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Taylor expansion of the density in a stochastic heat equation*

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Abstract

We prove a general result on asymptotic expansions of densities for families of perturbed Wiener functionals. As an application, we consider a stochastic heat equation driven by a space-time white noise $\varepsilon \dot{W}_{t,x}$, $\varepsilon \in (0,1]$. The main theorem describes the asymptotics, as $\varepsilon \downarrow 0$, of the density $p^{\varepsilon}_{t,x}(y)$ of the solution at a fixed point (t,x) for some particular value $y \in \mathbb{R}$, which, in the diffusion case, plays the role of the diagonal.

1. Introduction

Let $(\Omega, \mathcal{H}, P)$ be an abstract Wiener space and $\{F^{\varepsilon}, \varepsilon \in (0,1]\}$ a family of $\mathbb{R}^d$-valued Wiener functionals. Assume that, for any $\varepsilon \in (0,1]$, $P \circ (F^{\varepsilon})^{-1}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, denoted by p^{ε}. We are interested in the asymptotics of $p^{\varepsilon}(y)$, for a fixed $y \in \mathbb{R}^d$, as $\varepsilon \downarrow 0$.

For a family of diffusions obtained by perturbing the noise, this question is related with the short-time asymptotics of the heat kernel, which has been widely studied by many authors with both, analytic and probabilistic tools. A small but representative sample of references on the probabilistic approach are [1], [3], [4], [6], [10].

For general Wiener functionals the problem has been addressed in [14] and [12]. The method of these authors relies on asymptotic expansions of F^{ε}, as $\varepsilon \downarrow 0$, in the

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Sobolev spaces $\mathbb{D}^{k,p}$ of the Malliavin Calculus and their composition with Schwartz's distributions. The asymptotic expansion of a diffusion in $\mathbb{D}^{k,p}$ is obtained applying successively the Itô formula and the coefficients of the development have an explicit expression in terms of multiple iterated Itô-Wiener integrals.

In [8] a Wiener-Chaos approach to this problem has been developed, as follows. Consider an L^2 random vector F with Wiener-Chaos decomposition $F = E(F) + \sum_{n=1}^{\infty} I_n(f_n)$ and the family $F^\varepsilon = E(F) + \sum_{n=1}^{\infty} \varepsilon^n I_n(f_n)$, $\varepsilon \in (0,1]$. Assume $F \in \cap_{j=0}^{\infty} \mathbb{D}^{j,2}$, then there exists a version of $\{F^\varepsilon, \varepsilon \in (0,1)\}$ which is $\mathcal{C}^\infty$ in ε. Suppose in addition, $\sup_{\varepsilon\in(0,1]} \varepsilon^2 \|\Gamma_{F^\varepsilon}^{-1}\|_{L^p(\Omega)} \le C$, $p \in [1,\infty)$, where Γ_{F^ε} denotes the Malliavin matrix of F^ε, then the asymptotic expansion for $p^\varepsilon(y)$, $y = E(F)$, is obtained by passing to the limit as $n \to \infty$ the Taylor expansion of $\phi_n(F^\varepsilon)$, where $\{\phi_n, n \ge 1\}$ is a smooth approximation of the Dirac function at y (see also [6]). A precise description of the coefficients of the expansion in terms of the Itô-Wiener integrals $I_n(f_n)$ is also given.

This approach seems appropriate in situations where it is not clear how to apply an Itô formula, for example when F^ε is the solution to a stochastic partial differential equation (see [7] for related work). We study in [8] an example of *linear* hyperbolic stochastic partial differential equation. It does not seem possible to remove the linear assumption on the coefficients, because of the particular structure of $\{F^\varepsilon, \varepsilon \in (0,1]\}$.

The purpose of this paper is to obtain an asymptotic expansion, as $\varepsilon \downarrow 0$, for the densities of the following family of perturbed parabolic stochastic partial differential equations

$$\frac{\partial X^\varepsilon}{\partial t}(t,x) = \frac{\partial^2 X^\varepsilon}{\partial x^2}(t,x) + \varepsilon\sigma\Big(X^\varepsilon(t,x)\Big)\dot{W}_{t,x} + b\Big(X^\varepsilon(t,x)\Big), \tag{1.1}$$

$(t,x) \in [0,T] \times [0,1]$ with initial condition $X^\varepsilon(0,x) = X_0(x)$ and either Neumann or Dirichlet boundary conditions. The process $\{\dot{W}_{t,x}, (t,x) \in [0,T] \times [0,1]\}$ is a space-time white noise and the coefficients $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ and $b : \mathbb{R} \longrightarrow \mathbb{R}$ are smooth functions.

A solution of (1.1) is a process $\{X^\varepsilon(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ satisfying the evolution equation

$$\begin{aligned} X^\varepsilon(t,x) = &\int_0^1 G_t(x,y)\, X_0(y) dy \\ &+ \int_0^t\int_0^1 G_{t-s}(x,y)\Big\{\varepsilon\,\sigma\Big(X^\varepsilon(s,y)\Big)\,W(ds,dy) + b\Big(X^\varepsilon(s,y)\Big) ds\,dy\Big\}, \end{aligned} \tag{1.2}$$

where $G_t(x,y)$ is the fundamental solution of the heat equation on $[0,T]\times[0,1]$ with the above-mentioned boundary conditions, that means,

$$G_t(x,y) = \frac{1}{(2\pi t)^{1/2}} \sum_{n=-\infty}^{\infty} \Big\{ \exp\Big(-\frac{(y-x-2n)^2}{4t}\Big) + \gamma \exp\Big(-\frac{(y+x-2n)^2}{4t}\Big)\Big\},$$

with $\gamma = 1$ or $\gamma = -1$ for Neumann or Dirichlet boundary conditions, respectively (see for instance [13]).

