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Graded algebra automorphisms

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Abstract

We consider the Clifford algebra $C(q)$ of a regular quadratic space (V, q) over a field K with its structure of $\mathbb{Z}/2\mathbb{Z}$-graded K-algebra. We give a characterization of the group of graded automorphisms of $C(q)$.

In the last section we introduce the $\mathbb{Z}/n\mathbb{Z}$-graded algebras and we study as well as the group of graded automorphisms for some of them.

1. Introduction

We begin recalling the definitions.

Let K be a field with char$K \neq 2$. For a K-vector space V, let $T(V)$ denote its tensor algebra. We recall that

$$T(V) = \bigoplus_{i=0}^{\infty} T^i(V) \quad \text{for} \quad T^i(V) = V \otimes_K \overset{i)}{\cdots} \otimes_K V.$$

Given a quadratic form q over V, its *Clifford's algebra* $C(q)$ is the quotient algebra $T(V)/I(q)$ for $I(q)$ the two-sided ideal of $T(V)$ generated by elements of the form $x \otimes x - q(x) \in T(V), x \in V$. We note that $V = T^1(V)$ maps injectively into $C(q)$; we shall view this injection as an identification. From now on, multiplication in $C(q)$ will be expressed by juxtaposition. Note that V generates $C(q)$ as a K-algebra.

We can define a $\mathbb{Z}/2\mathbb{Z}$-gradation on $C(q)$. The even part of $C(q)$, which is the image of $\bigoplus_{i \text{ even}} T^i(V)$ under the quotient map, will be denoted by $C_0(q)$. Similarly, the odd part of $C(q)$ is the image of $\bigoplus_{i \text{ odd}} T^i(V)$ and will be denoted by $C_1(q)$. The subalgebra $C_0(q)$ is usually called the "even Clifford algebra" of q. The elements of $C_0(q)$ are called even elements and the elements of $C_1(q)$ are called odd elements. It can be proved that $C(q)$ is a central simple $\mathbb{Z}/2\mathbb{Z}$-graded algebra [3, Chap. 5, Th. 2.1].

Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis on V (with respect to q) such that $q(e_i) = a_i, \quad i = 1 \ldots n$. Then $C(q)$ is the K-algebra spanned by $\{e_1, \ldots, e_n\}$ with the relations:

$$e_i^2 = a_i, \quad 1 \leq i \leq n; \qquad e_i e_j = -e_j e_i, \quad 1 \leq i \neq j \leq n\,.$$

The products $e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n}$, where $\epsilon_i = 0$ or 1 constitute a basis for $C(q)$ as K-vector space. Then the dimension of $C(q)$ over K is 2^n.

Examples:

1. Let $q =< a >$ be the one-dimensional quadratic space with matrix (a), and basis $\{e\}$. Then $C(q) = K[x]/(x^2 - a)$.
2. Let $q =< a, b >$, $a, b \neq 0$ be a binary quadratic space relative to an orthogonal basis. Then, as graded algebras, $C(q) \simeq < \frac{a,b}{K} >$ where $< \frac{a,b}{K} >$ is the quaternion algebra.

We denote by $Z(A)$ the center of the algebra A.

A morphism between two graded algebras is called graded if it preserves the gradation.

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2. Automorphisms of $C(q)$

If $\dim(V)$ is odd, then $C_0(q)$ is a central simple algebra (CSA) over K. If $\dim(V)$ is even, then $C(q)$ is a central simple algebra over K. We put

$$C := \begin{cases} C_0(q) & \text{if } \dim(V) \text{ is odd} \\ C(q) & \text{if } \dim(V) \text{ is even} \end{cases}$$

So, C is a central simple algebra over K and the Skolem-Noether theorem [3, Chap. 4, Th. 1.8] determines the group of automorphisms of C. We have

Proposition 2.1

We can define a surjective group morphism:

$$C^* \xrightarrow{\varphi} Aut(C)$$
$$s \longrightarrow f_s$$

where $f_s(x) = sxs^{-1}$*. The kernel of this morphism is* K^**.*

So, $Aut(C) = C^*/K^*$*.*

We recall now the definition of the Clifford group [1. pag. 151].

DEFINITION 2.2. The Clifford group of q, denoted $G(q)$ (respectively the special Clifford group, $G^+(q)$), is the multiplicative group of invertible elements $s \in C(q)$ (resp. $s \in C_0(q)$) such that $sVs^{-1} = V$.

Note that $G^+(q) = G(q) \cap C_0(q)$.

We note that an element of $O(q)$ extends to a graded automorphism of $C(q)$ and conversely, the inner automorphism given by an element of $G(q)$ restricts to an element of $O(q)$.

