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Linear systems representing threefolds which are scrolls on a rational surface

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Dedicated to the memory of Ferran Serrano

Abstract

Let X be a scroll over a rational surface. We construct a linear system of surfaces in $\mathbb{P}^3$ yielding a birational map $\mathbb{P}^3 \dashrightarrow X$. We apply this construction to the scrolls of Bordiga and Palatini.

Introduction

Let $X \subset \mathbb{P}^r$ be a normal threefold which is a scroll over a rational surface. Then X is a rational variety. In this paper we will expose a method to determine a r-dimensional linear system $|\Sigma|$ of surfaces in $\mathbb{P}^3$ such that the rational map $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^r$ associated to $|\Sigma|$ maps $\mathbb{P}^3$ birationally onto X.

Our interest for this problem arose when studying smooth threefolds of $\mathbb{P}^5$. In [4] we considered smooth rational threefolds of $\mathbb{P}^5$ with rational hyperplane section: it is known that there are exactly five families of such varieties, of degrees d, with $3 \le d \le 7$. Precisely, for $d = 3$ there is $\mathbb{P}^2 \times \mathbb{P}^1$, for $d = 4$ the Del Pezzo threefold, complete intersection of two hyperquadrics, for $d = 5$ the Castelnuovo threefold, for $d = 6$ the Bordiga scroll, for $d = 7$ the Palatini scroll (for more reference, see [7]). We proved that for $3 \le d \le 6$ for any such threefold X there exists a line $L \subset X$ such that the projection centered at L from X to $\mathbb{P}^3$ is birational. In this way, we obtained a "uniform" description of the wanted linear systems $|\Sigma|$. Conversely, if

$d = 7$, i.e. if X is a Palatini scroll, we proved that a line with the above property never exists.

It remained the problem of finding a linear system $|\Sigma|$ in this last case. The method we have used for solving the problem is based on the fact that Palatini threefold is a scroll over a rational surface, precisely over a smooth cubic surface of $\mathbb{P}^3$. In fact a similar construction can be made for any scroll X over a rational surface V. The basic idea is classical: it is quoted e.g. in the paper [3] by Jongmans. Roughly speaking, it consists in establishing first an explicit birational correspondence between the lines of the scroll X and the lines of $\mathbb{P}^3$ passing through a fixed point P, and then a map between any two corresponding lines is given by projection from a fixed center.

More precisely: assume that X is contained in $\mathbb{P}^r$ and fix a general linear subvariety Λ of $\mathbb{P}^r$ of codimension 2. We compose the scroll map $X \to V$ with a fixed birational map $V \dashrightarrow \mathbb{P}^2$ to get a rational map $h : X \dashrightarrow \mathbb{P}^2$. Then we have an explicit birational map $\alpha : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ defined by $\alpha = (h, \pi_\Lambda)$, where $\pi_\Lambda : X \dashrightarrow \mathbb{P}^1$ is the projection centered at Λ. Finally, α has to be composed with the projection $\pi_L : \mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ from a line L, contained in one of the planes of $\mathbb{P}^2 \times \mathbb{P}^1$. Note that the projection π_L which is clearly birational, contracts such plane to a point P.

Here we perform a detailed study of such birational map $X \dashrightarrow \mathbb{P}^3$ and of its birational inverse from a modern point of view. The fundamental step is the complete determination of the exceptional divisors and the fundamental loci for both maps. From this it follows readily a rather satisfactory description of the linear system $|\Sigma|$ of surfaces of $\mathbb{P}^3$ defining the map $\mathbb{P}^3 \dashrightarrow X$, in particular of its base locus. Experience shows the importance of the knowledge of the *arithmetic* genus of the 1-dimensional components of $Bs(|\Sigma|)$; we show how this can be done by working-out the case of the Bordiga scroll.

It is interesting to note that the birational maps from $\mathbb{P}^3$ to the Bordiga scroll obtained in this paper and in [4] are different. Therefore, by composition, we obtain elements of the Cremona group of $\mathbb{P}^3$, given in general by surfaces of degree 5 and 7. For special choices made in the construction of the maps, we get classical cubo–cubic transformations.

The content of the various sections is as follows. Some basic facts concerning birational maps are recalled for the reader's convenience in § 1. In § 2 we study the birational map $g : X \dashrightarrow \mathbb{P}^3$ described above; in particular we find its exceptional divisors and the exceptional divisors of the inverse of g. § 3 is devoted to the study of the linear system $|\Sigma|$, and contains the main result of the paper: Theorem 3.1. The statement of this theorem concerns the determination of the irreducible components of $Bs(|\Sigma|)$, their mutual positions and their infinitesimal structure. Finally, in § 4 we apply Theorem 3.1 to the scrolls of Bordiga and Palatini. The results on $|\Sigma|$ obtained in these two cases are collected in the following table (where we denote by $A \cdot B$ the length of the zero dimensional scheme $A \cap B$):

Variety	**Degree**	deg Σ	**Base locus**
Bordiga scroll	6	5	• a curve B_2 of degree 14, geometric genus 3, arithmetic genus 23, with an ordinary multiple point P of multiplicity 10 • a line B_1 not containing P, with $B_1 \cdot B_2 = 4$
Palatini scroll	7	7	• a curve B_2 of degree 11, geometric genus 4, arithmetic genus 10, with an ordinary multiple point P of multiplicity 5 • a line B_1 not containing P, with $B_1 \cdot B_2 = 6$ • the first infinitesimal neighborhoods of six lines B_i, each containing P, with $B_1 \cdot B_i = 0$ and $B_2 \cdot B_i = 2$

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1. Some property of birational maps

We will always work over an algebraically closed field of characteristic 0. In this section we collect some classically known facts about birational maps. Moreover, we give the main features of a particular birational map, namely a projection $\mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$, which we shall need in the subsequent sections. For precise references to the literature and for proofs see [4].