In [2] the existence and smoothness of the density $p^{\varepsilon}_{t,x}(y)$ of the solution to (1.2) for fixed $(t,x)\in(0,T]\times(0,1)$, $\varepsilon\in(0,1]$, is proved, assuming the assumptions (H1), (H2) quoted in Section 3 of this paper. Moreover, the densities are strictly positive for any $y\in\mathbb{R}$.

We prove in Section 2 a general result on asymptotic expansions of densities, following ideas close to that developed in [8]. This result can be applied to situations not covered by our previous work, for instance we can get rid from the linearity of the coefficients in the example we have mentioned before.

Let $\{\psi_{X_0}(t,x), (t,x)\in[0,T]\times[0,1]\}$ be the solution of the deterministic evolution equation

$$\psi_{X_0}(t,x) = \int_0^1 G_t(x,y)X_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)b\Big(\psi_{X_0}(s,y)\Big)ds\,dy\,, \qquad (1.3)$$

In Section 3 we apply the general result established in Section 2 to the proof of a Taylor expansion of $p^{\varepsilon}_{t,x}(y)$ at $\varepsilon=0$ for $y=\psi_{X_0}(t,x)$. In the classical setting of diffusions this value of y corresponds to the initial condition (asymptotics on the diagonal).

2. Taylor expansion of the density

We start this section by quoting some notations and results on Malliavin Calculus.

Let $\phi:\Omega\longrightarrow\mathbb{R}^d$ be a Wiener functional. We denote by Γ_ϕ its Malliavin matrix. The random vector ϕ is said to be non-degenerate if $\phi\in\mathbb{D}^\infty(\mathbb{R}^d)$ and $\det\Gamma_\phi^{-1}\in\cap_{p\ge1}L^p(\Omega)$. Consider a non-degenerate random vector ϕ, $\psi\in\mathbb{D}^\infty$ and g any smooth function defined in $\mathbb{R}^d$. For any multiindex $\alpha\in\{1,\dots,d\}^k$, $\alpha=(\alpha_1,\dots,\alpha_k)$, $k\ge1$, there exists a random vector $H_\alpha(\phi,\psi)\in\mathbb{D}^\infty$ such that:

$$E\,[\,(\nabla^k_\alpha\, g)\,(\phi)\psi] = E\,[\,g(\phi)\,H_\alpha(\phi,\psi)]\,, \qquad (2.1)$$

with $\nabla_\alpha^k = \frac{\partial^k}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}$, where the elements $H_\alpha(\phi, \psi)$ are recurrently given by

$$H_{(i)}(\phi, \psi) = \sum_{j=1}^{d} \delta\left(\psi\, (\Gamma_\phi^{-1})^{ij}\; D\phi^j\right),$$
$$H_\alpha(\phi, \psi) = H_{(\alpha_k)}\left(\phi, H_{(\alpha_1,\dots,\alpha_{k-1})}\;(\phi, \psi)\right), \tag{2.2}$$

and δ denotes the Skorohod integral (see, for instance [5], [11]). If $d = 1$ we write $H_k(\phi, \psi)$ instead of $H_\alpha(\phi, \psi)$ for the index $\alpha = (1, \cdots, 1)$ of length k.

The following estimate is an easy extension of Proposition 3.2.2 in [11]. For any $p > 1$, $k \in \mathbb{N}$ and any multiindex α, there exists a constant $C\,(p, \alpha, k)$ and positive real numbers $k', a, a'\,, a'',\ b, b',\ d, d'$ depending on p, k and α such that

$$\|H_\alpha(\phi, \psi)\,\|_{k,p} \leq C\,(p, \alpha, k)\ \|\Gamma_\phi^{-1}\|_{k'}^{a}\ \|\phi\|_{d,b}^{a'}\ \|\psi\|_{d',b'}^{a''}, \tag{2.3}$$

where $\|\cdot\|_{k,p}$ denotes the norm of the Sobolev spaces $\mathbb{D}^{k,p}$ in the Malliavin Calculus.

Moreover, the Radon measure defined by

$$g \longmapsto E\Big(g\,(\phi)\,\psi\Big)$$

has a bounded $\mathcal{C}^\infty$ density $q(y)$ and

$$q(y) = E\left(1\!\!1_{\{\phi > y\}}\ H_{(1,\dots,d)}\ (\phi, \psi)\right). \tag{2.4}$$

The proof is a slight modification of Corollary 3.2.1 in [11].

DEFINITION 2.1. A family ϕ^ε, $\varepsilon \in [0, 1]$ of elements in $\mathbb{D}^\infty(\mathbb{R}^d)$ is said to be uniformly non-degenerate if the next conditions are satisfied:

(i) $\phi^0 = \lim_{\varepsilon \downarrow 0}\ \phi^\varepsilon$, a.s.

(ii) $\sup_{\varepsilon \in [0,1]}\ \|\Gamma_{\phi^\varepsilon}^{-1}\|_p \leq C$, for any $p \in [1, \infty)$.

Remark. $\phi^\varepsilon, \varepsilon \in [0, 1]$, have smooth densities.

Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$ function with compact support and assume that $\varepsilon \to \phi^\varepsilon$ is a $\mathcal{C}^\infty\big((0, 1); \mathbb{R}^d\big)$ mapping. Leibniz's formula yields, for any integer $j \geq 1$

$$\frac{d^j}{d\varepsilon^j}\ \Big(f(\phi^\varepsilon)\Big) = \sum^{(j)} (\nabla_\alpha^k f)\ (\phi^\varepsilon)\ \phi_{\beta_1}^{\varepsilon,\alpha_1} \cdot \ldots \cdot \phi_{\beta_k}^{\varepsilon,\alpha_k} \tag{2.5}$$

with $\phi_{\beta_i}^{\varepsilon,\alpha_i} := \frac{d^{\beta_i}\,\phi^{\varepsilon,\alpha_i}}{d\,\varepsilon^{\beta_i}}$. The symbol $\sum^{(j)}$ is a shorthand for

$$\sum_{k=1}^{j} \sum_{\substack{\beta_1+\ldots+\beta_k=j \\ \beta_1,\ldots,\beta_k\geq 1}} \sum_{\substack{\alpha\in\{1,\ldots,d\}^k \\ \alpha=(\alpha_1,\ldots,\alpha_k)}} c_j(\beta_1,\ldots,\beta_k) \tag{2.6}$$

and the coefficients $c_j(\beta_1,\ldots,\beta_k)$ are computed by induction.

