ_2-condition can be replaced by the δ_2^0-condition and (c) $\Rightarrow$ (b) as desired.

Lemma 1.4

The following statements hold true for the space h_ϕ.
(a) $\|f_m\|_\phi \uparrow 1$ implies that $I_\phi(f_m) \uparrow 1$, for sequences $f = (f_m)$ satisfying $0 \le |f_m| \le f$, where $f \in h_\phi$.
(b) h_ϕ does not contain an isomorphic copy of l_∞.
In particular, if $\|f\|_\phi = 1$, then $I_\phi(f) = 1$.

Proof. Since, by the definition, we have $I_\phi(\cdot) < +\infty$ on h_ϕ and moreover $I_\phi(f) \le \|f\|_\phi \le 1$, we conclude that $I_\phi(\cdot)$ is continuous at zero and consequently $I_\phi(\cdot)$ is continuous on h_ϕ. Hence, the last assertion of the lemma follows.

Next, let $\|f\|_\phi \uparrow 1$ and $\alpha_m = 1/\|f_m\|_\phi$. Then, $1 = I_\phi(\alpha_m f_m) = I_\phi((\alpha_m - 1)2f_m + (2-\alpha_m)f_m) \le (\alpha_m - 1)\ I_\phi(2f) + (2-\alpha_m)\ I_\phi(f_m)$. If for some subsequence we have $I_\phi(f_{m_k}) \le 1 - \epsilon$, for some $\epsilon \in (0,1)$, then we arrive to a contradiction, since $I_\phi(2f) < +\infty$. Thus $I_\phi(f_m) \uparrow 1$ and (a) follows.

We have $h_\phi \subset l_\phi$ as a closed sublattice. Thus h_ϕ possess a lattice norm which is order continuous. Therefore, by virtue of the well known result concerning general Banach lattices, h_ϕ does not contain an isomorphic copy of l_∞ ([20]) which proves (b) (cf. [11] and [17] for further results concerning copies of c_0 and l_∞ in l_ϕ). $\Box$

Remark. In the above lemma as well as in Lemma 1.6 we need not assume that $\phi < +\infty$.

Lemma 1.5 ([16])

The following statements are equivalent.
(a) For each $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that for all $f \in l_\phi$ satisfying $\|f\|_\phi \ge \epsilon$ there holds $I_\phi(f) \ge \eta(\epsilon)$; in other words the norm and the modular convergence coincide.
(b) ϕ satisfies the δ_2^0-condition and $\phi > 0$.

For the space h_ϕ one can prove a similar result.

Lemma 1.6

Let $\phi > 0$. If $f \geq g_m \geq 0$, where f, $g_m \in h_\phi$, then $I_\phi(g_m) \longrightarrow 0$ implies $\|g_m\|_\phi \longrightarrow 0$.

Proof. Assume to the contrary, that $\alpha_m = \|g_m\|_\phi \geq \epsilon$ for some $\epsilon \in (0,1]$, where $\alpha_m \leq 1$ (for the sake of simplicity we do not pass to the subsequence). From Lemma 1.4 it follows that $1 = I_\phi(g_m/\alpha_m)$. Moreover, $I_\phi(g_m/\alpha_m) \leq (1/\alpha_m - 1)\ I_\phi(2g_m) + (2 - 1/\alpha_m)\ I_\phi(g_m)$. Since $I_\phi(g_m) \longrightarrow 0$ and $\phi > 0$, we obtain that $g_{m_k} \longrightarrow 0$, $\mu-$ coordinatewise, with $0 \leq g_{m_k} \leq f$. Since $I_\phi(2f) < +\infty$, we conclude from the Lebesgue convergence theorem that $I_\phi(2g_{m_k}) \longrightarrow 0$. Now, since $\alpha_m \in (\epsilon, \|f\|_\phi]$, the above inequalities lead to a contradiction which finishes the proof. □

We call the modular $I_\phi(\cdot)$ $\phi-$uniformly monotone (resp. uniformly monotone), if for each $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that $f \geq g \geq 0$ in l_ϕ with $I_\phi(f) = 1$ and $I_\phi(g) \geq \epsilon$ (resp. with $\|f\|_\phi = 1$ and $\|g\|_\phi \geq \epsilon$) imply that $I_\phi(f - g) \leq 1 - \eta(\epsilon)$. Also $I_\phi(\cdot)$ is said to be $\phi-$strictly monotone (resp. strictly monotone), if $f \geq g \geq 0$ in l_ϕ with $I_\phi(f) = 1$ and $I_\phi(g) > 0$ (resp. $\|f\|_\phi = 1$ and $\|g\|_\phi > 0$) imply that $I_\phi(f - g) < 1$. We will write shortly that $I_\phi(\cdot)$ is a $\phi-$UM, UM, $\phi-$STM and STM modular, respectively. Let us point out that all these properties can be considered relatively to h_ϕ as well. Clearly, each UM modular is $\phi-$UM and therefore $\phi-$STM. Also, each STM modular $I_\phi(\cdot)$ is $\phi-$STM. However, thanks to the subadditivity of $I_\phi(\cdot)$ on the positive cone $(l_\phi)_+$, we have the following lemma ([23]), Proposition 1.4).