Let $f : Y \dashrightarrow X$ denote a birational map between *normal* varieties. A point $y \in Y$ is called *fundamental* for f if f is not regular at y. We will denote by F the set of the fundamental points for f, and we will call it the *fundamental locus* for f.

The importance of the fundamental locus for our investigation is due to the fact that, *if a birational map $f : \mathbb{P}^n \dashrightarrow X$ is defined by a linear system $|\Sigma|$ of hypersurfaces in $\mathbb{P}^n$, then the irreducible components of the fundamental locus of f are precisely those of the base locus $Bs(|\Sigma|)$ of $|\Sigma|$.*

To determine the fundamental locus the starting point is

Van der Waerden's Purity Theorem

With the notations introduced above, let $W \subset Y$ be an irreducible component of the fundamental locus of f. Let us denote by g the birational inverse of f (we will maintain this notation). Assume that $g(X) \cap W$ is dense in W and that W is not contained in the singular locus of Y. Then any component E of $g^{-1}(W)$ is of codimension 1 in X.

We will call any $E \subset X$ as above an *exceptional divisor* for g. Since the fundamental locus F for f is a closed subset of Y and $codim_Y(F) \geq 2$, every irreducible component of F is of the form $g(E)$ for some exceptional divisor E of g. Therefore, once we know the exceptional divisors for g, we will be able to describe $Bs(|\Sigma|)$.

Let us recall that a *characteristic curve* Γ of a linear system $|\Sigma|$ of surfaces of $\mathbb{P}^3$ is the free intersection of two general surfaces of $|\Sigma|$. If E is an exceptional surface for f, the birational map associated to $|\Sigma|$, and B is the corresponding base curve, then $\deg\ E = \#(B \cap \Gamma)$.

In the next section we will need the following example.

Let $X \subset \mathbb{P}^5$ be the image of $\mathbb{P}^2 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the Segre map s. For a fixed point $a \in \mathbb{P}^1$, we will denote $F_1 := s(\mathbb{P}^2 \times a)$, a plane on X. Moreover, let l be a line in $\mathbb{P}^2$ and set $F_2 := s(l \times \mathbb{P}^1)$. F_2 is a quadric surface on X. Finally, denote by L the line $F_1 \cap F_2$. Then:

Proposition 1.1

(i) The projection map $\pi_L : X \dashrightarrow \mathbb{P}^3$ with center L is birational;
(ii) the exceptional divisors of π_L are F_1 and F_2, in particular $\pi_L(F_1)$ is a single point P of $\mathbb{P}^3$, and $\pi_L(F_2)$ is a line $B \subset \mathbb{P}^3$ such that $P \notin B$;
(iii) the map π_L^{-1} is defined by the linear system of quadrics $|\Sigma|$ with base locus $B \cup P$;
(iv) the only exceptional divisor for π_L^{-1} is the plane $\Phi = \langle B \cup P \rangle$, which is contracted to the line L.

2. Construction of a birational map from $\mathbb{P}^3$ to X

Let $X \subset \mathbb{P}^r$ be a normal scroll of degree d over a rational surface V. Let $u : X \to V$ denote the scroll map. We will construct, now, a birational map $g : X \dashrightarrow \mathbb{P}^3$.

Let us fix a birational map $v : V \dashrightarrow \mathbb{P}^2$ and set $h := v \circ u : X \dashrightarrow \mathbb{P}^2$. Moreover, we fix inside $\mathbb{P}^r$ a linear subvariety $\Lambda \simeq \mathbb{P}^{r-2}$ such that Λ does not contain any line of the scroll X. Let $\pi_\Lambda : \mathbb{P}^r \dashrightarrow \mathbb{P}^1$ denote the projection map. Then we have a rational map