We now give the main result of this section.

Theorem 2.2

Let $\{F^\varepsilon,\ \varepsilon\in[0,1]\}$ be a uniformly non-degenerate family of R^d-valued random vectors satisfying the following conditions:
(a) There exists a version of $\{F^\varepsilon,\ \varepsilon\in(0,1)\}$ which is $\mathcal{C}^\infty$ in ε. Let $F_j^\varepsilon = \frac{d^j F^\varepsilon}{d\,\varepsilon^j}$, $j\in\mathbb{N}$. Then the limits

$$F_j^0 := \lim_{\varepsilon\downarrow 0} F_j^\varepsilon, \text{a.s.}$$

exist for any $j\in\mathbb{N}$,
(b) F_j^ε belongs to $\mathbb{D}^\infty(\mathbb{R}^d)$, for any $j\in\mathbb{N}$, $\varepsilon\in[0,1)$.

Then the density $p^\varepsilon(y)$ of F^ε has the Taylor expansion

$$p^\varepsilon(y) = p^0(y) + \sum_{j=1}^{N} \varepsilon^j\, \frac{1}{j!}\, p_j(y) + \varepsilon^{N+1}\, \tilde{p}_{N+1}^\varepsilon(y), \tag{2.7}$$

where $p^0(y)$ is the density of F^0,

$$p_j(y) = E\,\left\{ \mathbb{1}_{\{F^0>y\}}\ P_j \right\}\,,$$

with

$$P_j = \sum^{(j)} H_{(1,\ldots,d)}\left(F^0,\, H_\alpha\left(F^0,\, \prod_{\ell=1}^{k} F_{\beta_\ell}^{0,\alpha_\ell}\right)\right). \tag{2.8}$$

Moreover, if for any $j\in\mathbb{Z}^+,\ k\in\mathbb{N},\ p\in[1,\infty)$,

$$\sup_{\varepsilon\in(0,1]} \|F_j^\varepsilon\|_{k,p} \leq C, \tag{2.9}$$

then $\sup_{\varepsilon\in(0,1]}\left(|\tilde{p}_{N+1}^\varepsilon(y)\,|\right)$ is finite.

Proof of theorem 2.2. Let f be a $\mathcal{C}^\infty$ function with bounded support included in $\mathbb{R}^d$. The mapping $\varepsilon \in (0,1) \longmapsto f(F^\varepsilon)$ is $\mathcal{C}^\infty$, a.s. and a Taylor expansion yields

$$f(F^\varepsilon) = f(F^0) + \sum_{j=1}^{N} \varepsilon^j \, \frac{1}{j!} \, \frac{d^j}{d\,\varepsilon^j} \, \Big(f(F^\varepsilon)\Big)|_{\varepsilon=0} + \ \varepsilon^{N+1} \int_0^1 \frac{(1-t)^N}{N!} \, \frac{d^{N+1}}{d\,\eta^{N+1}} \, \Big(f(F^\eta)\Big)|_{\eta=\varepsilon t} \, dt\,.$$

Taking expectations and using (2.5) and (2.1) yields

$$\begin{aligned} E\big(f(F^\varepsilon)\big) &= E\big(f(F^0)\big) \\ &+ \sum_{j=1}^{N} \varepsilon^j \, \frac{1}{j!} \, E\Big\{f(F^0) \sum^{(j)} H_\alpha\Big(F^0, \prod_{\ell=1}^{k} F^{0,\alpha_\ell}_{\beta_\ell}\Big)\Big\} \\ &+ \ \varepsilon^{N+1} \int_0^1 \frac{(1-t)^N}{N!} \, E\Big\{f(F^{\varepsilon t}) \sum^{(N+1)} H_\alpha\Big(F^{\varepsilon t}, \prod_{\ell=1}^{k} F^{\varepsilon t,\alpha_\ell}_{\beta_\ell}\Big)\Big\}\, dt. \end{aligned} \tag{2.10}$$

Let

$$\begin{aligned} Q_j &= \sum^{(j)} H_\alpha\Big(F^0, \prod_{\ell=1}^{k} F^{0,\alpha_\ell}_{\beta_\ell}\Big), \quad j = 1, \cdots, N, \\ Q^{\varepsilon t}_{N+1} &= \sum^{(N+1)} H_\alpha\Big(F^{\varepsilon t}, \prod_{\ell=1}^{k} F^{\varepsilon t,\alpha_\ell}_{\beta_\ell}\Big). \end{aligned}$$

The assumptions on $\{F^\varepsilon, \varepsilon \in [0,1]\}$ ensure that the Radon measures defined by $E\left(f(F^\varepsilon)\right)$, $E\left(f(F^0)\right)$, $E\left(f(F^0)Q_j\right)$, $j = 1, \ldots, N$, and $E\left(f(F^{\varepsilon t})Q^{\varepsilon t}_{N+1}\right)$ have $\mathcal{C}^\infty$, bounded densities. Using (2.4) in (2.10) we obtain

$$p^\varepsilon(y) = p^0(y) + \sum_{j=1}^{N} \varepsilon^j \, \frac{1}{j!} \, E\left\{1\!\!1_{\{F^0>y\}} \, P_j\right\} + \varepsilon^{N+1} \, \tilde{p}^\varepsilon_{N+1}(y)\,,$$

with P_j given by (2.8) and

$$\tilde{p}^\varepsilon_{N+1}(y) = \int_0^1 \frac{(1-t)^N}{N!} \, E\left\{ 1\!\!1_{\{F^{\varepsilon t}>y\}} \, H_{(1,\ldots,d)}\left(F^{\varepsilon t}, \sum^{(N+1)} H_\alpha\Big(F^{\varepsilon t}, \prod_{\ell=1}^{k} F^{\varepsilon t,\alpha_\ell}_{\beta_\ell}\Big)\right)\right\} dt\,.$$