Lemma 1.7

The modular $I_\phi(\cdot)$ is always $\phi-$UM with $\eta(\epsilon) = \epsilon$. Consequently, the notions of $\phi-$UM and $\phi-$STM modulars coincide.

2. Main results

In the following it will be assumed that $\phi < +\infty$ whenever the subspace h_ϕ of l_ϕ is considered (in this case $h_\phi = l^a_\phi$). We begin with the following lemma.

Lemma 2.1

Let l_ϕ be an STM space. Then the following statements hold true.
(a) $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$,
(b) $\phi > 0$,
(c) ϕ *satisfies the* δ_2^0*-condition.*

If h_ϕ is an STM space (for h_ϕ we assume that $\phi < +\infty$) then, ϕ satisfy the conditions (a) and (b).

Proof. To prove (a) let us assume for a moment that $1 \notin \phi_n[\mathbb{R}_+]$ for some n. Applying the lsc of ϕ we get $\phi_n(\xi_0 x) < 1$ and $\xi_0 > 0$ where $\xi_0 = \sup\{r > 0 : \phi_n(rx) < 1\}$ and $x > 0$ is fixed but arbitrary. For $n \neq m$ there exists $y > 0$ such that $\phi_m(y) < 1 - \phi_n(\xi_0 x)$. Then, define $f = \xi_0 x e_n$ and $g = y e_n$. Clearly, $f,\, g \geq 0$. Since $I_\phi(f) < 1$ and $I_\phi(f/\lambda) > 1$ for $\lambda > 1$, we conclude that $\|f\|_\phi = 1$. On the other hand $I_\phi(f+g) = \phi_n(\xi_0 x) + \phi_n(y) < 1$ and $I_\phi((f+g)/\lambda) > 1$ whenever $\lambda < 1$. Hence it follows that $\|f+g\|_\phi = \|f\|_\phi = 1$. Now, applying the STM of l_ϕ we obtain $g = 0$, a contradiction.

To prove (b) let us notice that if there exists n and $x > 0$ such that $\phi_n(x) = 0$, then choose a sequence $h = (0, x_2, ...) \in l_\phi$ with $x_i \geq 0$ and $\|h\|_\phi = 1$ and a sequence $g = (x, 0, ...)$. Let $f = h + g$. Then, $I_\phi(f) = I_\phi(f - g) = 1$ where $f \geq g > 0$, a contradiction with the assumption that l_ϕ is STM.

The same reasoning applies for the space h_ϕ, so we omit the proof of (b) in this case. Let us point out that (a) is a consequence of the assumption $\phi < +\infty$.

To get (c), we will prove first that $\|h\|_\phi = 1$ implies $I_\phi(h) = 1$ in l_ϕ. Then, the proof will be completed by virtue of lemma 1.2. For this do assume that there exists $h = (x_n) \in l_\phi$ such that $\|h\|_\phi = 1$ and $I_\phi(h) < 1$. Without loss of generality let $h \geq 0$ and let $x_m > 0$ for some $m \in \mathbb{N}$. Applying (a) we have $\phi_m(\xi x_m) = 1$ for some $\xi > 0$. Thus the following function $t \longrightarrow \phi_m(t x_m)$ is continuous on $[0, \xi]$ and therefore there exists $t_0 \in (0, \xi)$ such that $\phi_m(t_0 x_m) + \Sigma_{n \neq m} \phi_n(x_n) = 1$ and $t_0 > 0$. Define $g = (t_0 - 1) x_m e_m$ and let $f = h + g$. Then $I_\phi(f) = 1$ and therefore $\|f\|_\phi = 1$. Moreover $\|f - g\|_\phi = 1$ with $f \geq g > 0$. On the other hand, by our assumption, $g = 0$, a contradiction with the choice of x_m. Now, applying Lemma 1.2, (c) follows. □

Remark. Since $\phi < +\infty$, in proof of (b), one can referee to Theorem 1.1 in [17] concerning (order) isometric copies of l_∞ in l_ϕ in the case when $\phi \notin \delta_2^0$.

Lemma 2.2

Let l_ϕ be a UM space. Then, the condition $(\star\star)$ is satisfied.