$$\alpha : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$$

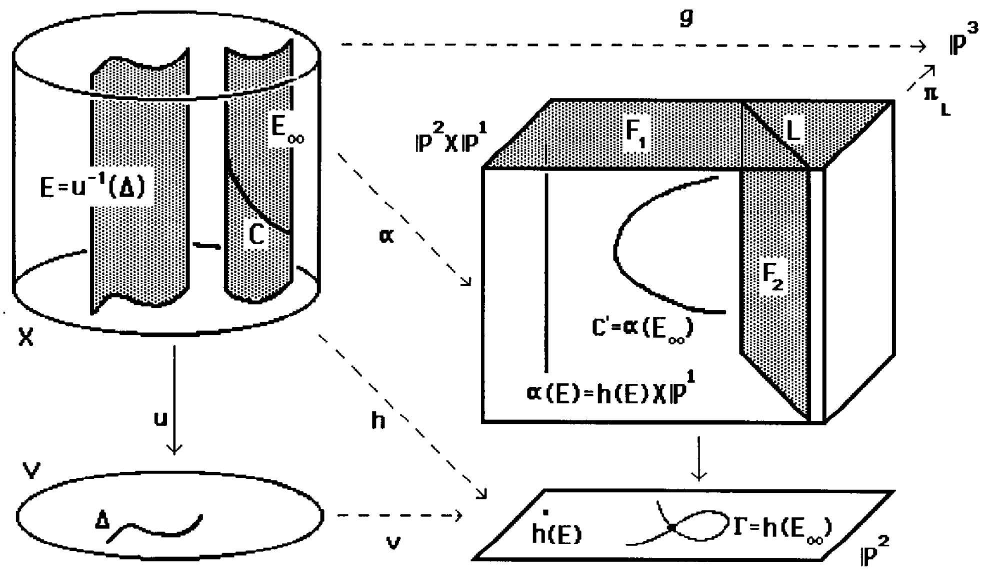
defined by

$$\alpha(x) := \big(h(x), \pi_\Lambda(x)\big)$$

which clearly results to be birational (see figure 1 below).

Let $L \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a line as in § 1. If we compose α with the projection π_L we get a birational map $g := \pi_L \circ \alpha : X \dashrightarrow \mathbb{P}^3$.

Let us remark that the lines of the scroll X correspond via the map g to the lines in $\mathbb{P}^3$ through the fixed point $P = \pi_L(F_1)$.



– Fig. 1 –

Let f be the birational inverse of g and let $|\Sigma|$ be the linear system of surfaces in $\mathbb{P}^3$ defining f. The main goal of this section is determination of all the exceptional divisors for g.

Since g is the composition of the birational maps α and π_L, to find out its exceptional divisors it is sufficient to determine the exceptional divisors for α and π_L, separately. For π_L this was already done in Proposition 1.1.

The exceptional divisors for α are of two different kinds, i.e. coming either from h or from π_Λ. First of all, if $\Delta \subset V$ is an exceptional divisor for the map $v : V \dashrightarrow \mathbb{P}^2$, then α contracts the surface $E := u^{-1}(\Delta)$ to the line $v(\Delta) \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$ (see Fig. 1).

On the other hand, α is not regular on the curve $C := X \cap \Lambda$. Nevertheless, it is clear that α is constant on any line of the scroll X which meets C; therefore α contracts the ruled surface $E_\infty := u^{-1}(u(C))$ to a curve $C' \subset \mathbb{P}^2 \times \mathbb{P}^1$. Clearly, there is no other exceptional divisor for α, and we conclude:

Proposition 2.1

Let F_1 and F_2 be the surfaces on $\mathbb{P}^2 \times \mathbb{P}^1$ defined in Proposition 1.1, and set $E_i := \alpha^{-1}(F_i)$ where $i = 1, 2$. Let $\Delta_3, \ldots, \Delta_s$ be the exceptional divisors for v on V, and set $E_i := u^{-1}(\Delta_i)$, for $i = 3, \ldots, s$. The exceptional divisors for g are the surfaces $E_1, E_2, E_3 \ldots, E_s, E_\infty$.

Note, in particular, that E_1 is a hyperplane section of X by construction of α.

We can determine the exceptional divisors for the map $f : \mathbb{P}^3 \dashrightarrow X$ by a similar "step by step" procedure, by determining the exceptional divisors for π_L^{-1} and for α^{-1} separately.

Recall from Proposition 1.1 that the unique exceptional divisor for π_L^{-1} is the plane generated by the point $P := \pi_L(F_1)$ and the line $B_1 := \pi_L(F_2)$.

As for the exceptional divisors for α^{-1} they arise either from h or from π_Λ. Let $T \subset \mathbb{P}^2$ be an exceptional divisor of $v^{-1} : \mathbb{P}^2 \dashrightarrow V$. Then α^{-1} contracts the surface $T \times \mathbb{P}^1$ to the line $u^{-1}(v^{-1}(T)) = h^{-1}(T)$.

There is one more exceptional divisor, arising from π_Λ. Let $\Gamma := h(C) \subset \mathbb{P}^2$ and set $U := \Gamma \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$ (note that the curve C' defined above lies on U; see Fig. 2). Finally, set $\Phi := \pi_L(U)$.

Proposition 2.2

The exceptional divisors for f are the plane $\langle P \cup B_1 \rangle$, the surfaces $\pi_L(T \times \mathbb{P}^1)$, where T is as above, and the surface Φ. Moreover, Φ is a cone with vertex P and degree n, where $n = \deg \Gamma$.

Proof. For any $b \in \Gamma$ the line $b \times \mathbb{P}^1$ is contained in U, and it is mapped by π_L to a line through P. Hence $\Phi = \pi_L(U)$ is a cone with vertex P. It remains to show that Φ is a cone of degree n. A line in $\mathbb{P}^2$ cuts Γ in n points. Therefore, a line $R \subset \mathbb{P}^2 \times a$ cuts U in n points. Since $\pi_L(R)$ is a line in $\mathbb{P}^3$, the degree of the cone Φ is n.

To prove that Φ is an exceptional divisor for f it is sufficient to show that α^{-1} contracts the surface U to the curve $C \subset X$ defined above. More precisely it is sufficient to show that α^{-1} contracts any line $Q \times \mathbb{P}^1$, where $Q \in \Gamma$, to a point of C. In fact, if $Q' = (Q, a) \in Q \times \mathbb{P}^1$ is a general (="not in C'") point of the line, then $\alpha^{-1}(Q')$ is the intersection of the line $h^{-1}(Q')$ of the scroll with a suitable hyperplane $H \subset \mathbb{P}^r$ containing Λ. But this intersection is clearly a point on C independent of the hyperplane H. $\square$

For further reference we make the following.