In order to give a uniform bound for $\tilde{p}^{\varepsilon}_{N+1}$, it suffices to check

$$\sup_{\varepsilon\in(0,1]} E\left\{ \left| H_{(1,\dots,d)} \Big(F^{\varepsilon},\, H_{\alpha}(F^{\varepsilon},\, G^{\varepsilon}) \Big) \right| \right\} \le C\,, \tag{2.11}$$

with $G^{\varepsilon} = \prod_{\ell=1}^{k} F^{\varepsilon,\alpha_\ell}_{\beta_\ell}$, for any $\alpha \in \{1,\dots,d\}^k$, $\beta_1+\cdots+\beta_k = N+1$, $k = 1,\cdots,N+1$.

This follows from (2.3), the non-degeneracy assumption $\sup_{\varepsilon\in(0,1]} \|\Gamma^{-1}_{F^{\varepsilon}}\|_p \le C$, for any $p\in(1,\infty)$, and (2.9). □

3. The stochastic heat equation

In this section we consider the family $\{X^{\varepsilon}(t,x), (t,x) \in [0,T]\times[0,1]\}$, $\varepsilon\in(0,1]$, defined by (1.2) and assume the following conditions on the coefficients:

(H1) $\sigma, b : \mathbb{R} \longrightarrow \mathbb{R}$ are $\mathcal{C}^{\infty}$ functions with bounded first order derivatives and higher order derivatives with polynomial growth,

(H2) There exists $C>0$ such that $\inf\{\,|\sigma(y)|;\ y\in\mathbb{R}\} \ge C$.

Fix $t\in(0,T]$, $x\in(0,1)$ and let $p^{\varepsilon}_{t,x}(y)$ be the density of $X^{\varepsilon}(t,x)$. Our purpose is to apply Theorem 2.2 to obtain a Taylor expansion at $\varepsilon = 0$ of $p^{\varepsilon}_{t,x}(y)$ with $y = \psi_{X_0}(t,x)$ (see (1.3)). Define $X^0(t,x) = \psi_{X_0}(t,x)$. Notice that $X^{\varepsilon}(t,x), \varepsilon \in [0,1]$, does not satisfy the conditions of Definition 2.1, because for $\varepsilon = 0$, $X^{\varepsilon}(t,x)$ is deterministic. Instead, we consider the random variables defined by

$$\hat{X}^{\varepsilon}(t,x) = \frac{X^{\varepsilon}(t,x) - \psi_{X_0}(t,x)}{\varepsilon}\,, \qquad 0<\varepsilon\le 1\,. \tag{3.1}$$

We shall prove that $\hat{X}^0(t,x) := \lim_{\varepsilon\to 0} \hat{X}^{\varepsilon}(t,x)$ exists a.s. and the family $\hat{X}^{\varepsilon}(t,x)$, $\varepsilon\in[0,1]$ satisfies the hypotheses of Theorem 2.2. The Taylor expansion for $p^{\varepsilon}_{t,x}(y)$ will be obtained taking into account that, for $y = \psi_{X_0}(t,x)$,

$$p^{\varepsilon}_{t,x}(y) = \frac{1}{\varepsilon}\, \hat{p}^{\varepsilon}_{t,x}(0), \tag{3.2}$$

where $\hat{p}^{\varepsilon}_{t,x}(y)$ denotes the density of $\hat{X}^{\varepsilon}(t,x)$.

We introduce some notation. Let $X^{\varepsilon}_j(t,x)$, $X^0_j(t,x), j\ge 1, \varepsilon\in(0,1]$, be the solutions of the following stochastic differential equations:

$$\begin{aligned} X^{\varepsilon}_1(t,x) = \int_0^t\!\!\int_0^1 G_{t-s}(x,y)\Big\{&\sigma\big(X^{\varepsilon}(s,y)\big)\, W(ds,dy) \\ &+\ \varepsilon\,\sigma'\big(X^{\varepsilon}(s,y)\big)\, X^{\varepsilon}_1(s,y)\, W(ds,dy) \\ &+\ b'\big(X^{\varepsilon}(s,y)\big)\, X^{\varepsilon}_1(s,y)\, ds\, dy\Big\}, \end{aligned} \tag{3.3}$$

$$X_1^0(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \Big\{\sigma\big(\psi_{X_0}(s,y)\big)\, W(ds,dy) + b'\big(\psi_{X_0}(s,y)\big) X_1^0(s,y)\, ds\, dy\Big\} \tag{3.4}$$

and, for $j \geq 2$,

$$X_j^\varepsilon(t,x) = I_{j-1}^\varepsilon(t,x) + \int_0^t \int_0^1 G_{t-s}(x,y)\; c_j(j)\; \Big\{\varepsilon\, \sigma'\big(X^\varepsilon(s,y)\big) \times X_j^\varepsilon(s,y)\, W(ds,dy) + b'\big(X^\varepsilon(s,y)\big) X_j^\varepsilon(s,y)\, ds\, dy\Big\}, \tag{3.5}$$

$$X_j^0(t,x) = I_{j-1}^0(t,x) + \int_0^t \int_0^1 G_{t-s}(x,y)\; c_j(j)\; b'\big(\psi_{X_0}(s,y)\big) X_j^0(s,y)\, ds\, dy, \tag{3.6}$$