Proof. By virtue of Lemma 2.1 there exists a positive sequence (r_n) such that $\phi_n(r_n) = 1$. If the condition $(\star\star)$ is not satisfied, then there exists $\epsilon \in (0,1)$ and a sequence (θ_n) in the interval $(0,1)$ such that $\theta_n \longrightarrow 1$ and $\phi_n(\theta_n r_n) \leq 1 - \epsilon$ (we omit passing to the subsequence). Choose (s_n) such that $1 = \phi_{2n}(\theta_{2n} r_{2n}) + \phi_{2n+1}(s_{2n+1})$. Clearly $\phi_{2n+1}(s_{2n+1}) \geq \epsilon$. Next, define sequences (f_n), (g_n) in l_ϕ where $f_n = \theta_{2n} r_{2n} e_{2n} + s_{2n+1} e_{2n+1}$ and $g_n = (1 - \theta_{2n}) r_{2n} e_{2n} + s_{2n+1} e_{2n+1}$. We assume that n is large enough such that $\theta_n > 1/2$. Then, $I_\phi(f_n) = 1$ and hence $\|f_n\|_\phi = 1$. Also, $0 \leq g_n \leq f_n$ for n large. Moreover, $I_\phi(f_n - g_n) = \phi_{2n}(r_{2n}) = 1$, so that $\|f_n - g_n\|_\phi = 1$. Finally, since $I_\phi(g_n) = \phi_{2n+1}(s_{2n+1}) \geq \epsilon$, we have $\|g_n\|_\phi \geq \epsilon$, a contradiction since l_ϕ is a UM space. □

The property STM (and hence the local properties LUM, WLUM, CWLUM; Theorem 2.4 below) and the property UM for the space l_ϕ can be expressed in terms of the respective properties of the modular $I_\phi(\cdot)$.

Theorem 2.3

The following, respective, pairs ((a), (b)) *of statements are equivalent.*
(a) l_ϕ *is an STM space (resp.* l_ϕ *is an UM space).*
(b) (i) $I_\phi(\cdot)$ *is an STM modular (resp.* $I_\phi(\cdot)$ *is a UM modular),*
(ii) $\|f\|_\phi = 1$ *implies* $I_\phi(f) = 1$ *(resp.* $\|f_n\|_\phi \uparrow 1$ *implies* $I_\phi(f_n) \uparrow 1$*).*

Proof. (The STM case). Implications (b) $\Rightarrow$ (a) and (a) $\Rightarrow$ (b)(i) follow from the definitions. We will prove that (a) $\Rightarrow$ (b)(ii). If l_ϕ is STM then, by virtue of Lemma 2.1, $1 \in \phi_n[\mathbb{R}_+]$, $\phi > 0$ and $\phi \in \delta_2^0$. In turn, applying Lemma 1.2, we get that (b)(ii) is satisfied as desired.

(The UM case). The implication (b) $\Rightarrow$ (a) follow immediately from the definitions. To prove (a) $\Rightarrow$ (b)(i) we apply the corresponding implication for the STM case. Consequently, $\|f\|_\phi = 1$ implies $I_\phi(f) = 1$ and, by virtue of Lemma 2.1, $\phi > 0$. Thus, by Lemma 1.7, $I_\phi(\cdot)$ is $\phi-$UM and consequently UM. To prove (a) $\Rightarrow$ (b)(ii) we apply that by virtue of Lemma 2.2 ϕ satisfies the condition ($\star\star$). Again, since each UM space is an STM space we conclude that (Lemma 2.1) ϕ satisfies the δ_2^0-condition and $1 \in \phi_n[\mathbb{R}_+]$ ($n \in \mathbb{N}$). To finish the proof it suffices to apply Lemma 1.3 ((c) $\Rightarrow$ (a)). $\square$

In theorems below characterizations of the STM, LUM, WLUM, CWLUM and UM Musielak-Orlicz spaces $l_\phi$in terms of the function ϕ are given.

Theorem 2.4

Let us consider the following statements (a), (b) and (c) (it is not necessary that $\phi < +\infty$*).*
(a) l_ϕ *is a LUM space.*
(b) l_ϕ *is an STM space.*
(c) (i) $\phi > 0$,
(ii) $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$,
(iii) ϕ *satisfies the* δ_2^0*-condition.*
Then, (a) $\Rightarrow$ *(b) and (b)* $\iff$ *(c). If* $\phi < +\infty$ *then also (b)* $\Rightarrow$ *(a) and in this case the properties LUM, WLUM, CWLUM, STM for* l_ϕ *coincide.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the definitions. Next, the implications (b) $\Rightarrow$ (c)(i)(ii)(iii) follow from Lemma 2.1.

We will prove that (c) $\Rightarrow$ (b). Let $f \geq g \geq 0, \|f\|_\phi = \|f-g\|_\phi$ and let us assume for a moment that $\|g\|_\phi > 0$. From (c)(ii),(iii) and Lemma 1.2 it follows that $I_\phi(f) = 1$ and $I_\phi(f-g) = 1$. To finish the proof we apply that each modular is $\phi-$STM (Lemma 1.7). Thus $I_\phi(g) = 0$ and consequently, by virtue of the condition (c)(i), $g = 0$, a contradiction.