Remark 2.3. The planes $\mathbb{P}^2 \times a$ on $\mathbb{P}^2 \times \mathbb{P}^1$ correspond via the map π_L to the planes W_a in $\mathbb{P}^3$ containing the line B_1 and via the map α to the sections S_a of X with the hyperplanes H_a containing Λ.

3. Description of the linear system defining $f : \mathbb{P}^3 \dashrightarrow X$

The construction of the birational maps f and g made in the previous section depends on several choices. We will make now some harmless "transversality" assumptions; under them we will get a very precise description of $Bs(|\Sigma|)$.

First of all, we can assume $C \cap Sing(X) = \emptyset$. In fact, X is normal, hence $codim_X Sing(X) \geq 2$.

Now, let us consider the rational surfaces E_i for $i = 3, \ldots, s$ (see Proposition 2.1). Assume that E_i has degree δ_i in $\mathbb{P}^r$. In particular, being C a curve section of X, we have $C \cdot E_i = \delta_i$. By moving Λ, hence C, it is harmless to assume that $\bigcup_i u(C \cap E_i)$ are $\sum_i \delta_i$ distinct points on V.

Moreover, we will assume that the hyperplane $H \subset \mathbb{P}^r$ such that $E_1 = X \cap H$ does not contain any line of the scroll X through a point of $C \cap E_i$, for any $i = 3, \ldots, s$.

We can state now our main result:

Theorem 3.1

Let $X \subset \mathbb{P}^r$ be a normal scroll of degree d over a rational surface. Let f be the birational map from $\mathbb{P}^3$ to X defined in §2, and let $|\Sigma|$ be the linear system of surfaces defining f.

Keeping the notations introduced in §2, under the transversality assumptions made at the beginning of this section, we have:

(i) The 1-dimensional components of $Bs(|\Sigma|)$ are the line $B_1 := \pi_L(F_2)$, the lines $B_i := \pi_L(v(\Delta_i) \times \mathbb{P}^1)$ for $i = 3, \ldots, s$ and the curve $B_2 := \pi_L(C') \subset \Phi$. The unique 0-dimensional component of $Bs(|\Sigma|)$ is the point $P = \pi_L(F_1)$.

(ii) The curve B_2 intersects the line B_1 at n points (where $n = \deg(\Gamma)$) and is δ_i-secant to any line B_i, $i = 3, \ldots, s$. B_2 is unisecant all the other lines on Φ.

(iii) The degree of B_2 is equal to $\deg\ E_\infty - d + n$ and its geometric genus is equal to the sectional genus of X. The point P is a multiple point of multiplicity $\deg\ E_\infty - d$ for B_2. The tangent lines of B_2 at P are distinct and the branches of B_2 at P are nonsingular (for the terminology concerning singular points we follow [5]; see also § 4). If the sectional genus of X is strictly positive, then B_2 is smooth outside P.

(iv) The surfaces Σ are monoids of degree $n+1$ whose point of multiplicity n is P. The scheme structure of the 0-dimensional (embedded) component P of the base locus $Bs(|\Sigma|)$ is exactly that of $(n-1)$-th infinitesimal neighborhood of P. Let S be a hyperplane section of X corresponding to $\Sigma \in |\Sigma|$, and set $C_1 := S \cap E_1$. Then the tangent cone of Σ at P is $\pi_L(h(C_1) \times \mathbb{P}^1)$.

(v) The base locus of $|\Sigma|$ contains the $(\delta_i - 1)$-th infinitesimal neighborhood of B_i, for any $i = 3, \ldots, s$.

Proof. (i) follows from Propositions 2.1 and 1.1. In particular, by definition, $B_2 = g(E_\infty) = \pi_L(C')$. As remarked above, we have $C' \subset U$, hence B_2 lies on Φ.

On $\mathbb{P}^2 \times \mathbb{P}^1$ we have $F_2 = L \times \mathbb{P}^1$. Then, from $\deg\ \Gamma = n$ it follows that $C' \cdot F_2 = n$ and B_2 intersects the line B_1 at n points. To see that B_2 is δ_i-secant to B_i, it is sufficient to remark that all the δ_i points of $C \cap E_i$ have the same image in the map $h : X \dashrightarrow \mathbb{P}^2$, by definition of E_i (see Fig. 2, right).