We have:

1. If the dimension of V is even, then $\varphi(G(q)) = O(q)$ by [1. Th. 5.4], so

$$G(q)/K^* \cong O(q)\,.$$

We obtain
$$\frac{C(q)^*/K^*}{G(q)/K^*} \cong C(q)^*/G(q) \cong \frac{Aut(C(q))}{O(q)}.$$

2. If the dimension of V is odd, $\varphi(G^+(q)) = SO(q)$ by [1, Th. 5.4], so

$$G^+(q)/K^* \cong SO(q)\,.$$

We obtain

$$\frac{(C_0(q))^*/K^*}{G^+(q)/K^*} \cong (C_0(q))^*/G^+(q) \cong \frac{Aut(C_0(q))}{SO(q)}\,.$$

3. Graded automorphisms for $\dim(V)$ odd

In this section we want to study the structure of the group of graded automorphisms of $C(q)$ when $\dim(V)$ is odd.

Let (V, q) be a regular quadratic space over K with $\dim(V)$ odd. Let $f : C(q) \longrightarrow C(q)$ be a graded K-algebras automorphism.

As $C(q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, we have $C(q) = C_0(q) \oplus C_1(q)$ and $C_0(q)$ is a central simple algebra over K. By the structure theorem for central simple graded algebras of odd type [3, Chap. 4 Th. 3.6], there exists an element $z \in Z(C(q)) \cap C_1(q)$ such that:

- $C_1(q) = C_0(q)z$.
- $z^2 = a \in K^*$ (the square class of a does not depend on the choice of $z \in Z(C(q)) \cap C_1(q) - \{0\}$).
- $Z(C(q)) = K \oplus Kz$.

So, as $C_1(q) = C_0(q)z$, knowing f is equivalent to knowing $f|C_0(q)$ and $f(z)$.

As f is a graded automorphism, then:

$$f|C_0(q) : C_0(q) \longrightarrow C_0(q)$$

is an automorphism of $C_0(q)$, a central simple algebra. So, by the Skolem-Noether theorem, there exists an $s \in C_0(q)^*$ such that $f(v) = svs^{-1} \quad \forall v \in C_0(q)$ and s is determined up to a factor in K^*.

We determine now $f(z)$.

Proposition 3.1

With the preceding notations $f(z) = \pm z$ and the sign of $f(z)$ is independent of the chosen z.

Proof. The element $z \in Z(C(q)) \cap C_1(q)$. So, $f(z) \in Z(C(q)) = K \oplus Kz$ and, as f is graded, $f(z) \in Kz$. Then, $f(z) = \alpha z$ for some $\alpha \in K$.

Since $z^2 = a$, we have that $a = f(z^2) = (f(z))^2 = \alpha z \alpha z = \alpha^2 z^2 = \alpha^2 a$. Then $\alpha^2 = 1$ and $\alpha = \pm 1$. So, $f(z) = \pm z$.

Let z' be another element with the same properties of z. Since $z' \in Z(C(q)) \cap C_1(q) = (K \oplus Kz) \cap C_1(q)$, then $z' = \lambda z$ with $\lambda \in K$. So, $f(z') = \lambda f(z)$.

If $f(z) = z \Rightarrow f(z') = \lambda z = z'$ and if $f(z) = -z \Rightarrow \quad f(z') = -\lambda z = -z'$. $\square$

We observe that if $g : C_0(q) \longrightarrow C_0(q)$ is an automorphism of $C_0(q)$, it extends to two graded automorphisms of $C(q)$ sending z to z and $-z$, respectively.

Let $Autgr(C(q))$ denote the group of graded automorphisms of $C(q)$.

Proposition 3.2

We can define a surjective group morphism:

$$Autgr\big(C(q)\big) \xrightarrow{\varphi} \big(C_0(q)\big)^*/K^*$$
$$f \longrightarrow s$$

where s is the element corresponding to the inner automorphism $f|C_0(q)$. The kernel of this morphism is a group isomorphic to $\{\pm 1\}$.

Hence, we have the exact sequence:

$$0 \to \{\pm 1\} \to Autgr\big(C(q)\big) \xrightarrow{\varphi} \big(C_0(q)\big)^*/K^* \to 0$$

and then, the isomorphism:

$$\frac{Autgr(C(q))}{\{\pm 1\}} \cong \frac{(C_0(q))^*}{K^*}\,.$$

Proof.

1. φ is a morphism if and only if $\varphi(ff') = \varphi(f)\varphi(f')$. Let s, s' be the corresponding elements to the inner automorphisms $f|C_0(q)$ and $f'|C_0(q)$ respectively. Then $\varphi(f)\varphi(f') = ss'$.
   We have to study $ff'|C_0(q)$. Let $u \in C_0(q)$. Then, $(f\circ f')(u) = f(s'us'^{-1}) = ss'us'^{-1}s^{-1} = (ss')u(ss')^{-1}$.
   So $\varphi(ff') = ss' = \varphi(f)\varphi(f')$.
2. It is surjective, because for each s we have one (actually two) graded automorphism of $C(q)$:
$$f(u) = sus^{-1} \text{ if } \quad u \in C_0(q) \text{ and } f(z) = z$$
$$f(u) = sus^{-1} \text{ if } \quad u \in C_0(q) \text{ and } f(z) = -z\,.$$
3. We now compute the kernel: Two graded automorphism f, f' are mapped to the same s by φ if and only if $f|C_0(q) \equiv f'|C_0(q)$, that is, if and only if
$$f \equiv f' \text{ or}$$
$$f|C_0(q) \equiv f'|C_0(q) \text{ and } f(z) = z,\ f'(z) = -z\,,$$
   i.e. if they are equal or if they have the same restriction to $C_0(q)$ and differ in the sign of the image of z. □