To prove that in the case $\phi < +\infty$ (c) $\Rightarrow$ (a), let us assume that (a) does not hold true, i.e. there exist $\epsilon > 0$, $f \in l_\phi$ with $\|f\|_\phi = 1$ and a sequence (g_n) in l_ϕ satisfying $f \geq g_n \geq 0$ with $\|g_n\|_\phi \geq \epsilon$, such that $\|f - g_n\|_\phi \longrightarrow 1$. First, we need to prove that $I_\phi(f-g_n) \longrightarrow 1$. Since $\phi < +\infty$, and ϕ satisfies the δ_2^0-condition, we conclude that l_ϕ coincide with h_ϕ (see section 1). Therefore, we can apply Lemma 1.4. Hence, it follows that $I_\phi(f-g_n) \longrightarrow 1$ as desired. Next, applying Lemma 1.6 (cf. also Lemma 1.5), by virtue of (c)(i)(iii), we have $I_\phi(g_n) \geq \eta(\epsilon)$ for some $\eta(\epsilon) > 0$. In h_ϕ (cf. also Lemma 1.2), we always have $I_\phi(f) = 1$. On the other hand, from Lemma 1.7, we know that $I_\phi(\cdot)$ is automatically ϕ-UM and hence ϕ-LUM, i.e. $I_\phi(f) = 1,\ f \geq g_n \geq 0$ with $I_\phi(f-g_n) \longrightarrow 1$ imply that $I_\phi(g_n) \longrightarrow 0$. Thus, we arrived to the contradiction. Thus (c) $\Rightarrow$ (a) and the proof is finished. □

From Theorem 2.5 below it will be seen that the LUM and UM properties for the Musielak-Orlicz spaces l_ϕ does not coincide in general. In the case of nonatomic measure space the situation is quite different (cf. [23], Theorem 2.7).

Theorem 2.5

The following statements are equivalent.

(a) l_ϕ *is a UM space.*

(b) (i) $\phi > 0$,

(ii) $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$,

(iii) ϕ *satisfies the condition* $(\star\star)$,

(iv) ϕ *satisfies the condition* δ_2^0 *or, equivalently, the norm* $\|\cdot\|_\phi$ *is* σ*-order continuous.*

Proof. Since the UM property implies the STM property, the implications (a) $\Rightarrow$ (b)(i)(ii)(iii) follow in the same way as in the proof of Theorem 2.4, i.e. we apply Lemma 2.1. Moreover, from Lemma 2.2 it follows that (a) $\Rightarrow$ (b)(iii). Thus (a) $\Rightarrow$ (b).

To prove that (b) $\Rightarrow$ (a) let $f_n \geq g_n \geq 0, \|f_n\|_\phi = 1$ and $\|f_n - g_n\|_\phi \longrightarrow 1$ as $n \longrightarrow +\infty$. From Lemma 1.2 and Lemma 1.3 it follows that $I_\phi(f_n) = 1$ and $I_\phi(f_n - g_n) \longrightarrow 1$. Since each modular $I_\phi(\cdot)$ is $\phi-$UM (Lemma 1.7) we obtain that $I_\phi(g_n) \longrightarrow 0$. Applying Lemma 1.5, we conclude that $\|g_n\|_\phi \longrightarrow 0$, i.e. (a) follows whenever ϕ satisfies the δ_2^0-condition or, equivalently, $\|\cdot\|_\phi$ is order continuous (cf. Theorem 2.6 below). □

Remark. From the above characterization and the nature of the $(\star\star)$-condition it follows that the STM and the UM property coincide for sequential Musielak-Orlicz spaces l_ϕ whenever $\phi_n(\cdot)$ does not depend on n.

The following proposition seems to be well known, however we do not know any reference to a complete proof in the case under consideration. We apply in the proof (in essential way) ideas from [15].

Theorem 2.6

The following statements are equivalent.
(a) $l_\phi = l_\phi^a$, i.e. the norm $\|\cdot\|_\phi$ is $(\sigma-)$order continuous.
(b) ϕ satisfies the δ_2^0-condition.

Proof. (b) $\Rightarrow$ (a). Let $0 \leq f_k \leq f$, $(f_k) \downarrow 0$ where f, $f_k = (x_n^k) \in l_\phi$. Without loss of generality we can assume that $I_\phi(f) = 1$. We will prove that $I_\phi(\xi f_k) \longrightarrow 0$, for $k \longrightarrow +\infty$. Let ξ, $\epsilon > 0$ be arbitrary. Let β, $\gamma > 0$, $m \in \mathbb{N}$ and (α_n) be from the δ_2^0-condition. Then, given any $k_0 \in \mathbb{N}$ there exists m large enough such that $\Sigma_{n\geq m}\phi_n(x_n^{k_0}) < \epsilon/(3\gamma)$, $\phi_n(x_n^{k_0}) \leq \beta$, $\Sigma_{n\geq m}\alpha_n \leq \epsilon/3$. Moreover, for $k \geq k_0$ large enough there holds $\Sigma_{n=1}^{m}\phi_n(x_n^k) < \epsilon/3$. Hence, for such $k \geq k_0$ we have

$$I_\phi(\xi f_k) \leq \Sigma_{n=1}^{m-1}\phi_n(\xi x_n^k) + \gamma\Sigma_{n\geq m}\phi_n(x_n^k) + \Sigma_{n\geq m}\alpha_n \leq \epsilon.$$

Finally, since ξ is arbitrary (positive), we conclude that $\|f_k\|_\phi \longrightarrow 0$ as desired.