where we set

$$\begin{aligned} I_{j-1}^\varepsilon(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \Bigg\{ &\sum_{k=1}^{j-1} \sum_{\substack{\beta_1+\ldots+\beta_k=j-1\\ \beta_1,\ldots,\beta_k\geq 1}} k_{j-1}(\beta_1,\ldots,\beta_k) \\ &\times \sigma^{(k)}\big(X^\varepsilon(s,y)\big) \prod_{\ell=1}^{k} X_{\beta_\ell}^\varepsilon(s,y)\, W(ds,dy) \\ &+ \sum_{k=2}^{j} \sum_{\substack{\beta_1+\ldots+\beta_k=j\\ \beta_1,\ldots,\beta_k\geq 1}} c_j(\beta_1,\ldots,\beta_k) \Big[\varepsilon\, \sigma^{(k)}\big(X^\varepsilon(s,y)\big) \prod_{\ell=1}^{k} X_{\beta_\ell}^\varepsilon(s,y)\, W(ds,dy) \\ &+ b^{(k)}\big(X^\varepsilon(s,y)\big) \prod_{\ell=1}^{k} X_{\beta_\ell}^\varepsilon(s,y)\, ds\, dy\Big]\Bigg\}, \end{aligned}$$

$$\begin{aligned} I_{j-1}^0(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \Bigg\{ &\sum_{k=1}^{j-1} \sum_{\substack{\beta_1+\ldots+\beta_k=j-1\\ \beta_1,\ldots,\beta_k\geq 1}} k_{j-1}(\beta_1,\ldots,\beta_k) \\ &\times \sigma^{(k)}\big(\psi_{X_0}(s,y)\big) \prod_{\ell=1}^{k} X_{\beta_\ell}^0(s,y)\, W(ds,dy) \\ &+ \sum_{k=2}^{j} \sum_{\substack{\beta_1+\ldots+\beta_k=j\\ \beta_1,\ldots,\beta_k\geq 1}} c_j(\beta_1,\ldots,\beta_k) \\ &b^{(k)}\big(\psi_{X_0}(s,y)\big) \prod_{\ell=1}^{k} X_{\beta_\ell}^0(s,y)\, ds\, dy\Bigg\}. \end{aligned} \tag{3.7}$$

The coefficients $c_j(\beta_1, \ldots, \beta_k)$ -the same as in (2.6)- and $k_j(\beta_1, \ldots, \beta_k)$ are defined by induction.

We use the convention $X_0^\varepsilon(t,x) = X^\varepsilon(t,x)$ and $X_0^0(t,x) = \psi_{X_0}(t,x)$.

The next proposition is one of the ingredients for checking that $X^\varepsilon(t,x), \varepsilon \in (0,1)$, satisfy the hypothesis (a) of Theorem 2.2.

Proposition 3.1

Assume (H1). There exists a version of $\{X^\varepsilon(t,x), \varepsilon \in (0,1)\}$ *which is a* $\mathcal{C}^\infty$ *function with respect to* ε *and, for any* $j \in \mathbb{N}$, $\frac{d^j X^\varepsilon}{d\varepsilon^j}(t,x) = X_j^\varepsilon(t,x)$. *Moreover,*

$$\text{a.s.} - \lim_{\varepsilon \downarrow 0} X_j^\varepsilon(t,x) = X_j^0(t,x),\ j \in \mathbb{Z}^+.$$

Remark. The following property of the heat kernel will be used along the proofs:

$$G_t(x,y) \le \frac{C}{\sqrt{t}} \exp\Big(- \frac{(y-x)^2}{4t} \Big). \tag{3.8}$$

Proof of Proposition 3.1. We first establish the continuity property. Using (3.8), Burkholder's, Hölder's and Gronwall's inequalities, it is easy to check

$$\sup_{0 \le \varepsilon \le 1} \sup_{x,t} E\,|X^\varepsilon(t,x)|^p \le C\,, \tag{3.9}$$

for any $p \in (1,\infty)$ and some positive constant C. Burkholder's and Hölder's inequalities together with (3.8) and (3.9) yield

$$E\,|X^{\varepsilon+\xi}(t,x) - X^\varepsilon(t,x)|^p \le C\,|\xi|^p + C \int_0^t \sup_x E\,|X^{\varepsilon+\xi}(s,x) - X^\varepsilon(s,x)|^p\,ds,$$

for any $p \in (1,\infty)$, $\varepsilon \in [0,1]$ and ξ such that $0 \le \varepsilon + \xi \le 1$. Thus Gronwall's lemma implies

$$\sup_{x,t} E\,|X^{\varepsilon+\xi}(t,x) - X^\varepsilon(t,x)|^p \le C\,|\xi|^p. \tag{3.10}$$

The existence of a continuous version of $\{X^\varepsilon(t,x), \varepsilon \in [0,1]\}$ follows from Kolmogorov's theorem. Moreover,

$$\lim_{\varepsilon \downarrow 0} X^\varepsilon(t,x) = \psi_{X_0}(t,x), \text{ a.s.}$$

We now check differentiability of order $j = 1$. For any $\varepsilon \in (0,1)$, $\xi \in \mathbb{R} - \{0\}$ such that $0 \le \varepsilon + \xi \le 1$, we define

$$Z_\xi^\varepsilon(t,x) = \frac{X^{\varepsilon+\xi}(t,x) - X^\varepsilon(t,x)}{\xi}\,.$$