By determining the graded automorphisms of $C(q)$ such that the corresponding s lies in $G^+(q)$ we obtain:

Proposition 3.3

$$\frac{Autgr(C(q))/\{\pm 1\}}{O(q)/\{\pm 1\}} \cong \frac{(C_0(q))^*/K^*}{G^+(q)/K^*}$$

and then,

$$\frac{Autgr(C(q))}{O(q)} \cong \frac{(C_0(q))^*}{G^+(q)}\,.$$

Proof. From the group homomorphism of [1, Th. 5.4] we can define the following group homomorphism: If $f \in SO(q)$, then there exists $s \in (G^+(q))^*$ such that:

$$\begin{aligned} f : V &\longrightarrow V \\ v &\longrightarrow f(v) = svs^{-1}. \end{aligned}$$

So, we can define:

$$\begin{aligned} SO(q) &\longrightarrow G^+(q)/K^* \\ f &\longrightarrow s \end{aligned}$$

where s is the preceding.

Given $s \in G^+(q)$, we have two automorphisms f in $Autgr(C(q))$ such that $\varphi(f) = s$.

1. If $f(z) = z$ and $f|C_0(q) \equiv s()s^{-1}$, then

$$\begin{aligned} f|V : V &\longrightarrow V \\ v &\longrightarrow svs^{-1}. \end{aligned}$$

2. If $f(z) = -z$ and $f|C_0(q) \equiv s()s^{-1}$, then

$$\begin{aligned} f|V : V &\longrightarrow V \\ v &\longrightarrow -svs^{-1}. \end{aligned}$$

We note that $f|V$ is in $O(q)$ in the two cases, but in $SO(q)$ in exactly one case, since $\dim(V)$ is odd.

We have, then

$$\frac{O(q)}{\{\pm 1\}} \cong \frac{G^+(q)}{K^*}$$

and connecting with the results of Proposition 3.2 we finish the proof. □

Now, we want to give another characterization for $Autgr(C(q))$.

If t is an homogeneous element of $(C(q))^*$, we can write $t = z^{\partial(t)}s$ where $s \in C_0(q)$ with $\partial(t) = 0$ if $t \in C_0(q)$ and $\partial(t) = 1$ if $t \in C_1(q)$.

Proposition 3.4

The two graded automorphisms of $C(q)$, $f_s \equiv s()s^{-1}$ and $f_t \equiv t()t^{-1}$ are equal.

Proof. If $x \in C(q)$, then $txt^{-1} = z^{\partial(t)}sx(z^{\partial(t)}s)^{-1} = z^{\partial(t)}sxs^{-1}z^{-\partial(t)} = z^{\partial(t)-\partial(t)}sxs^{-1} = sxs^{-1}$ because $z \in Z(C(q))$. □

So, we can define the map:

$$\psi : \{\text{Homogeneous elements of } (C(q))^*\} \longrightarrow Autgr(C(q))$$
$$s \longrightarrow f_s : C(q) \to C(q)$$

where $f_s|C_0(q) \equiv s()s^{-1}$ and $f_s(z) = (-1)^{\partial(s)}z$. If we put $I(s) = (-1)^{\partial(s)}$, we have equivalently:

$$f_s : C(q) \longrightarrow C(q)$$
$$x(\text{homogeneous}) \longrightarrow I(s)^{\partial(x)}sxs^{-1}.$$

Proposition 3.5

ψ is a surjective group morphism and $ker\psi = K^*$.
We have, then, the exact sequence:

$$1 \to K^* \to \{\textit{Homogeneous elements of } C(q)^*\} \to Autgr(C(q)) \to 1\,.$$

Proof.

1. ψ is a morphism if and only if $\psi(ss') = \psi(s) \circ \psi(s')$.
   Let $x \in C(q)$ be homogeneous.
   $\psi(ss')(x) = I(ss')^{\partial(x)}ss'x(ss')^{-1} = [I(s)I(s')]^{\partial(x)}ss'xs'^{-1}s^{-1}$.
   $\psi(s)[\psi(s')(x)] = \psi(s)(I(s')^{\partial(x)}s'xs'^{-1}) = I(s)^{\partial(x)}I(s')^{\partial(x)}ss'xs'^{-1}s^{-1} = [I(s)I(s')]^{\partial(x)}ss'xs'^{-1}s^{-1}$.
   And the two, are the same.
2. ψ is surjective, obviously.
3. We now compute the kernel. Let $s \in C(q)^*$, homogeneous.
   $s \in ker(\psi) \Leftrightarrow f_s(x) = x \quad \forall x \in C(q) \Leftrightarrow I(s)^{\partial(x)}sxs^{-1} = x \quad \forall x \in C(q)$.
   In particular, if $x \in C_0(q)$, we have $sx = xs \Rightarrow s \in C_{C(q)}(C_0(q)) = Z(C(q)) = K \oplus Kz$ [3, Chap. 4,Th. 3.6].
   Then, $x = f_s(x) = I(s)^{\partial(x)}sxs^{-1} = I(s)^{\partial(x)}x \; \forall x \in C(q)$. So,
   $I(s) = 1 \Rightarrow \partial(s) = 0 \Rightarrow s \in C_0(q)$. Since $s \in Z(C(q)) \cap C_0(q) \Rightarrow s \in K$.
   And it is invertible. □

4. Graded automorphisms for $\dim(V)$ even

In this section we want to study the structure of graded automorphisms of $C(q)$ when $\dim(V)$ is even.