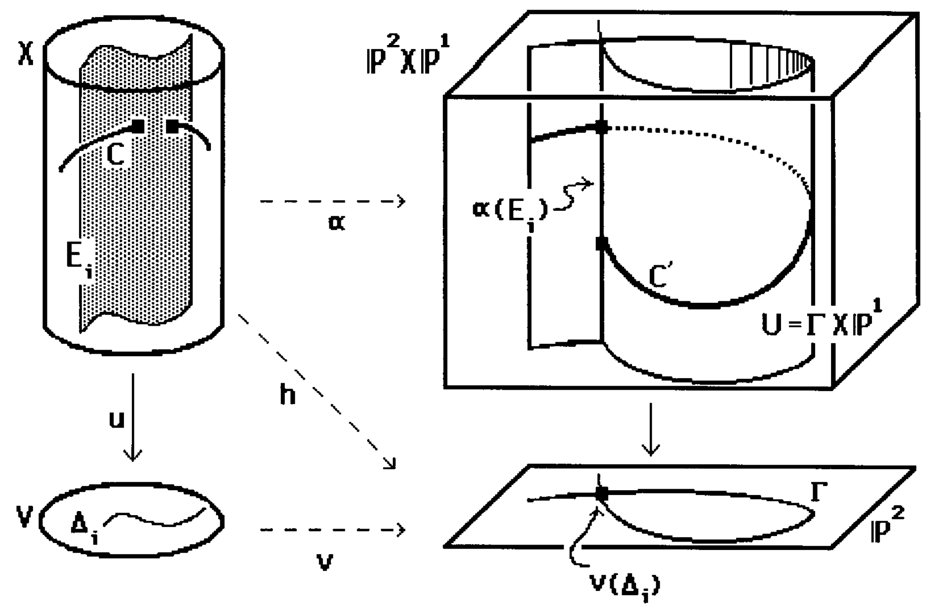
The fact that B_2 is unisecant to any line on the cone Φ which is different from $B_3, \ldots, B_s$ is easily checked, and this completes the proof of (ii).

To compute the degree of B_2, let us consider a plane $W \subset \mathbb{P}^3$ such that $B_1 \subset W$. By Remark 2.3, the points of $W \cap B_2$ outside B_1 correspond to points of $(\mathbb{P}^2 \times a) \cap C'$, hence they correspond to lines on $S_a \cap E_\infty$. Since $C \subset H_a$, from the definition of E_∞ we get $H_a \cap E_\infty = C \cup l_1 \cup \ldots \cup l_p$, where every l_j is a line of the ruling. By Bezout: $p = \deg\ E_\infty - d$. Since B_1 meets B_2 at n points, we get the announced degree for B_2.

From the discussion above it follows that any plane $\mathbb{P}^2 \times a$ on $\mathbb{P}^2 \times \mathbb{P}^1$ cuts C' at $\deg\ E_\infty - d$ points. In particular, taking $\mathbb{P}^2 \times a = F_1$ we conclude that P is a multiple point of B_2 of multiplicity $\deg E_\infty - d$.

To study the singular point P of B_2 we start by considering the common points of C' and F_1. These points are in one-to-one correspondence with the lines on E_∞

contained in the hyperplane $H \subset \mathbb{P}^r$ such that $E_1 = X \cap H$. We let $P_1, \ldots, P_r \in \mathbb{P}^2$ denote the image in h of such lines. Since the curve C is smooth and any line of the scroll E_∞ intersects C in exactly one point, *the curve C' is smooth.* The singular locus of the ruled surface $U \subset \mathbb{P}^2 \times \mathbb{P}^1$ is formed by the lines $b \times \mathbb{P}^1$, where b is a singular point of the curve $\Gamma = h(C)$. We have already seen that these points are exactly the points $v(\Delta_i)$. Therefore, by our assumption that the hyperplane $H \subset \mathbb{P}^r$ such that $E_1 = X \cap H$ does not contain any line of the scroll X through a point of $C \cap E_i$, for any $i = 3, \ldots, s$, it follows that the P_i's are smooth points for Γ. Hence, the points $c_i := P_i \times a$ of $C' \cap F_1$ ($F_1 = \mathbb{P}^2 \times a$) are smooth for both C' and U. Let T_i denote the tangent plane to U at c_i; then T_i contains the tangent line t to C' at c_i, and the line $P_i \times \mathbb{P}^1$. We remark that T_i and F_1 intersect along the line $T_{\Gamma,P_i} \times a$. Finally, assume that the line L on F_1 does not contain any point c_i. Then, the linear spans $\langle t \cup L \rangle$ and $\langle (T_{\Gamma,P_i} \times a) \cup L \rangle$ are the same. Now, it is easily seen by a direct computation that, if the lines R, S of the scroll $\mathbb{P}^2 \times \mathbb{P}^1$ are different, then for any plane M on $\mathbb{P}^2 \times \mathbb{P}^1$ the linear spans $\langle R \cup M \rangle$ and $\langle S \cup M \rangle$ are also different. From all this it follows that at P the curve B_2 has deg $E_\infty - d$ *different* tangent lines.



– Fig. 2 –

We can consider $\pi_L : C' \to B_2$ as the normalization of the curve B_2, since C' is smooth and π_L is birational. Moreover, the map π_L is unramified at every point c_i; in fact, in $\mathbb{P}^3 = \langle t \cup L \rangle$, where t denotes the tangent line to C' at c_i, the planes $\langle P \cup t \rangle$ and $\langle P \cup L \rangle$ are different. From this it follows that *the branches of B_2 at P are nonsingular* ([5]).

Since α induces a birational map between the curves C and C' and since π_L does the same between the curves C' and B_2, the geometric genus of B_2 is equal to the geometric genus of C, namely to the sectional genus of X.

To complete the proof of (iii) it remains to show that B_2 is smooth outside P. We start by remarking that *the cone $\pi_L(U) = \Phi \subset \mathbb{P}^3$ depends only on the choice of the plane F_1 on $\mathbb{P}^2 \times \mathbb{P}^1$, and not on the center of projection L on F_1*. In fact, let $L' \subset F_1$ be another line. Then, for the general line $b \times \mathbb{P}^1$ on the ruled surface U we have that the linear span of L and $b \times \mathbb{P}^1$ coincides with the linear span of L' and $b \times \mathbb{P}^1$. Therefore, on Φ we have the family of curves $\{\pi_L(C')\}$ with L varying in $\check{F}_1$. Moreover, it is clear that all these curves form a linear system Ω on Φ.