By the mean value theorem,

$$\begin{aligned}
Z_\xi^\varepsilon(t,x) =& \int_0^t \int_0^1 G_{t-s}(x,y)\Big\{\sigma\big(X^{\varepsilon+\xi}(s,y)\big)\, W(ds,dy) \\
&+ \varepsilon\Big[\int_0^1 \sigma'\Big(X^\varepsilon(s,y) + \lambda\big(X^{\varepsilon+\xi}(s,y) - X^\varepsilon(s,y)\big)\Big)\,d\lambda\Big] Z_\xi^\varepsilon(s,y)\, W(ds,dy) \\
&+ \Big[\int_0^1 b'\Big(X^\varepsilon(s,y) + \lambda\big(X^{\varepsilon+\xi}(s,y) - X^\varepsilon(s,y)\big)\Big) d\lambda\Big] Z_\xi^\varepsilon(s,y)\, ds\, dy\Big\}. \quad (3.11)
\end{aligned}$$

Clearly, from (3.9), for any $p \in (1,\infty)$ and some positive constant C,

$$\sup_{\substack{0\le\varepsilon\le1 \\ \xi\ne0,\,0\le\xi+\varepsilon\le1}} \ \sup_{x,t} E\,|Z_\xi^\varepsilon(t,x)|^p \le C. \quad (3.12)$$

For any $p \in (1,\infty)$,

$$E\,|Z_\xi^\varepsilon(t,x) - Z_{\xi'}^{\varepsilon'}(t,x)|^p \le C \sum_{i=1}^6 A_i^{\varepsilon,\varepsilon',\xi,\xi'}(t,x)\,,$$

with

$$\begin{aligned}
A_1^{\varepsilon,\varepsilon',\xi,\xi'}(t,x) =&\ E\Big|\int_0^t\int_0^1 G_{t-s}(x,y)\Big[\sigma\big(X^{\varepsilon+\xi}(s,y)\big) \\
&- \sigma\big(X^{\varepsilon'+\xi'}(s,y)\big)\Big] W(ds,\,dy)\Big|^p, \\
A_2^{\varepsilon,\varepsilon',\xi,\xi'}(t,x) =&\ E\Big|\int_0^t\int_0^1 G_{t-s}(x,y)(\varepsilon-\varepsilon')\Big[\int_0^1 \sigma'\Big(X^{\varepsilon'}(s,y) \\
&+ \lambda\big(X^{\varepsilon'+\xi'}(s,y) - X^{\varepsilon'}(s,y)\big)\Big) d\lambda\Big] Z_{\xi'}^{\varepsilon'}(s,y)\, W(ds,\,dy)\Big|^p, \\
A_3^{\varepsilon,\varepsilon',\xi,\xi'}(t,x) =&\ E\Big|\int_0^t\int_0^1 G_{t-s}(x,y)\,\varepsilon\Big[\int_0^1 \Big\{\sigma'\Big(X^\varepsilon(s,y) \\
&+ \lambda\big(X^{\varepsilon+\xi}(s,y) - X^\varepsilon(s,y)\big)\Big) - \sigma'\Big(X^{\varepsilon'}(s,y) \\
&+ \lambda\big(X^{\varepsilon'+\xi'}(s,y) - X^{\varepsilon'}(s,y)\big)\Big)\Big\} d\lambda\Big] Z_\xi^\varepsilon(s,y)\, W(ds,\,dy)\Big|^p,
\end{aligned}$$

$$\begin{aligned}
A_4^{\varepsilon,\varepsilon',\xi,\xi'}(t,x) = \ & E\Big|\int_0^t\int_0^1 G_{t-s}(x,y)\ \Big[\int_0^1 \Big\{b'\Big(X^{\varepsilon}(s,y) \\
& +\ \lambda\big(X^{\varepsilon+\xi}(s,y)-X^{\varepsilon}(s,y)\big)\Big) - b'\Big(X^{\varepsilon'}(s,y) \\
& +\ \lambda\big(X^{\varepsilon'+\xi'}(s,y)-X^{\varepsilon'}(s,y)\big)\Big)\Big\}d\lambda\Big] Z_{\xi}^{\varepsilon}(s,y)\,ds\,dy\Big|^p, \\
A_5^{\varepsilon,\varepsilon',\xi,\xi'}(t,x) = \ & E\Big|\int_0^t\int_0^1 G_{t-s}(x,y)\ \varepsilon\Big[\int_0^1 \sigma'\Big(X^{\varepsilon'}(s,y)+\lambda\big(X^{\varepsilon'+\xi'}(s,y) \\
& -\ X^{\varepsilon'}(s,y)\big)\Big)d\lambda\Big]\Big(Z_{\xi'}^{\varepsilon'}(s,y)-Z_{\xi}^{\varepsilon}(s,y)\Big)W(ds,\,dy)\Big|^p, \\
A_6^{\varepsilon,\varepsilon',\xi,\xi'}(t,x) = \ & E\Big|\int_0^t\int_0^1 G_{t-s}(x,y)\ \Big[\int_0^1 b'\Big(X^{\varepsilon'}(s,y)+\lambda\big(X^{\varepsilon'+\xi'}(s,y) \\
& -\ X^{\varepsilon'}(s,y)\big)\Big)d\lambda\Big]\Big(Z_{\xi'}^{\varepsilon'}(t,x)-Z_{\xi}^{\varepsilon}(t,x)\Big)ds\,dy\Big|^p.
\end{aligned}$$

The assumption (H1), the estimates (3.10), (3.12) and Burkholder's and Hölder's inequalities, imply

$$\sum_{i=1}^{4}\sup_{x,t}\ A_i^{\varepsilon,\varepsilon',\xi,\xi'}(t,x) \le C\left\{|\varepsilon-\varepsilon'|^p + |\xi-\xi'|^p\right\}. \tag{3.13}$$

The remaining terms are bounded as follows:

$$\sup_x\ A_5^{\varepsilon,\varepsilon',\xi,\xi'} + \sup_x\ A_6^{\varepsilon,\varepsilon',\xi,\xi'} \le C\int_0^t \sup_x\ E\,|Z_{\xi}^{\varepsilon}(s,x)-Z_{\xi'}^{\varepsilon'}(s,x)|^p ds\,. \tag{3.14}$$