Let (V,q) be a regular quadratic space over K with $\dim(V)$ even. In this case, it is known that $C(q)$ is a central simple algebra over K, so, every automorphism of $C(q)$ is an inner automorphism and then

$$Aut\big(C(q)\big) \cong C(q)^*/K^*.$$

We now study the graded automorphisms.

Let $f \in Autgr(C(q))$. In particular $f \in Aut(C(q))$ and so, there exists an $s \in C(q)^*$ such that $f(x) = sxs^{-1}\,\forall x \in C(q)$. We are going to characterize the elements s giving graded automorphisms.

Proposition 4.1

If $f_s \equiv s()s^{-1}$ is graded, then s is an homogeneous element of $C(q)$.

Proof. We put $s = s_0 + s_1$ where $s_i \in C_i(q)$. Let $\{e_1,\dots,e_n\}$ be an orthogonal basis with respect to the quadratic space and denote $f(e_i) = v_i$. We note that $v_i \in C_1(q)$, $i = 1,\dots,n$ (because e_i is odd and f is graded). As $f(e_i) = v_i\ \forall i$, we have $se_is^{-1} = v_i \Rightarrow se_i = v_is \Rightarrow (s_0+s_1)e_i = v_i(s_0+s_1) \Rightarrow s_0e_i+s_1e_i = v_is_0+v_is_1$. In the last equality, $s_0e_i, v_is_0 \in C_1(q)$ and $s_1e_i, v_is_1 \in C_0(q)$. Then $s_0e_i = v_is_0$ and $s_1e_i = v_is_1$. We can write $s_0e_i = v_is_0 = (se_is^{-1})s_0$. Then $s^{-1}s_0e_i = e_is^{-1}s_0\forall i$. So $s^{-1}s_0 \in Z(C(q)) = K$ and $s^{-1}s_0 = \lambda$. Then $s_0 = \lambda s$.

1. If $\lambda = 0 \Rightarrow s_0 = 0 \Rightarrow s = s_1 \in C_1(q)$.
2. If $\lambda \neq 0 \Rightarrow s = \frac{1}{\lambda}s_0 \Rightarrow s_1 = 0 \Rightarrow s \in C_0(q)$. □

Proposition 4.2

We can define a surjective group morphism:

$$\big\{\textit{Homogeneous elements of } C(q)^*\big\} \xrightarrow{\varphi} Autgr\big(C(q)\big)$$
$$s \longrightarrow f_s$$

where $f_s(x) = sxs^{-1}\ \forall x \in C(q)$. The kernel of this morphism is K^*. Hence, we have the exact sequence:

$$1 \to K^* \to \big\{\textit{Homogeneous elements of } C(q)^*\big\} \xrightarrow{\varphi} Autgr\big(C(q)\big) \to 1\,.$$

Proof.

1. To prove that φ is morphism just a similar argument as before is needed.
2. φ is surjective because in this case, $C(q)$ is central simple algebra over K and then we can take the s corresponding to $f \in Aut(C(q))$. We know that s is homogeneous by 4.1.
3. We now compute the kernel of φ. Let $s \in C(q)^*$, since $s \in ker(\varphi) \Leftrightarrow f_s = Id \Rightarrow x = f_s(x) = sxs^{-1}\ \forall x \in C(q) \Rightarrow sx = xs\ \forall x \in C(q) \Rightarrow$ $s \in Z(C(q)) \cap C(q)^* \Rightarrow s \in K^*$. Then $ker(\varphi) = K^*$. □

5. Graded automorphisms of some $\mathbb{Z}/n\mathbb{Z}$-graded algebras

We will study now the automorphisms of some $\mathbb{Z}/n\mathbb{Z}$-graded algebras. So, we begin with the definition of these algebras.

We note that for central simple $\mathbb{Z}/2\mathbb{Z}$-graded algebras, we have the structure theorem. This is an essential result to study the automorphisms of these algebras. For $\mathbb{Z}/n\mathbb{Z}$-graded algebras a similar theorem is not yet known. So, in the examples studied here, we will determine some properties of these particular algebras that allow us to study its graded automorphisms.

5.1 Some examples of $\mathbb{Z}/n\mathbb{Z}$-graded algebras

Let n be a fixed integer number $n \geq 2$ and K be a field of characteristic different to p, $\forall p \mid n$, which contains the group of nth roots of unity. We fix ω a primitive nth root of unity.