Now, C' intersects F_1 in at least two points because, otherwise, C' would be birationally isomorphic to $\mathbb{P}^1$, hence rational. But this contradicts our assumption that the sectional genus of X is > 0. So, we can choose as L a chord of C' contained in F_1.

The curve $\pi_L(C')$ is smooth outside P. In fact, a singular point would correspond to a chord M of C' intersecting L. In this case the plane $\langle M \cup L \rangle$ has four points in common with $\mathbb{P}^2 \times \mathbb{P}^1$, hence it is contained in $\mathbb{P}^2 \times \mathbb{P}^1$, since $\deg(\mathbb{P}^2 \times \mathbb{P}^1) = 3$. Then $\langle M \cup L \rangle = F_1$ because the only planes on $\mathbb{P}^2 \times \mathbb{P}^1$ are those of the ruling, and we are done.
Therefore, for the general line $L \subset F_1$ the curve $\pi_L(C')$ is smooth outside P.

By Proposition 2.2, to get the surface of $|\Sigma|$ representing the hyperplane section S of X with a hyperplane through Λ, we have just to add the cone Φ to the plane $g(S)$. Hence, the degree of the surfaces Σ is $n + 1$.

Let S be an arbitrary hyperplane section of X. Note that $S \cap E_1 \neq \emptyset$, hence $P \in \Sigma = g(S)$. Since the lines $R \subset \mathbb{P}^3$ through P correspond to the lines of the scroll, a general R intersects a fixed surface Σ outside P in only one point and Σ is a monoid with P as n-ple point.

The assertion concerning the tangent cone of Σ at P is easily checked and the proof of (iv) is complete.

Consider the restriction of g to E_i, namely $g : E_i \to B_i$; if we restrict further on to a hyperplane section of E_i, we get a covering of B_i of degree δ_i : this proves the last assertion. $\square$

Remark 3.2. As observed in the course of the above proof, the sections S_a of X with the hyperplanes H_a of $\mathbb{P}^r$ containing Λ correspond to the surfaces of $|\Sigma|$ which break into the cone Φ and a plane W_a containing the line B_1 (see also Remark 2.3). The surfaces $\Sigma \in |\Sigma|$ cut W_a along curves of degree $n+1$ all having B_1 as a component. By eliminating this fixed component we get a linear system $|\Sigma'|$ of plane curves of degree n on W_a. This linear system defines the inverse of the birational map $g : S_a \dashrightarrow W_a = \mathbb{P}^2$. In the case $|\Sigma'|$ is well understood and unique (see, e.g. [1],[2]), we get useful informations about $|\Sigma|$.

We will show how this remark can be used in the last section, where we will deduce from it the completeness of the linear system $|\Sigma|$ constructed above in the case of the Palatini scroll.

Remark 3.3. The restriction of $h : X \dashrightarrow \mathbb{P}^2$ to the hyperplane section E_1 of X is a birational map $h : E_1 \dashrightarrow \mathbb{P}^2$, whose inverse is given by a linear system $|nH_{\mathbb{P}^2} - Z|$, where Z denotes the base locus.

If the general hyperplane section S of X is linearly normal (i.e.: if the linear system cut out on S by the hyperplanes of $\mathbb{P}^r$ is complete), then $h(D) \in |nH_{\mathbb{P}^2} - Z|$ for every curve section D of X.

In fact, let H be a hyperplane of $\mathbb{P}^r$ containing D and set $S := H \cap X$ and $D' := H \cap E_1$. On S the curves D and D' are linearly equivalent, hence $h(D)$ and $h(D')$ are linearly equivalent on $\mathbb{P}^2$. From this it follows that the surfaces $h^{-1}h(D)$ and $h^{-1}h(D')$ are linearly equivalent on X, and therefore $D' = E_1 \cap h^{-1}h(D')$ is linearly equivalent to $E_1 \cap h^{-1}h(D)$ on E_1. Since D' is a hyperplane section of E_1, by our assumption $E_1 \cap h^{-1}h(D)$ is also a hyperplane section of E_1, and we can conclude $h(D) = h(E_1 \cap h^{-1}h(D)) \in |nH_{\mathbb{P}^2} - Z|$.

4. Examples: the scrolls of Bordiga and Palatini

In this section we will apply Theorem 3.1 to the scrolls of Bordiga and Palatini in $\mathbb{P}^5$.

Let $X \subset \mathbb{P}^5$ be a Bordiga scroll. It has degree 6, sectional genus 3 and it is a scroll over $\mathbb{P}^2$, the scroll map $u : X \to \mathbb{P}^2$ being the adjunction map. The hyperplane sections of X are Bordiga surfaces; in particular, $S \subset \mathbb{P}^4$ is the image of a rational map (birational onto S) $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ defined by the linear system $|4H_{\mathbb{P}^2} - \sum_{1 \le i \le 10} y_i|$ ([1]). The inverse of ϕ is defined on the whole S and is the adjunction map.

