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On the quantum cohomology of the plane, old and new, and a K3 analogue

Z. Ran

Department of Mathematics, University of California, Riverside, California 929521, USA

E-mail: ziv@ucrmath.ucr.edu

Abstract

We describe a method for counting maps of curves of given genus (and variable moduli) to $\mathbb{P}^2$, essentially by splitting the $\mathbb{P}^2$ in half; then specialising to the case of genus 0 we show that the method of quantum cohomology may be viewed as the "mirror" of the former method where one splits the $\mathbb{P}^1$ rather than the $\mathbb{P}^2$; finally we indicate an analogue of the former method where $\mathbb{P}^2$ is replaced by a K3 quartic.

Recent work on Mirror Symmetry and Quantum Cohomology has contributed to a revival of interest in problems of a classical nature in Enumerative Geometry (cf. [2] and references therein). These problems involve (holomorphic) maps

$$f : C \to X \tag{1}$$

where X is a fixed variety and C is a compact Riemann surface whose moduli are sometimes fixed ("Gromov-Witten") but here will not be, unless otherwise stated. While the case $\dim X = 1$ is not entirely without interest (cf. [1]), the problem begins in earnest with $\dim X = 2$ and naturally the simplest such X is $\mathbb{P}^2$. Here the problem specifically is to count the images $f(C)$ of maps (1) where C has genus g, $f(C)$ has degree d and passes through $3d+g-1$ fixed points in $\mathbb{P}^2$. This problem has already, in essence, been solved in the author's earlier paper [3] by means of a recursive method (we note however that the formula in [3], (3c.1), (3c.3) is trivially misstated and the factor $c(\tilde{K}_1, \tilde{K}_2)$ should not be present).

Our purpose here is twofold. In Section 1 we give a partial exposition of the method of [3] and illustrate it on a couple of new examples, namely the curves of degree d and genus $g = \frac{(d-1)(d-2)}{2} - 2$ (i.e. with 2 nodes); and the rational quartics. We recover classical formulae due, respectively, to Roberts [4] and Zeuthen [6]. Hopefully, this will help make the method of [3] more accessible. In Section 2 we show that the method of Kontsevich et al., at least as exposed in [2], may be viewed as none other than the "dual" of that of [3] for the case of rational curves, "dual" meaning "interchanging source and target".

This paper owes its existence to the unfailing encouragement of Bill Fulton, who believed all along in [3]; it is indeed a pleasure to thank him here.

1. Old

We find it technically convenient here to work with possibly reducible curves; the modifications or "correction terms" needed to treat the irreducible case are a routine matter.

Consider the locus $V_{d,\delta}$ of (not necessarily irreducible) curves of degree d in $\mathbb{P}^2$ having δ ordinary nodes. This is well known to be a smooth locally closed subvariety of pure codimension δ in $\mathbb{P}^{\binom{d+2}{2}-1}$ and we are interested in its degree as such, which may be interpreted as the number of curves of $V_{d,\delta}$ passing through $\binom{d+2}{2} - \delta - 1$ general points in $\mathbb{P}^2$, a number which we denote by $N_{d,\delta}$. The idea is to get at $N_{d,\delta}$ by a recursive procedure, based on specializing $\mathbb{P}^2$ to a surface (called a "fan")

$$S_0 = S_1 \cup S_2$$

where $S_1 = Bl_0(\mathbb{P}^2)$ (the "bottom" component), $S_2 = \mathbb{P}^2$ (the "top" component) and $E = S_1 \cap S_2$ (the "axis") embedded in S_i with self-intersection $2i-3, i = 1, 2$. Corresponding to this is a specialization

$$V_{d,\delta} \to \sum m(\pi) V_{(d,e),(\delta_1,\delta_2),\pi}\,, \tag{2}$$

where $V_{(d,e),(\delta_1,\delta_2),\pi}$ is a family of Cartier divisors on S_0 whose general member C_0 may be described as follows:

- $C_0 = C_1 \cup C_2$,
- $C_1 \in |dH - eE|_{S_1}, C_2 \in |eE|_{S_2}$ nodal curves with δ_1 (resp. δ_2) nodes, smooth near E,
- the divisor $D = C_1.E = C_2.E$ has shape π, i.e. π is a partition having ℓ_i blocks of size i (to be written as $\pi = [\ell_i]$) and $D = \sum_{i=1}^{r} \sum_{j=1}^{\ell_i} iQ_{ij}, Q_{ij} \in E$

distinct. Moreover $m(\pi) = \prod_1^r i^{\ell_i}$ and the sum is extended over all data $((d,e),(\delta_1,\delta_2),\pi)$ satisfying

$$\delta_1 + \delta_2 + \sum_{i=1}^{r}(i-1)\ell_i = \delta \tag{3}$$

(i.e. each i-tacnode iQ_{ij} "counts as $i-1$ nodes").