(a) $\Rightarrow$ (b). Assume to the contrary that the δ_2^0-condition is not satisfied. As in [15] (pp. 142–143) one can prove the existence of disjoint partition $(\mathbb{N}_k)$ $(k \in \mathbb{N})$, of the set $\mathbb{N}$ of natural numbers and sequences (x_n), (r_k) of nonnegative numbers where $r_k \uparrow 1$ such that $\Sigma_{n\in\mathbb{N}_k}\phi_n(x_n/r_k) \geq 1$ with $\phi_n(x_n) \leq b_k = 1/a_k$ and $\phi_n(x_n/r_k) - a_k\phi_n(x_n) \geq 0$, where $n \in \mathbb{N}_k$ and $a_k = 2^{k+2}$. Let $u_m = \Sigma_{k\geq m}\Sigma_{n\in\mathbb{N}_k}x_n e_n$. Then, $u_m \downarrow 0$. On the other hand $I_\phi(u_m) = \Sigma_{k\geq m}\Sigma_{n\in\mathbb{N}_k}\phi_n(x_n) \leq \Sigma_{k\geq 1}1/2^{k+1} < 1$. Next, let m be large enough such that if $0 < r < 1$, then for $k \geq m$ there holds $1/r_k \leq 1/r$. Hence, it follows that $I_\phi(u_m) = \Sigma_{k\geq m}\Sigma_{n\in\mathbb{N}_k}\phi_n(x_n/r) \geq \Sigma_{k\geq m}\Sigma_{n\in\mathbb{N}_k}\phi_n(x_n/r_k) = +\infty$. Therefore, $\|u_m\|_\phi = 1$ for m large, a contradiction with (b) and the proof is finished. □

Theorem 2.7

The following respective pairs $((a),\ (b))$ of statements are equivalent (it is not necessary that $\phi < +\infty$).
(a) h_ϕ is an STM space (resp. h_ϕ is a LUM space).
(b) $I_\phi(\cdot)$ is an STM modular on h_ϕ (resp. $I_\phi(\cdot)$ is a LUM modular on the space h_ϕ).

Proof. (a) $\Rightarrow$ (b). Since $\|u\|_\phi \leq \alpha \leq 1$ implies $I_\phi(u) \leq \alpha$ the implications follow by virtue of the the corresponding definitions.

(b) $\Rightarrow$ (a). Let $I_\phi(\cdot)$ be an STM modular but let $\|f-g\|_\phi = 1$ with $0 < g \leq f, \|f\|_\phi = 1$. From Lemma 1.4 it follows that $I_\phi(f-g) = 1$ so we get a contradiction with the STM of $I_\phi(\cdot)$.

If $I_\phi(\cdot)$ is LUM but h_ϕ is not LUM then, for some $f \geq 0$ with $\|f\|_\phi = 1$ and $\epsilon > 0$ there exists a sequence (g_m) satisfying $0 \leq g_m \leq f, \|g_m\|_\phi \geq \epsilon, \|f-g_m\|_\phi \longrightarrow 1$ as $m \longrightarrow +\infty$. From Lemma 1.4 we obtain that $I_\phi(f-g_m) \longrightarrow 1$, a contradiction with the LUM property of $I_\phi(\cdot)$ which finishes the proof. □

Theorem 2.8

The following statements are equivalent for the space h_ϕ.
(a) h_ϕ is a LUM space.
(b) h_ϕ is an STM space.
(c) (i) $1 \in \phi_n[\mathbb{R}_+] \quad (n \in \mathbb{N})$,
(ii) $\phi > 0$.

Proof. The implication (a) $\Rightarrow$ (b) follows from the definitions. By virtue of Lemma 2.1 (c) follows immediately. To prove that (c) $\Rightarrow$ (a) we apply Lemma 1.7. Next, by virtue of Lemma 1.4, we get that $I_\phi(\cdot)$ is LUM. Now, applying Lemma 1.4 and Theorem 2.7 we conclude that h_ϕ is LUM as desired. □

Remark. We cannot expect that h_ϕ is UM in general since every UM space is monotonically complete. Indeed, h_ϕ cannot be monotonically complete since h_ϕ is order dense in l_ϕ ([31]).

3. Monotonicity properties for $L_\phi(\mu)$ and induced copies of l_ϕ

To explain differences in characterizations of the UM property in the case of $L_\varphi(\mu)$ for μ nonatomic and μ purely atomic let us recall (cf. [23]) the following characterization.

Theorem 3.1

The following statements are equivalent for μ nonatomic.
(a) $L_\varphi(\mu)$ is an STM space.
(b) $L_\varphi(\mu)$ is a UM space.
(c) (i) $\varphi > 0$,
(ii) φ satisfies the Δ_2-condition (i.e. $\|\cdot\|_\phi$ is order continuous).