Definition 5.1. A $\mathbb{Z}/n\mathbb{Z}$-graded K-algebra A is a finite-dimensional K-algebra given in the form $A_0 \oplus \cdots \oplus A_{n-1}$, such that

$$K = K \cdot 1 \subseteq A_0$$
$$A_i A_j \subseteq A_{i+j} \text{ where the subscripts are taken modulo } n\,.$$

In particular, A_0 is a subalgebra. Sometimes we say just graded algebras if n is clear.

A $\mathbb{Z}/n\mathbb{Z}$-graded algebra A is said to be *concentrated at degree 0* if $A_i = 0 \quad \forall i \in \{1, \ldots, n-1\}$.

For a graded algebra A as above, the elements in $h(A) = A_0 \cup \ldots \cup A_{n-1}$ will be called the *homogeneous elements of* A. If $a \in h(A)$, we write $\partial(a) = i$ if $a \in A_i$.

A subspace $S \subset A$ is called *graded* if it is the direct sum of the intersections $S_i = S \cap A_i$. This means if $s \in S$ and $s = s_0 + \cdots + s_{n-1} \quad s_i \in A_i$, then each $s_i \in S$. *Graded ideal* has the obvious meaning.

DEFINITION 5.2. Let A be a $\mathbb{Z}/n\mathbb{Z}$-graded algebra.

1. We shall call A a central graded algebra (CGA) over K if $Z(A) \cap A_0 = K$ where $Z(A)$ is the center of A as non-graded algebra.
2. A is called a simple graded algebra (SGA) over K if A has no proper graded (two-sided) ideals.
3. If A is central graded algebra and simple graded algebra, we say A is central simple graded algebra.

DEFINITION 5.3. Let $A = A_0 \oplus \cdots \oplus A_{n-1}$ be a $\mathbb{Z}/n\mathbb{Z}$-graded algebra and $f : A \longrightarrow A$ be an automorphism. We say f is a graded automorphism if $f(A_i) \subset A_i \quad \forall i \in \{0, \ldots, n-1\}$.

We shall now look at some examples of central simple graded algebras. We will afterwards study its automorphisms.

EXAMPLE 1: Consider $A = K(\sqrt[n]{a})$ a extension of K with $a \in K$ such that $a \notin K^d$, $\forall d \mid n$. So, $[A : K] = n$ (degree of the extension). We can make A into a $\mathbb{Z}/n\mathbb{Z}$-graded K-algebra by declaring

$$A_0 = K,\, A_1 = K\sqrt[n]{a}, \ldots, A_i = K\sqrt[n]{a}^i, \ldots, A_{n-1} = K\sqrt[n]{a}^{n-1}.$$

We shall use the notation $A = K < \sqrt[n]{a} >$ to indicate the fact that A is made into a graded algebra in this way.

– A is commutative, so $Z(A) = A$. We have $Z(A) \cap A_0 = A \cap A_0 = K$.
Then A is a central graded algebra.

– Since A is a field, it is simple.

It follows that A is, in fact, a central simple graded algebra.

EXAMPLE 2: For $a, b \in K^*$, let $A = (\frac{a,b}{K})_\omega$ be the K-algebra which is generated by elements $\{i, j\}$ which satisfy $\{\, i^n = a, j^n = b, ij = \omega ji\}$. A basis for A as vector space over K consists of $\{i^r j^s : 0 \leq r, s < n\}$. So A has dimension n^2 as K-algebra. You can find the definition and properties of these algebras, for example in [4, Section 15.4] and [2, Exercise 4.28]. This is a generalization of the quaternion algebras.

We can make A into a $\mathbb{Z}/n\mathbb{Z}$-graded K-algebra by setting

$$A_l =< i^k j^m \mid k + m \equiv l \pmod{n} >_K .$$

We shall use the notation $A = K < \frac{a,b}{K} >_\omega$ to indicate the fact that A is made into a graded algebra in this way. As we have fixed the field K and the nth root of unity ω , we sometimes just write $A =< a, b >$.

In [2, Exercise 4.28] it is proved that A is central simple over K. So it is a central simple graded algebra.

EXAMPLE 3: For $a_1, a_2, a_3 \in K^*$, let $A = (a_1, a_2, a_3)$ be the K-algebra which is generated by elements $\{e_1, e_2, e_3\}$ which satisfy

$$\left\{e_i^n = a_i \ \ i = 1, 2, 3, \quad e_ie_j = \omega e_je_i \quad \text{if} \quad i < j\right\}.$$

We can make A into a $\mathbb{Z}/n\mathbb{Z}$-graded K-algebra by setting

$$A_l =< e_1^{\varepsilon_1} e_2^{\varepsilon_2} e_3^{\varepsilon_3} \mid \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \equiv l \pmod{n}, \quad 0 \leq \varepsilon_i \leq n-1 >$$

We shall use the notation $A = K < a_1, a_2, a_3 >$ to indicate the fact that A is made into a graded algebra in this way.