Now to apply the specialization (2) to the degree question, we specialize our point set on $\mathbb{P}^2$ to a collection of points on S_0, which a priori we may distribute at will among S_1 and S_2, with each distribution giving rise to some formula which, however, may or may not be usable. For the purpose of the present discussion we will make the important simplifying assumption

$$\delta < d\,,$$

and put $d+1$ points on S_1 and the remaining $\binom{d+1}{2} - 1 - \delta$ on S_2. It is then easy to see that the only limit components $V.$ that will contribute to the resulting formula will be ones with

$$e = d - 1\,.$$

For those, we can write

$$C_1 = C_{1,0} + \sum_{i=1}^{\delta_1} R_i \tag{4}$$

with $C_{1,0}$ a smooth (rational) curve of "type" $(d-\delta_1, d-\delta_1-1)$ (i.e. $C_{1,0} \in |(d-\delta_1)H - (d-\delta_1-1)E|$) and R_i distinct rulings.

Now let us say that a partition $\pi' = [\ell_i'] \le \pi = [\ell_i]$ if $\ell_i' \le \ell_i \forall i$, in which case we may define the complementary partition $\pi - \pi' = [\ell_i - \ell_i']$; also put $|\pi| = \sum i\ell_i$, $s(\pi) = \sum \ell_i$, $n(\pi) = \frac{s(\pi)!}{\ell_1!\cdots\ell_r!}$. Counting the degree of a limit component $V_1 = \{C_1 \cup C_2\}$ in terms of those of $\{C_1\}$ and $\{C_2\}$ is basically a matter of decomposing the "diagonal" condition $C_1.E = C_2.E$ correspondingly to the standard Kunneth decomposition of the diagonal class on the product of $\Pi\mathbb{P}^{\ell_i}$ with itself; this leads to a sum of conditions corresponding to partitions $\pi' \le \pi$, each amounting to fixing the location on E of a portion D' of $C_1.E$ corresponding to π' and the complementary portion D'' of $C_2.E$ corresponding to $\pi - \pi'$. The resulting formula is as follows.

$$N_{d,\delta} = \sum_{|\pi|=d-1} m(\pi) \sum_{\substack{\pi'=[\ell_i'] \\ \le \pi = [\ell_i]}} m(\pi-\pi')n(\pi-\pi')N_{d-1,\delta-s(\pi-\pi')+s(\pi)-d+1,\pi-\pi',\pi'}$$

$$\times \sum_{j=0}^{\ell_1'} \binom{\ell_1'}{j}\binom{d+1}{s(\pi-\pi')-j}. \tag{5}$$

Here $N_{e,\delta_2,\pi'',\pi'}$ denotes the degree of the locus of nodal curves of degree e with δ_2 nodes meeting a fixed line E in a fixed divisor of shape π'' plus a divisor of shape π'. We have used the fact that $\delta_1 = s(\pi - \pi')$, which comes from the observation that the number of "axis" conditions on the bottom curve C_1, i.e. $|\pi| - s(\pi - \pi') = d - 1 - s(\pi - \pi')$, plus the number of "interior" points imposed, i.e. $d+1$, must equal the dimension of the family (4), i.e. $2d - \delta_1$. Also, the factor $m(\pi - \pi')n(\pi - \pi')$ is simply the degree of the "discriminant" variety of divisors of shape $\pi - \pi'$ on $E = \mathbb{P}^1$, while the binomial factors correspond to letting j of the rulings go through some of the multiplicity -1 part of D' with the remaining $\delta_1 - j$ going through some of the $d+1$ interior points.

Now of course in general the formula (5) is not by itself sufficient as one needs a recursive formula starting and ending with the $N_{e,\delta_2,\pi'',\pi'}$ or something similar. Such a formula is indeed given in [3], and it is not our purpose to reproduce it here. In the examples worked out below the necessary further recursion is relatively straightforward, and will be indicated.

EXAMPLE 1: $N_{d,2}$

There are seven relevant limit components and we proceed to list them and their contributions.

A. $V_{(d,d-1),(0,2),[d-1]}$; multiplicity $m = 1$; contribution $N_{d-1,2}$

B. $V_{(d,d-1),(1,1),[d-1]}$; $m = 1$. As $\delta_1 = 1$ we must take $\pi' = [d-2], \pi - \pi' = [1]$ so $j = 0$ or 1 and the contribution is $(d + 1 + d - 2).N_{d-1,1,[1],[d-2]} = 3(2d-1)(d-2)^2$.