Recall (eg. [27], [28], [31]) $L_\varphi(\mu)$ consists of all μ-measurable functions on T such that $I_\varphi(\lambda u) = \int_T \varphi(\lambda|u(t)|, t)d\mu < +\infty$ for some λ depending on u. Now, $\varphi : \mathbb{R}_+ \times T \longrightarrow \overline{\mathbb{R}}_+$ is measurable and such that $\varphi(\cdot, t)$ is convex, lsc, continuous at zero with $\varphi(0, t) = 0$ and nontrivial, for $\mu-$a.e. $t \in T$. We write $\varphi \in \Delta_2$, if φ satisfies a Δ_2-condition, i.e. if there exist $K > 0$ and an integrable nonnegative and measurable function h such that $\varphi(2x, t) \le K\varphi(x, t) + h(t)$ for all $x \in \mathbb{R}_+$ and $t \in T$ except a set of zero measure.

Let us consider a sequential Musielak-Orlicz space $l_\phi$for ϕ given by the formula:

$$\phi_n(x) = \int_{T_n} \varphi(|x|, t)d\mu, \quad (x \in \mathbb{R}, \quad n \in \mathbb{N}),$$

where (T_n) is a partition of T satisfying $0 < \mu(T_n) < +\infty$, $T \overset{\mu}{=} \cup_{n=1}^{\infty} T_n$, $T_n \bigcap T_m \overset{\mu}{=} \emptyset$ whenever $n \neq m$. Such partition exists since $\varphi(\cdot, t)$ is continuous at zero (this continuity is equivalent to the decomposability of $L_\varphi(\mu) : L_\infty(T_n, \mu) \subset L_\varphi(\mu)$ for all n, see [24] for further bibliography). We endow l_ϕ with the Luxemburg norm $\|\cdot\|_\phi$. Let us consider the following mapping

$$j \ : \ \ l_\phi \ni f = (x_n) \longrightarrow j(f) = \Sigma_{n=1}^{\infty} \, x_n 1_{T_n}.$$

Clearly $u = j(f) \in L_\varphi(\mu)$ and j defines an order isometry of l_ϕ into $L_\varphi(\mu)$. From Theorem 3.1 we obtain

Corollary 3.2

If $L_\varphi(\mu)$ is an STM space then the space l_ϕ under consideration is a UM space.

In view of Theorem 2.5 this observation shows that the Δ_2-condition must be stronger than the δ_2^0-condition with $\phi = (\phi_n(\cdot))$ for $\phi_n(\cdot) = \varphi(\cdot, n)$. In fact, if $\varphi \in \Delta_2$ then $\phi \in \delta_2^0$ for $\gamma = K$, $\beta = +\infty$, $m = 1$ and $c_n = \int_{T_n} h(t)d\mu$ (see the definition of the δ_2^0-condition). In particular, we have $\phi < +\infty$.

Proposition 3.3

If $\varphi \in \Delta_2$, then for ϕ defined above we have $\phi \in \delta_2$. Moreover, $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$and ϕ satisfies the condition $(\star\star)$.

4. Applications to dominated best approximation

Some applications to dominated best approximation were given in [23] (see also [14] for further results). In this section we complete results from [23] in the case of the counting measure μ and for the Luxemburg norm. For the case of the Amemiya norm see [14].

Let K be a subset of a normed lattice X. If $f \in X$ then let $P_K(f)$ be the set of all best approximations to f with respect to K. The problem of finding an element $g \in P_K(f)$ where $K \leq f$ we call a dominated best approximation. As in the paper [23], following to the identity (cf. [14], Lemma 4.1)

$$\operatorname{dist}(f, K) = \inf_{g \in K \geq f} \|f - g\| = \inf_{v \in V \leq f} \|f - v\|,$$

where $V = 2f - K$, the term “dominated” can be also applied in the case $K \geq f$.

Recall, a subset $K \subset X$ is called a sublattice, if $x, y \in K$ implies that $x \vee y \in X$.

Proposition 4.1 ([23], Proposition 3.1)

The following statements are equivalent.
(a) X is an STM space.
(b) For all $f \in X$ and all sublattices $K \subset X$ such that $K \leq f$ there holds Card $(P_K(f)) \leq 1$ (i.e. the dominated best approximation with respect to sublattice is unique).

This equivalence is still true when one consider the order intervals $[u, v]$ instead of sublattices K.

The existence problem for the dominated best approximation is solved by virtue of the following result (cf. also [23], Proposition 3.3 and [14], Theorem 4.3).

Theorem 4.2

The following are equivalent for any Banach lattice X.
(a) X has order continuous norm.
(b) All order intervals $[g, h]$ are weakly compact (hence, $P_{[g,h]}(f) \neq \emptyset$ whenever $f \leq g$ or $h \leq f$, for each order interval $[g, h]$ and $f \in X$).
(c) $P_K(f) \neq \emptyset$ for every $f \in X$ and each closed sublattice K such that $K \leq f$.