Proposition 5.4

Let A be the algebra of the Example 3. Then

1. *A is a central simple $\mathbb{Z}/n\mathbb{Z}$-graded algebra over K.*
2. *Let $z = e_1e_2^{-1}e_3 \in A$. Then $z \in Z(A) \cap A_1$ and we have*
   $Z(A) = K \oplus Kz \oplus \cdots \oplus Kz^{n-1}$. We put $a = z^n = (-1)^{n-1}a_1a_2^{-1}a_3$.
3. *$A_i = A_0z^i \quad \forall i \in \{0, \ldots, n-1\}$.*
4. *A_0 is a central simple algebra over K.*

Proof. 1. We prove first $Z(A) \cap A_0 = K$. Given an element $e \in Z(A) \cap A_0$ we want to show that $e \in K$. We can put it in the form

$$e = \sum_{\varepsilon_1+\varepsilon_2+\varepsilon_3 \equiv 0(n)} \lambda_{\varepsilon_1\varepsilon_2\varepsilon_3} e_1^{\varepsilon_1} e_2^{\varepsilon_2} e_3^{\varepsilon_3}.$$

As $e \in Z(A)$, in particular $ee_1 = e_1e$. So

$$\lambda_{\varepsilon_1\varepsilon_2\varepsilon_3} e_1^{\varepsilon_1+1} e_2^{\varepsilon_2} e_3^{\varepsilon_3} = \lambda_{\varepsilon_1\varepsilon_2\varepsilon_3} e_1^{\varepsilon_1} e_2^{\varepsilon_2} e_3^{\varepsilon_3} e_1 = \lambda_{\varepsilon_1\varepsilon_2\varepsilon_3} \omega^{-(\varepsilon_2+\varepsilon_3)} e_1^{\varepsilon_1+1} e_2^{\varepsilon_2} e_3^{\varepsilon_3}.$$

Then $\varepsilon_2 + \varepsilon_3 \equiv 0 \pmod{n}$ and so, $\varepsilon_1 \equiv 0 \pmod{n} \Rightarrow \varepsilon_1 = 0$.

In the same way, we prove $\varepsilon_2 = \varepsilon_3 = 0$ and then $e \in K$.

We want to prove now that A is simple graded. Let $I \neq 0$ be a two-sided graded ideal in A. Our goal is to show that $1 \in I$. Each homogeneous element $x \in I$ of degree k can certainly be put in the form $x = \sum_{i=1}^{r} \lambda_i e_1^{\varepsilon_1^i} e_2^{\varepsilon_2^i} e_3^{\varepsilon_3^i}$ with $\varepsilon_1^i + \varepsilon_2^i + \varepsilon_3^i \equiv k \pmod{n}$. Among all nonzero homogeneous elements in I, let us pick x as above, such that r is as small as possible. Multiplying this by e_1^{-k} we can suppose $x \in A_0$. We consider now the element

$$x' = \frac{\rho}{\lambda_1} e_1^{-\varepsilon_1^1} e_2^{-\varepsilon_2^1} e_3^{-\varepsilon_3^1} x = 1 + \sum_{i=2}^{r} \lambda_i' e_1^{\delta_1^i} e_2^{\delta_2^i} e_3^{\delta_3^i} \in I$$

where ρ is a suitable nth root of unity and $\delta_j^i = \varepsilon_j^i - \varepsilon_j^1$.

We calculate now $e_1 x' - x' e_1 = \sum_{2}^{r} \lambda_i' e_1^{\delta_1^i + 1} (1 - \omega^{-(\delta_2^i + \delta_3^i)} e_2^{\delta_2^i} e_3^{\delta_3^i} \in I$. By the choice of r we conclude that $\lambda_i'(1 - \omega^{-(\delta_2^i + \delta_3^i)}) = 0$. If $\delta_1^i \neq 0$ then $\lambda_i' = 0$. If $\delta_1^i = 0$ we can proceed in the same way by calculating $e_j x' - x' e_j$, if $\delta_j^i \neq 0$. We conclude $\lambda_i' = 0, \quad \forall i$ and then $1 \in I$.

2. Let $z = e_1 e_2^{-1} e_3 \in A_1$. We want to see that $z \in Z(A)$ and to this end, we need just to check $e_i z = z e_i$ for $i = 1, 2, 3$. It is a simple computation. So, $z \in Z(A) \cap A_1$ and then $z^k \in Z(A) \cap A_k$.
Therefore, we have $K \oplus Kz \oplus \cdots \oplus Kz^{n-1} \subset Z(A)$. Now if $y \in Z(A) \cap A_i$, we have $yz^{n-i} \in Z(A) \cap A_0 = K$. So, $y = kz^i$ for some $k \in K$.

Computing z^n, we obtain $z^n = (-1)^{n-1} a_1 a_2^{-1} a_3 \in K$.

3. $A_i = A_i z^n = A_i z^{n-i} z^i \subseteq A_0 z^i$. The another inclusion is clear.