C. $V_{(d,d-1),(2,0),[d-1]}$; $m = 1$; $\pi' = [d-3]; j = 0, 1, 2$, contribution $= (\binom{d-3}{2} + (d-3)(d+1) + \binom{d+1}{2}).N_{d-1,0,[2],[d-3]} = 2d^2 - 5d + 3$.

D. $V_{(d,d-1),(0,1),[d-3,1]}$; $m = 2, \delta_1 = 0 \Rightarrow \pi' = \pi$, so contribution is $2N_{d-1,1,0,[d-3,1]}$.

By an easier but simpler recursion (involving 1 node and 1 tangency), the latter evaluates to $12(d-1)(d-2)(d-3)$.

E. $V_{(d,d-1),(1,0),[d-3,1]}; m = 2$. $\pi' = [d-3]$ or $[d-4, 1]$, contribution $= 8(d-1)(d-3)$.

F. $V_{(d,d-1),(0,0),[d-4,0,1]}, m = 3, \pi' = \pi$, contribution $9d - 27$.

G. $V_{(d,d-1),(0,0),[d-5,2]}, m = 4, \pi' = \pi$, contribution 4.4. $\binom{d-3}{2} = 9d^2 - 56d + 96$.

Summing up, we get

$$N_{d,2} - N_{d-1,2} = 18d^3 - 81d^2 + 84d + 12\,.$$

Moreover it is easy to see that $N_{3,2} = \binom{7}{2} = 21$ so by integrating we get

$$N_{d,2} = \frac{9}{2}d^4 - 18d^3 + 6d^2 + \frac{81}{2}d - 33\,.$$

This is a classical formula due to S. Roberts [4], which has been given modern treatment by I. Vainsencher [5]. Note that the curves are automatically irreducible if $d \geq 4$.

Example 2: $N_{4,3}$

Here we have seven limit components.

A. $V_{(4,3),(0,3),[3]}, m = 1$, contribution 15.

B. $V_{(4,3),(1,2),[3]}, m = 1$ contribution $21.7 = 147$.

C. $V_{(4,3),(2,1),[3]}, m = 1$, contribution $15N_{3,1,[2],[1]} = 180$.

D. $V_{(4,3),(3,0),[3]}, m = 1$, contribution $\binom{5}{2} = 10$.

E. $V_{(4,3),(1,1),[1,1]}, m = 2, \pi' = [1]$ or $[0,1]$. Contribution $2.2N_{3,1,[0,1],[1]} + 2.5.N_{3,1,[1],[0,1]}$.

By a similar but simpler recursion the latter N's evaluate respectively to 10, 16, so the total contribution is 200.

F. $V_{(4,3),(0,2),[1,1]}, m = 2, \pi = \pi' = [1,1]$, contribution $2.15.2 = 60$.

G. $V_{(4,3),(0,1),[0,0,1]}, m = 3, \pi = \pi' = [0,0,1]$, contribution $3.N_{3,1,[0],[0,0,1]}$.

By a similar but simpler recursion, the latter N is 21, so the contribution is 63.

Summing up, we get

$$N_{4,3} = 675 = 5^2.3^3.$$

As the $\{cubic + line\}$ locus clearly has degree $\binom{11}{2} = 55$, we obtain 620 as the number of irreducible rational quartics through 11 points, (cf. [6]).

Remark. Much progress on the computational aspect of $N_{d,\delta}$ was recently made by Y. Choi (UCR dissertation, to appear).

2. New

The new approach works for maps from a fixed curve C, say to $\mathbb{P}^2$. For simplicity we will assume $C = \mathbb{P}^1$. Considering rational curves of degree d in $\mathbb{P}^2$ amounts to considering curves of bidegree $(1, d)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, and the old method to count them is by specialising the $\mathbb{P}^2$ factor to a fan; the new approach on the other hand is to specialise the $\mathbb{P}^1$ factor to a "1-dimensional fan", i.e. to

$$C_0 = C_1 \cup C_2, C_i = \mathbb{P}^1, C_1 \cap C_2 = \{x\}\,.$$

Because $\mathbb{P}^1$ is simpler than $\mathbb{P}^2$ this approach works better in this case; on the other hand it is apparently unknown how to make it work when the source curve is allowed to vary with moduli.