Then, (3.13), (3.14), Gronwall's lemma and Kolmogorov's theorem show that $X^{\varepsilon}(t,x)$ has a differentiable version. Moreover, it is easy to prove

$$L^2-\lim_{\xi\downarrow 0} Z_{\xi}^{\varepsilon}(t,x) = X_1^{\varepsilon}(t,x),$$

and therefore $\frac{dX^{\varepsilon}}{d\varepsilon}(t,x) = X_1^{\varepsilon}(t,x)$, a.s. The standard argument used before also proves that $\{X_1^{\varepsilon}(t,x), \varepsilon\in[0,1]\}$ satisfies the requirements of Kolmogorov's continuity criterion. Thus

$$\text{a.s.}-\lim_{\varepsilon\downarrow 0}\ X_1^{\varepsilon}(t,x)\ =\ X_1^0(t,x).$$

We finish the proof using induction on j. $\square$

Corollary 3.2

Assume (H1). Then, for any $j \in \mathbb{Z}^+$,

$$E\{\sup_{0\le\varepsilon\le 1} |X_j^\varepsilon(t,x)|^p\} < \infty .$$

Proof. It is an immediate consequence of Kolmogorov's theorem. □

The next lemma states the relationship between the derivatives of the processes $X^\varepsilon(t,x)$ and $\hat{X}^\varepsilon(t,x)$, respectively. As before, for any natural number j, we set $\hat{X}_j^\varepsilon(t,x) = \frac{d^j\, \hat{X}^\varepsilon(t,x)}{d\,\varepsilon^j}$ and $\hat{X}_0^\varepsilon(t,x) = \hat{X}^\varepsilon(t,x)$.

Lemma 3.3

Assume (H1). Then, for any $j \in \mathbb{Z}^+$, $\varepsilon \in (0,1)$,

$$\hat{X}_j^\varepsilon(t,x) = \frac{1}{j+1}\left\{X_{j+1}^0(t,x) + \varepsilon\int_0^1 (1-\xi^{j+1})\; X_{j+2}^{\varepsilon\xi}\,(t,x)\,d\xi\right\}. \tag{3.15}$$

Proof. We proceed by induction on j. For $j = 0$, (3.15) follows from a Taylor expansion of $X^\varepsilon(t,x)$ at $\varepsilon = 0$. Suppose that (3.15) is satisfied for any $0 \le k \le j$. Then, for every $\delta \in \mathbb{R} - \{0\}$ with $0 \le \varepsilon + \delta \le 1$,

$$\begin{aligned}
&\frac{1}{\delta}\left\{\hat{X}_j^{\varepsilon+\delta}(t,x) - \hat{X}_j^\varepsilon(t,x)\right\}\\
&= \frac{1}{\delta}\left\{\frac{\varepsilon+\delta}{j+1}\int_0^1 (1-\xi^{j+1})\; X_{j+2}^{(\varepsilon+\delta)\xi}(t,x)\,d\xi\right.\\
&\left. - \frac{\varepsilon}{j+1}\int_0^1 (1-\xi^{j+1})\; X_{j+2}^{\varepsilon\xi}(t,x)\,d\xi\right\}\\
&= \frac{1}{j+2}\, X_{j+2}^0(t,x) + B_1^{\varepsilon,\delta}(t,x) + B_2^{\varepsilon,\delta}(t,x)\,,
\end{aligned} \tag{3.16}$$

with

$$\begin{aligned}
B_1^{\varepsilon,\delta}(t,x) &= \frac{\varepsilon}{j+1}\int_0^1 (1-\xi^{j+1})\; \frac{X_{j+2}^{(\varepsilon+\delta)\xi}(t,x) - X_{j+2}^{\varepsilon\xi}(t,x)}{\delta}\; d\xi\,,\\
B_2^{\varepsilon,\delta}(t,x) &= \frac{1}{j+1}\int_0^1 (1-\xi^{j+1})\; \left(X_{j+2}^{(\varepsilon+\delta)\xi}(t,x) - X_{j+2}^0(t,x)\right) d\xi\,.
\end{aligned}$$

Corollary 3.2 and the bounded convergence theorem yield, a.s.

$$\lim_{\delta\to 0} B_1^{\varepsilon,\delta}(t,x) = \frac{\varepsilon}{j+1}\int_0^1 (\xi - \xi^{j+2})\; X_{j+3}^{\varepsilon\xi}(t,x)\,d\xi. \tag{3.17}$$

Using again the bounded convergence theorem, Taylor's expansion, a change of variable and Fubini's theorem, we obtain

$$\begin{aligned}\lim_{\delta\to 0}\ B_2^{\varepsilon,\delta}(t,x) &= \frac{1}{j+1}\int_0^1 (1-\xi^{j+1})\int_0^{\varepsilon\xi} X_{j+3}^{\beta}(t,x)\,d\beta\,d\xi\\ &= \frac{\varepsilon}{j+1}\int_0^1 \Big(\frac{j+1}{j+2}-\xi+\frac{\xi^{j+2}}{j+2}\Big) X_{j+3}^{\varepsilon\xi}\,(t,x)\,d\xi\,. \end{aligned} \tag{3.18}$$

Then (3.16)-(3.18) prove the formula (3.15) for the integer $j+1$. □

Assume (H1). As a consequence of Lemma 3.3 and Corollary 3.2 we have, a.s.

$$\hat{X}_j^0(t,x) := \lim_{\varepsilon\downarrow 0}\ \hat{X}_j^{\varepsilon}(t,x) = \frac{1}{j+1}\ X_{j+1}^0(t,x), \tag{3.19}$$

for any $j\in\mathbb{Z}^+$. Then (3.15), (3.19) and the equations satisfied by $X_j^{\varepsilon}(t,x)$, $\varepsilon\in[0,1]$, $j\in\mathbb{Z}^+$, given in (3.3)-(3.6), allows to check as in [2] that $\hat{X}_j^{\varepsilon}(t,x), \varepsilon\in[0,1]$, $j\in\mathbb{Z}^+$, belong to $\mathbb{D}^\infty$.