Proof. The equivalence $(a) \iff (b)$ is well known (cf. [20], section X4.1). In the case (a) $\Rightarrow$ (c) the proof is the same as the proof of the implication (a) $\Rightarrow$ (b) of Proposition 3.3 in [23] (c) $\Rightarrow$ (a). We proceed as in the proof of Proposition 3.3 in [23]. In this case, however, we must apply Dini's theorem (cf. Theorem 4.3, [14]). Namely, let for a contrary (a) does not hold true, i.e. $f_n \downarrow 0$ but $\inf_n\{\|f_n\|\} = \alpha$ for some sequence (f_n) and $\alpha > 0$. Replacing f_n by $g_n = (1+1/n)f_n$ we obtain $\|g_{n+1}\| < \|g_n\|$. Let $K = \{g_n\}$. Then $f = 0 \leq K$ and $P_K(f) = \emptyset$. Clearly K is a sublattice. To prove that K is norm closed let for a moment $\|f_{n_k} - g\| \longrightarrow 0$ with f_{n_k} in K and $g \notin K$. Then, $f_{n_k} \longrightarrow g$ weakly with (f_{n_k}) nonincreasing. Applying Dini's theorem we conclude that $g = \inf_{n_k}\{f_{n_k}\} = 0$, a contradiction. □

In [14] we have proved (Theorem 4.3) that *X is CWLUM space*, equivalently: *X is STM and order continuous, if and only if the dominated best approximation problem is uniquely solvable for any closed sublattice of X.*

By virtue of the Proposition 4.1 and Theorem 2.4, it follows the following result for the Musielak-Orlicz sequence space l_ϕ.

Theorem 4.3

The following statements are equivalent for l_ϕ.
(a) The dominated best approximation problem with respect to closed sublattices is unique.
(b) (i) $\phi > 0$,
(ii) $1 \in \phi_n[\mathbb{R}_+]$ $(n \in \mathbb{N})$,
(iii) ϕ satisfies the δ_2^0-condition.

By virtue of Theorem 4.2 and Theorem 2.6 we obtain a characterization of the solvability in terms of the δ_2^0-condition.

Theorem 4.4

The following statements are equivalent.
(a) The dominated best approximation problem with respect to closed sublattices is solvable.
(b) ϕ satisfies the δ_2^0-condition.

It is worth noticing that the solvability of the dominated best approximation problem in l_ϕ is a consequence of the unicity.

Finally, we will give a characterization theorem concerning best approximation for the spaces l_ϕ. We can apply a general characterization theorem (Theorem 3.6) from [23] for ideal Banach function spaces $\mathrm{E}(\mu)$. The characterization of STM property for l_ϕ (Theorem 2.4) enables us to prove more complete result.

Theorem 4.5

Let ϕ satisfies the δ_2^0-condition or, equivalently, l_ϕ has an order continuous norm. Let $f = (x_n) \in l_\phi \backslash K$, where K is a (nonempty) convex subset in l_ϕ and let $f_0 = (x_n^0)$. The following statements are equivalent.

(a) $f_0 \in P_K(f)$.

(b) There exists a sequence $g = (\xi_n) \in l_{\phi^*}$ (ϕ^* - the Young conjugate to ϕ) with ξ_i satisfying

(i) $|\xi_i| \in \partial\phi\left(\frac{|x_i - x_i^0|}{\|f - f_0\|_\phi}\right)$, *for all* $i \in \mathbb{N}$,

(ii) $\mathrm{sign}(\xi_i) = \mathrm{sign}(x_i - x_i^0)$ *for all* $i \in \mathbb{N}$ *such that* $\xi_i(x_i - x_i^0) \neq 0$.

(iii) $\Sigma_{i=1}^{+\infty} \xi_i(x_i - x_i^0) \geq 0$, *for all* $h = (y_n) \in K$.

Moreover:

1. If l_ϕ is STM, i.e. $\phi > 0$ and $1 \in \phi_n[\mathbb{R}_+]$, then for each $i \in \mathbb{N}$ either $\xi_i = 0$ or, in the opposite case, $\mathrm{sign}(\xi_i) = \mathrm{sign}(x_i - x_i^0)$ whenever $x_i - x_i^0 \neq 0$ (in this case the sign of ξ_i can be undefined for some i).

2. If ϕ is smooth at zero then for each $i \in \mathbb{N}$ either $\xi_i = 0$ or, in the opposite case, $\mathrm{sign}(\xi_i) = \mathrm{sign}(x_i - x_i^0)$.

We omit the proof since we can proceed as in the proof of Theorem 3.7 in [23] assuming that μ is the counting measure. Now, the role of the Δ_2-condition is replaced by the δ_2^0 condition. This fact yields in particular that the dual $(l_\phi)^*$ to l_ϕ can be identified with the associated space l_{ϕ^*} and that the formula $l(x) = \Sigma_{n=1}^{+\infty} x_n s_n$, with $s = (s_n) \in l_{\phi^*}$, gives a general form of the linear and continuous functional on l_ϕ (cf. [17]). Moreover, the δ_2^0-condition implies the continuity of the modular $I_\phi(\cdot)$. The rest of the proof runs in a standard way.