4. As $A_i = A_0 z^i$ we have that $Z(A_0) = Z(A) \cap A_0 = K$. So A_0 is a central algebra.

Let $0 \neq I \subseteq A_0$ be a two-sided ideal of A_0. Then $J = I \oplus Iz \oplus \cdots \oplus Iz^{n-1}$ is a two-sided graded ideal in A, and so equals A. Thus $I = A_0$, proving that A_0 is simple. □

5.2. Graded automorphisms of $K < a, b >$

We want to study now the structure of the group of graded automorphisms of the algebra $A = K < a, b >$ when $a \in K, b \in K$. This case is similar to the case of Section 4. We change the notation and we put $e_1 = i$ and $e_2 = j$.

In this case, A is a central simple algebra over K. Let $f \in Autgr(A)$, in particular $f \in Aut(A)$ and by the Skolem-Noether theorem, there exists an $s \in A^*$ such that $f(x) = sxs^{-1}, \quad x \in A$.

Proposition 5.5

If $f_s \equiv s()s^{-1}$ is graded, then s is an homogeneous element of A.

We put $s = s_0 + \cdots s_{n-1}$ where $s_i \in A_i$ and $f(e_i) = se_is^{-1} = v_i$ for $i = 1, 2$. As $e_i \in A_1$ and f is graded, $v_i \in A_1$. We have for $i = 1, 2$, $se_is^{-1} = v_i \Rightarrow se_i = v_is \Rightarrow (s_0 + \cdots + s_{n-1})e_i = v_i(s_0 + \cdots + s_{n-1}) \Rightarrow s_0e_i + \cdots s_{n-1}e_i = v_is_0 + \cdots v_is_{n-1}$. In this equality, the parts of the same degree have to be equal, so $s_ke_i = v_is_k \quad \forall k\in \{0, \ldots, n-1\}$, $i = 1, 2$.

In particular $s_0e_i = v_is_0$ and by definition $v_i = se_is^{-1}$. Then, $s_0e_i = se_is^{-1}s_0$ and $s^{-1}s_0e_i = e_is^{-1}s_0, \quad i = 1, 2$. So, $s^{-1}s_0 \in Z(A) = K \Rightarrow s_0 = \lambda_0 s$ with $\lambda_0 \in K$. If $\lambda_0 \neq 0$, then $s = \frac{1}{\lambda_0}s_0 \in A_0$ is homogeneous of degree 0. If $\lambda_0 = 0$, then $s_0 = 0$ and $s = s_1 + \cdots s_{n-1}$. We can do the same process for $k = 1, 2, \ldots$ until we find a $\lambda_k \neq 0$. If $\lambda_k = 0, \quad \forall k = 0, \ldots, n-2$, then $s = s_{n-1}$ which is homogeneous of degree $n - 1$. □

Theorem 5.6

We can define a surjective group morphism

$$\{Homogeneous\ elements\ of\ A^*\} \xrightarrow{\varphi} Autgr(A)$$
$$s \longrightarrow f_s$$

where $f_s(x) = sxs^{-1}\, \forall x \in A$. *The kernel of this morphism is* K^*. *Hence, we have the exact sequence:*

$$1 \to K^* \to \{Homogeneous\ elements\ of\ A^*\} \xrightarrow{\varphi} Autgr(A) \to 1.$$

Proof. Is similar to the proof of 4.2.

5.3. Graded automorphisms of $K < a_1, a_2, a_3 >$

We want to study now the structure of the group of graded automorphisms of the algebra $A = K < a_1, a_2, a_3 >$ when $a_i \in K$. This case is similar to the case of Section 3.

Let $f : A \longrightarrow A$ be a graded automorphism (as K-algebras). By the Proposition 5.4, A_0 is a central simple algebra over K and putting $z = e_1e_2^{-1}e_3$, we have $A_i = A_0z^i \quad \forall i\in \{0, \ldots, n-1\}$. As f is graded, $f(A_i) \subset A_i$, so knowing f is equivalent to knowing $f \mid A_0$ and $f(z)$.

As f is a graded automorphism, then:

$$f|A_0 : A_0 \longrightarrow A_0$$

is an automorphism of A_0, a central simple algebra. So, by the Skolem-Noether theorem, there exists an $s \in A_0^*$ such that $\forall v \in A_0$, $f(v) = svs^{-1}$ and s is determined up to a factor in K^*.

We determine now $f(z)$.

Proposition 5.7

With the preceding notations $f(z) = \varrho_n z$ *with* $\varrho_n^n = 1$.

Proof. Similarly to the case $n = 2$, we just have to observe that since $z \in Z(A) \cap A_1$, then $f(z) \in Z(A) \cap A_1 = Kz$. So, $f(z) = \alpha z$ with $\alpha \in K$. Since $z^n = (\sqrt[n]{a})^n = a$, we have that $a = f(z^n) = f(z)^n = \alpha^n a$. So, $\alpha^n = 1$. We put $\alpha = \varrho_n$ to note that it depends on n. □

We observe that if $g : A_0 \longrightarrow A_0$ is an automorphism of A_0, it extends to n graded automorphisms of A sending z to $\omega^i z$ for $i \in \{0, \dots, n-1\}$.