To be precise, fix a pair of points y_1, y_2 and a pair of lines L_3, L_4 in $\mathbb{P}^2$ and 4 points $x_1, ...x_4 \in \mathbb{P}^1 = C$ and consider curves of bidegree $(1, d)$ in $C \times \mathbb{P}^2$ containing $(x_1, y_1), (x_2, y_2)$ and meeting $x_3 \times L_3, x_4 \times L_4$, as well as a further collection of $3d - 4$ "horizontal" lines $C \times z_j$. We then specialise this to $C_0 \times \mathbb{P}^2$ in two ways: (A) $x_1..., x_4$ specialise to $x_{1,1}, x_{2,1} \in C_1, x_{3,2}, x_{4,2} \in C_2$; (B) x_1, x_3, x_2, x_4 specialise to $x_{1,1}, x_{3,1} \in C_1, x_{2,1}, x_{4,2} \in C_2$. In the (A) limit it is possible to have a component of bidegree $(1, 0)$ in $C_2 \times (L_3 \cap L_4)$, while in the (B) limit all curves have bidegrees $(1, d_1) \cup (1, d_2), d_1 + d_2 = d, d_i > 0$. Thus letting n_d denote the number of rational curves in $\mathbb{P}^2$ through $3d - 1$ points, writing $(A) = (B)$ we get an equation of the form

$$n_d + f(n_1, ..., n_{d-1}) = g(n_1, ..., n_{d-1})$$

for suitable quadratic expressions f, g, which may be solved for n_d.

EXAMPLE: $d = 4$

$$f = \binom{8}{2}.12.1.1.1.3 + \binom{8}{3}.1.1.2.2.4 + 1.1.12.3.3.3 = 2228$$

with the summands corresponding to $d_1 = 3, 2, 1$ and, e.g. in the first product the factors corresponding to: choosing 2 of the 8 points z_j for the image of C_2 to go through; the number of possible images of C_1, C_2, x_3, x_4, x;

$$g = 8.12.1.3.1.3 + \binom{8}{4}.1.1.2.2.4 + 8.1.12.1.3.3 = 2848$$

$$n_4 = 620.$$

3. K3

We consider a general smooth quartic surface $S \subset \mathbb{P}^3$ and wish to count rational, i.e. trinodal or tritangent plane sections of S. For this we degenerate S to

$$S_0 = S_1 \cup_E S_2$$

where S_1 is the blow-up of $\mathbb{P}^2$ at a general $(3, 4)$ complete intersection $A \cap B = \{p_1, ...p_{12}\}$ with E the proper transform of A, S_2 is a cubic surface with E as hyperplane section, and E having opposite normal bundles in S_1 and S_2. It is easy to see that S_0 embeds as a divisor of type $(4, 3)$ in a limit of $\mathbb{P}^3$ analogous to the

limit of $\mathbb{P}^2$ considered above, so that S_0 is a limit of a K3 quartic. Now consider limit plane sections of S_0

$$C_0 = C_1 \cup C_2.$$

These come in 2 types, (1,0) and (1,1), depending on whether the plane contains the blown-up point.

Type (1,0).

Here C_2 is empty while C_1 corresponds to a plane quartic through $\{p_1, ..., p_{12}\}$ or equivalently $\{p_1, ..., p_{11}\}$. The rational (trinodal) curves C_1 coincide with a generic fibre (corresponding to B) of the projection of the Severi variety $V_{4,3}$ from the $\mathbb{P}^2$ of quartics containing A. As this $\mathbb{P}^2$, however non-generic, is disjoint from $V_{4,3}$, the fibre consists of 675 generic, multiplicity-1 points, of which 620 correspond to irreducible curves.

Type (1,1).

Here C_2 is a plane section of S_2 while C_1 corresponds to a line L in $\mathbb{P}^2$. Depending on the position of L relative to A, we get six limit components corresponding exactly to those configurations in Example 2 above which are of "irreducible type", i.e. where the off-axis nodes do not disconnect the curve.

A. L is general while C_2 is one of the 45 plane triangles on S_2. Contribution 45.
B. L contains exactly one p_i while C_2 contains one of the 27 lines on S_2. Contribution 324.
C. L contains exactly two p_i's; C_2 is a nodal (off E) member of a pencil of plane sections on S_2, of which there are 12. Contribution 792.
E. L contains p_i and is tangent to A elsewhere (12.4 such), C_2 is a singular (off E) member of the corresponding pencil (10 such). Contribution 960.
F. C_2 contains a line M on S_2 and L contains $M \cap E$ and is elsewhere tangent to A. Contribution 216.
G. L is flex-tangent to A, C_2 is one of the 9 members of the corresponding pencil singular off A. Contribution 243.

For total we obtain the classically-known number 3200.

Remark. Recently Yau and Zaslow (hep-th/9512121) have found a remarkable formula for the number of rational hyperplane sections on any K3 surface.

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