References

1. M.A. Akcoglu and L. Sucheston, On uniform monotonicity of norms and ergodic theorems in function spaces, *Rend. Circ. Mat. Palermo* **8**(2) Suppl. (1985), 325–335.
2. G. Alherk and H. Hudzik, Copies of l^1 and c_0 in Musielak-Orlicz sequence spaces, *Comment. Math. Univ. Carolinae* **35**(1) (1994), 9–19.
3. G. Birkhoff, *Lattice Theory*, Providence, RI, 1967.
4. B. Bru and H. Heinich, Approximation dans les espaces de Köthe, *C.R. Acad. Sci. Paris* **296** (1983), 773–775.
5. B. Bru and H. Heinich, Monotonies des espaces d'Orlicz, *C.R. Acad. Sci. Paris* **301** (1985), 893–894.
6. B. Bru and H. Heinich, Applications de dualité dans les espaces de Köthe, *Studia Math.* **93** (1989), 41–69.

7. S. Chen and H. Hudzik, On some properties of Musielak-Orlicz spaces and their subspaces of order continuous elements, *Periodica Math. Hungarica* **25**(1) (1992), 13–20.
8. D.F. Cudia, The geometry of Banach spaces: Smoothness, *Trans. Amer. Math. Soc.* **110**(2) (1964), 41–69.
9. H. Hudzik, Uniform convexity of Musielak-Orlicz spaces with Luxemburg's norm, *Commentat. Math. (Prace. Mat.)* **24** (1983), 21–32.
10. H. Hudzik, On some equivalent conditions in Musielak-Orlicz spaces, *Comment. Math. (Prace Mat.)* **24** (1984), 57–64.
11. H. Hudzik and Y. Ye, Support functionals and smoothness in Musielak-Orlicz sequence spaces endowed with the Luxemburg norm, *Comment. Math. Univ. Carolinae* **31**(4) (1990), 661–684.
12. H. Hudzik and A. Kamińska, Some remarks on convergence in Orlicz space, *Comment. Math. (Prace Mat.)* **21** (1979), 81–88.
13. H. Hudzik and L. Maligranda, Orlicz spaces in which the Luxemburg norm and the Orlicz norm are proportional are Lebesque spaces, (in preparation).
14. H. Hudzik and W. Kurc, Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices, (to appear in *J. Approx. Theory*).
15. A. Kamińska, Rotundity of Musielak-Orlicz sequence spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **29** (1981), 137–144.
16. A. Kamińska, Uniform rotundity of Musielak-Orlicz sequence spaces, *J. Approx. Theory* **6**(4) (1986), 302–322.
17. A. Kamińska, Flat Orlicz-Musielak sequence spaces, *Bull. Acad. Polon. Sci. Math.* **30**(7-8) (1982), 347–352.
18. A. Kamińska, Some convexity properties of Musielak-Orlicz spaces of Bochner type, *Suppl. Rendiconti Circolo di Palermo, Serie II* **10** (1985), 63–73.
19. A. Kamińska and W. Kurc, Weak uniform rotundity in Orlicz spaces, *Comment. Math. Univ. Carolinae* **27**(4) (1986), 644–651.
20. L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Moscow 1984 (in Russian).
21. W. Kurc, On a class of proximinal subspaces in Banach spaces of vector-valued functions, *Function Spaces, Proceed. Intern. Conf., Poznań, 1986 Teubner-Texte zür Mathematik, Band* **103**, 105–111.
22. W. Kurc, Characterization of some monotonicity properties of a lattice norm in Musielak-Orlicz spaces, *Acta Univ. Carolinae* **30**(2) (1990), 91–94.
23. W. Kurc, Strictly and uniformly monotone Musielak-Orlicz spaces and applications to best approximation, *J. Approx. Theory*, **69**(2) (1992), 173–187.
24. W. Kurc, Extreme points of the unit ball in Orlicz spaces of vector-valued functions with the Amemiya norm, *Math. Japonica* **38**(2) (1993), 277–288.
25. W. Kurc, A dual property to uniform monotonicity in Banach lattices, *Collect. Math.* **44** (1993),155–165.
26. W. Kurc, Monotonicity, order smoothness and duality for convex functionals, *Collect. Math.* **48** (1997),635–655.
27. J. Musielak, *Orlicz spaces and modular spaces*, in "Lecture Notes in Math.", 1034, Springer- Verlag, New York-Berlin, 1983.
28. M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, in "Pure and Applied Math.", Marcel Dekker 1991.

29. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Publ. House Acad. Soc. Rep. Romania and Springer-Verlag, Berlin-Heidelberg-New York 1970.
30. M.A. Smith, Some examples concerning rotundity in Banach spaces, *Math. Ann.* **233** (1978), 155–161.
31. W. Wnuk, Representation of Orlicz lattices, *Dissertationes Math.*, **235** (1984).