Therefore, for the Bordiga scroll the map h is equal to the scroll map u. In particular, there are no exceptional divisors Δ for the map $v : V \to \mathbb{P}^2$. Hence, the only exceptional divisors for $g : X \dashrightarrow \mathbb{P}^3$ are E_1, E_2, and E_∞, and the irreducible components of $Bs(|\Sigma|)$ are P, B_1, and B_2.

The surfaces Σ have degree 5. Since the scroll map $u : X \to \mathbb{P}^2$ is the adjunction map and $\deg \Gamma = 4$, we can compute the degree of E_∞ as follows:

$$\deg\ E_\infty = E_\infty \cdot H_X^2 = u^{-1}(4H_{\mathbb{P}^2}) \cdot H_X^2 = 4(2H_X + K_X) \cdot H_X^2\,.$$

But $K_X \cdot H_X^2 = -8$ ([7]), and we conclude $\deg\ E_\infty = 16$. Therefore, the degree of B_2 is 14, and the point P has multiplicity 10 for B_2.

We recall that the exceptional divisor E_1 for g is a hyperplane section of X. On E_1 there are exactly ten lines of the scroll, and the adjunction map h contracts these lines to the points $P_1, \dots, P_{10} \in \mathbb{P}^2$. By [4], Proposition 4.1, we can assume that these points satisfy any kind of "general position" assumption. Moreover, observe that, by construction of the map g, the set $\{P_1, \dots, P_{10}\} \subset \mathbb{P}^2$ is isomorphic to the plane section of the tangent cone to B_2 at P.

The first conclusion we can draw from these remarks is that *a surface of degree* 5 *in* $\mathbb{P}^3$ *containing* B_2 *is necessarily a monoid of vertex* P (consider the Taylor expansion at P of the equation of the surface).

In view of computing the arithmetic genus of B_2 starting from its geometric genus $p_g(B_2) = 3$, it is advisable to have more precise informations on the structure of the singular point P. We let A denote the local ring of B_2 at P, $\mathcal{M}$ its maximal ideal, and, finally, we let G denote the associated graded ring. *G is reduced.* In fact, we can apply again the above remarks and assume that the points $P_1, \dots, P_{10} \in \mathbb{P}^2$ are in "generic position" (definition in [6]). Then, the conclusion follows from (*iii*) of Theorem 2.4. and [6], Theorem 4.2.

Now, let $\overline{A}$ denote the integral closure of A into its field of fractions. From [6], Theorem 4.4 we conclude that $\mathcal{M}^3$ is the conductor of A into $\overline{A}$. Therefore

$$\dim\left(\frac{A}{\mathcal{M}^3}\right) = 10 \qquad \text{and} \qquad \dim\left(\frac{\overline{A}}{\mathcal{M}^3}\right) = 10 \cdot 3 = 30$$

where 10 is the multiplicity of the point P on B_2. Hence $\delta = \dim(\overline{A}/A) = 20$, and we conclude

$$p_a(B_2) = p_g(B_2) + \delta = 23.$$

Remark 4.1. In [4] we got a different linear system $|\Sigma'|$ of quintic surfaces in $\mathbb{P}^3$ defining a birational map $f' : \mathbb{P}^3 \dashrightarrow X$. The inverse of f' is the projection centered at a suitable line $L' \subset X$. If we compose $f' : \mathbb{P}^3 \dashrightarrow X$ with $g : X \dashrightarrow \mathbb{P}^3$ we clearly get an element T of the Cremona group of $\mathbb{P}^3$, which depends on the choices of L' for f' and of Λ and L for g.

For general choices, T is of type $(7, 5)$, i.e. the surfaces defining T have degree 7 and those defining T^{-1} have degree 5. But, for special choices, this degrees can decrease. In particular, if $L' \subset \Lambda$, we get a cubo–cubic transformation.

Let us analyze briefly the situation in the cases $\Lambda \cap L' = \emptyset$ and $L' \subset \Lambda$.

In the first case, T^{-1} is defined by the subsystem of $|\Sigma|$ of surfaces containing $g(L')$. It is easy to prove that this curve is a skew cubic. Let M be a plane in $\mathbb{P}^3$: the surface $f(M)$ has degree $5^2 - \deg(B_1 \cup B_2) = 10$. Since $\sharp(f(M) \cap L') = \sharp(M \cap g(L') = 3$, $T(M)$ is a surface of degree 7.

If $L' \subset \Lambda$, L' is a component of $C = X \cap \Lambda$, hence it is part of the fundamental locus of g. The adjunction map $u : X \to \mathbb{P}^2$ takes both L' and its residual in C to conics. Hence the surface Φ (Proposition 2.2) splits in two quadric cones. One of them is contained in all surfaces Σ corresponding to hyperplane sections of X containing L'. This shows that T^{-1} is defined by cubics. Conversely, it is rather easy to see that the image of a line via T is a skew cubic ([4], Proposition 3.11), so also T is defined by cubic surfaces.