Proposition 5.8

We can define a surjective group morphism:

$$Autgr(A) \xrightarrow{\varphi} (A_0)^*/K^*$$
$$f \longrightarrow s$$

where s is the element corresponding to the inner automorphism $f|A_0$. The kernel of this morphism is a group isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Hence, we have the exact sequence:

$$0 \to \mathbb{Z}/n\mathbb{Z} \to Autgr(A) \xrightarrow{\varphi} (A_0)^*/K^* \to 0$$

and then, the isomorphism:

$$\frac{Autgr(A)}{\mathbb{Z}/n\mathbb{Z}} \cong \frac{(A_0)^*}{K^*}.$$

Proof.

1. It is similar to 1 in the proof of Proposition 3.2.
2. It is surjective, because for each s we have one (actually n) graded automorphism of A:

$f(u) = sus^{-1}$ if $u \in A_0$ and $f(z) = \omega^i z$ for $i \in \{0, \dots, n-1\}$

3. We now compute the kernel: Two graded automorphism f, f' are mapped to the same s by φ if and only if $f|A_0 \equiv f'|A_0$, that is, if and only if

$$f|A_0 \equiv f'|A_0 \text{ and } f(z) = \omega^i f'(z)$$

for some $i \in \{0, \dots, n-1\}$ i.e. if they are equal or if they have the same restriction to A_0 and differ in the image of z by a nth root of unity. □

Now, we want to give another characterization for $Autgr(A)$.

If t is an homogeneous element of A^*, we can write $t = z^{\partial(t)} s$ where $s \in A_0$ with $\partial(t) = i$ if $t \in A_i$.

Proposition 5.9

The two graded automorphisms of A, $f_s \equiv s()s^{-1}$ and $f_t \equiv t()t^{-1}$ are equal.

Proof. If $x \in A$, then $txt^{-1} = z^{\partial(t)} sx(z^{\partial(t)} s)^{-1} = z^{\partial(t)} sxs^{-1} z^{-\partial(t)} = z^{\partial(t)-\partial(t)} sxs^{-1} = sxs^{-1}$ because $z \in Z(A)$. □

So, we can define the map:

$$\begin{aligned} \psi : \{\text{Homogeneous elements of } A^*\} &\longrightarrow Autgr(A) \\ s &\longrightarrow f_s : A \to A \end{aligned}$$

where $f_s|A_0 \equiv s()s^{-1}$ and $f_s(z) = \omega^{\partial(s)} z$. If we put $I(s) = \omega^{\partial(s)}$, we have equivalently:

$$\begin{aligned} f_s : A &\longrightarrow A \\ x(\text{ homogeneous }) &\longrightarrow I(s)^{\partial(x)} sxs^{-1}. \end{aligned}$$

Theorem 5.10

ψ is a surjective group morphism and $ker\psi = K^*$.
We have, then, the exact sequence:

$$1 \to K^* \to \{\textit{Homogeneous elements of } A^*\} \to Autgr(A) \to 1\,.$$

Proof.

1. ψ is a morphism similarly to the proof of Proposition 3.5.
2. ψ is surjective, obviously.
3. We now compute the kernel. Let $s \in A^*$, homogeneous. We have $s \in ker(\psi) \Leftrightarrow f_s(x) = x \quad \forall x \in A \Leftrightarrow I(s)^{\partial(x)} sxs^{-1} = x \quad \forall x \in A$.

In particular, if $x \in A_0$, we have $sx = xs$. So $s \in Z(A) = K \oplus Kz \oplus \cdots \oplus Kz^{n-1}$. Then, $x = f_s(x) = I(s)^{\partial(x)} sxs^{-1} = I(s)^{\partial(x)} x \quad \forall x \in A$. So, $I(s) = 1 \Rightarrow \partial(s) = 0 \Rightarrow s \in A_0$. Since $s \in Z(A) \cap A_0 \Rightarrow s \in K$. And it is invertible. □

These examples can be seen as a generalization of Clifford algebras. In a similar way as in Example 3, we could define a generalized Clifford algebra with m generators, $e_1, \ldots, e_m$.

In the same way that the structure theorem for central simple $\mathbb{Z}/2\mathbb{Z}$-graded algebras, in particular for Clifford algebras [3, Chap. 5 Thm. 2.4, 2.5], implies that the group of graded automorphisms is determined according to the parity of the number of generators as K-algebra, an analogous structure theorem for $\mathbb{Z}/n\mathbb{Z}$-graded algebras, should give, using the techniques described above, the same assertion for the group of graded automorphisms for generalized Clifford algebras.

References

1. N. Bourbaki, *Algèbre. Chapitre 9: Formes sesquilinéaires et formes quadratiques*, Hermann, Paris, 1959.
2. B. Farb and R.K. Dennis, *Noncommutative Algebra*, Springer-Verlag (Graduate Texts in Mathematics, 144), New York, 1993.
3. T.Y. Lam, *The algebraic theory of quadratic forms*, W.A. Benjamin, Inc.(Advanced Book program), Massachusetts, 1973.
4. R. Pierce, *Associative Algebras*, Springer-Verlag (Graduate Texts in Mathematics, 88), New York, 1982.