Let $X \subset \mathbb{P}^5$ be a Palatini scroll; it has degree 7 and sectional genus 4. A Palatini scroll is a scroll over a smooth cubic surface $V \subset \mathbb{P}^3$, and the scroll map $u : X \to V$ is the adjunction map. If S denotes a general hyperplane section of X, then the restriction of u to S is the adjunction map for S. The natural choice for the map $v : V \to \mathbb{P}^2$ in our case is the blow-up of $\mathbb{P}^2$ at the points $x_1, \ldots, x_6 \in \mathbb{P}^2$. The composition $S \to \mathbb{P}^2$ of these maps is the blow-up of $\mathbb{P}^2$ at eleven points $x_1, \ldots, x_6, y_1, \ldots, y_5$; more precisely, the linear system of curves in $\mathbb{P}^2$ defining the birational inverse of $S \to \mathbb{P}^2$ is $|6H_{\mathbb{P}^2} - \sum_{1\le i\le 6} 2x_i - \sum_{1\le j\le 5} y_j|$ (see [1]). Therefore, for the Palatini scroll we have $n = 6$ and the surfaces Σ will have degree 7.

A computation similar to that performed above for the Bordiga scroll shows that deg $E_\infty = 12$. Hence, the degree of B_2 is 11 and the multiplicity of P for B_2 is 5.

The exceptional divisors for the map $v : V \to \mathbb{P}^2$ are six lines on V. It is shown in [4] that, for any line W on V, the surface $u^{-1}(W)$ is a quadric on X. Therefore, as components of the base locus of $|\Sigma|$ we have also the first infinitesimal neighborhoods of six lines $B_3, \ldots, B_8$, each containing P. Let us denote by D the first infinitesimal neighborhood of $B_3 \cup \ldots \cup B_8$. Note that for any surface in $\mathbb{P}^3$ containing D the point P has multiplicity at least 6. In particular, any surface of degree 7 containing D is already a monoid.

The curve B_2 intersects the line B_1 at 6 points and the lines $B_3, \ldots, B_8$ are chords for B_2.

The geometric genus of B_2 is 4; it is possible to perform a computation similar to that for Bordiga to show that $p_a(B_2) = 10$. From this one can compute $p_a\big(Bs(|\Sigma|)\big) = 97$.

We want to prove now that

Proposition 4.2

The linear system $|\Sigma|$ constructed above to represent the Palatini scroll is complete.

Proof. The cone Ψ projecting B_2 from P has degree $11 - 5 = 6$, where 5 is the multiplicity of P on B_2. Moreover, Ψ is clearly singular along any line $B_3, \ldots, B_8$, hence Ψ contains D. As remarked above, for any surface containing D the point P has multiplicity at least 6. Then, a surface of degree 6 containing D is necessarily a cone with vertex P, and we conclude that Ψ *is the only surface of degree* 6 *containing* $D \cup B_2$.

Since E_1 contains $C = X \cap \Lambda$, and is a hyperplane section of X it corresponds to a reducible surface Σ having as a component a plane W containing the line B_1 (see Remark 3.2). Moreover, the hyperplane sections of E_1 correspond to the curves of degree 6 of a suitable linear system $|\Sigma'|$ on W. We denote by B and Z the base loci of $|\Sigma|$ and $|\Sigma'|$ respectively.

Then we can define a linear map

$$\varphi : H^0\big(\mathbb{P}^3, \mathcal{J}_B(7)\big) \to H^0\big(W, \mathcal{J}_Z(6)\big)$$

by setting $\varphi(F) := \overline{F} \cdot h^{-1}$, where $\overline{F}$ denotes the restriction of F to W, and h denotes the equation on W of the curve $W \cap \Psi$. The kernel of φ is the one dimensional vector space generated by the equation of $W \cup \Psi$.

We know that the linear system $|6H_W - Z|$ is complete and of dimension 4, namely $h^0\big(W, \mathcal{J}_Z(6)\big) = 5$ ([2]). Therefore $h^0\big(\mathbb{P}^3, \mathcal{J}_B(7)\big) = 6$ and the proof is complete. $\square$

We conclude with the following remark.

Remark 4.3. The scrolls of Bordiga and Palatini are both suitable projections into $\mathbb{P}^5$ of the Veronesean embedding of $\mathbb{P}^3$ by the linear system of surfaces of degree 7.

Let X be a Bordiga scroll. By Theorem 3.1 it is sufficient to construct a rational map $h : X \dashrightarrow \mathbb{P}^2$ such that for the general curve section D of X the plane curve $h(D)$ has degree 6.

Since the general hyperplane section $S \subset \mathbb{P}^4$ of X is linearly normal, by Remark 3.3 we have $h(D) \in |4H_{\mathbb{P}^2} - \sum_{1 \le i \le 10} x_i\,|$ for every curve section D of X.

Let $w : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ denote the standard quadratic transformation centered at x_1, x_2, x_3, and let y_1, y_2, y_3 denote the points in which the three lines $x_i x_j$ are contracted by w. If we compose w with u we get a new rational map $X \dashrightarrow \mathbb{P}^2$ that we will denote again by h, such that for every curve section D of X we have $h(D) \in |5H_{\mathbb{P}^2} - 2\sum_{1 \le i \le 3} y_i - \sum_{4 \le i \le 10} x_i\,|$.

We repeat this procedure by composing the rational map $X \dashrightarrow \mathbb{P}^2$ obtained above with the standard quadratic transformation $t : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ centered at y_3, x_4, x_5. For the new rational map $h : X \dashrightarrow \mathbb{P}^2$ we have $h(D) \in |6H_{\mathbb{P}^2} - 3z - 2\sum_{1\leq i\leq 4} y_i - \sum_{6\leq i\leq 10} x_i \mid$ (with a suitable labeling of the points), for every curve section D of X.

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