Assume (H1) and (H2). We have proved in [9]

$$\sup_{0<\varepsilon\le 1}\ E\Big(|\det \Gamma^{-1}_{\hat{X}^{\varepsilon}(t,x)}|^p\Big) \le C. \tag{3.20}$$

for any $p\in(1,\infty)$ and some positive constant C. The same arguments as in Lemma 2.5 [9] show that the zero-mean Gaussian variable $\hat{X}^0(t,x) = X_1^0(t,x)$ satisfies $E\,|X_1^0(t,x)|^2>0$.

Hence we have proved that $\{\hat{X}^{\varepsilon}(t,x), \varepsilon\in[0,1]\}$ is a family of uniformly non-degenerate random variables satisfying the conditions (a) and (b) of Theorem 2.2. The next proposition shows that (2.9) is also fulfilled.

Proposition 3.4

Assume (H1). Then, for any $j\in\mathbb{Z}^+,\ k\in\mathbb{N},\ p\in(1,\infty)$,

$$\sup_{0<\varepsilon\le 1}\ \sup_{x,t}\ \|\hat{X}_j^{\varepsilon}(t,x)\|_{k,p}\le C\,.$$

Proof. Due to the equality (3.15), it suffices to check

$$\sup_{0\le\varepsilon\le 1}\ \sup_{x,t}\ \|X_j^{\varepsilon}(t,x)\,\|_{k,p}\le C\,, \tag{3.21}$$

for any $j\in\mathbb{Z}^+,\ k\in\mathbb{N},\ p\in(1,\infty)$ and some positive constant C.

For $j = 0$, (3.21) has been proved in [2]. The processes $\{X_j^\varepsilon(t,x), (t,x) \in [0,T] \times [0,1]\}$, $\varepsilon \in [0,1]$, $j \in \mathbb{Z}^+$, are solutions of the stochastic evolution equations (3.3)-(3.6). Using the same method as for the proof of (3.9), one checks, for any $j \in \mathbb{Z}^+$,

$$\sup_{0\le\varepsilon\le1} \sup_{x,t} E\big(|X_j^\varepsilon(t,x)|^p\big) \le C. \tag{3.22}$$

The Malliavin derivatives of arbitrary order of the processes $\{X_j^\varepsilon(t,x), (t,x) \in [0,T] \times [0,1]\}$, $\varepsilon \in [0,1]$, $j \in \mathbb{Z}^+$, also satisfy stochastic evolution equations. Therefore, the standard method based on Burkholder's, Hölder's and Gronwall's inequalities can be applied to complete the proof. For the matter of illustration we consider the Malliavin derivative of $X_1^\varepsilon(t,x)$. Using (3.3) and the rules of Malliavin Calculus, we obtain

$$D_{rz}\, X_1^\varepsilon(t,x) = \mathbb{1}_{(r<t)} \left\{ \sum_{i=1}^{6} M_i^\varepsilon(r,z;t,x) \right\},$$

$(r,z) \in [0,T] \times [0,1]$, with

$$\begin{aligned}
M_1^\varepsilon(r,z;t,x) &= G_{t-r}(x,z)\Big\{\sigma\big(X^\varepsilon(r,z)\big) + \varepsilon\,\sigma'\big(X^\varepsilon(r,z)\big)\; X_1^\varepsilon(r,z)\Big\},\\
M_2^\varepsilon(r,z;t,x) &= \int_r^t \int_0^1 G_{t-s}(x,y)\,\sigma'\big(X^\varepsilon(s,y)\big)\, D_{rz}\, X^\varepsilon(s,y)\, W(ds,dy),\\
M_3^\varepsilon(r,z;t,x) &= \int_r^t \int_0^1 G_{t-s}(x,y)\,\varepsilon\,\sigma''\big(X^\varepsilon(s,y)\big)\, D_{rz}\, X^\varepsilon(s,y)\\
&\quad \times\; X_1^\varepsilon(s,y)\, W(ds,dy),\\
M_4^\varepsilon(r,z;t,x) &= \int_r^t \int_0^1 G_{t-s}(x,y)\,\varepsilon\,\sigma\big(X^\varepsilon(s,y)\big)\, D_{rz}\, X_1^\varepsilon(s,y)\, W(ds,dy).\\
M_5^\varepsilon(r,z;t,x) &= \int_r^t \int_0^1 G_{t-s}(x,y)\, b''\big(X^\varepsilon(s,y)\big)\, D_{rz}\, X^\varepsilon(s,y)\, X_1^\varepsilon(s,y)\, dsdy,\\
M_6^\varepsilon(r,z;t,x) &= \int_r^t \int_0^1 G_{t-s}(x,y)\, b'\big(X^\varepsilon(s,y)\big)\, D_{rz}\, X_1^\varepsilon(s,y)\, dsdy.
\end{aligned}$$

Then, for any $p \in (2,\infty)$.

$$\begin{aligned}
&E\Big|\int_0^t \int_0^1 |D_{rz}\; X_1^\varepsilon(t,x)\,|^2 dr\, dz\Big|^{p/2}\\
&\quad \le\; C \sum_{i=1}^{6}\; E\Big|\int_0^t \int_0^1 |M_i^\varepsilon(r,z;t,x)\,|^2 dr\, dz\Big|^{p/2}.
\end{aligned}$$

(3.9) and (3.22) yield

$$\sup_{0\leq\varepsilon\leq 1}\ \sup_{x,t}\ E\Big|\int_0^t\int_0^1 |M_1^\varepsilon(r,z;t,x)|^2 dr\,dz\Big|^{p/2} \leq C\,. \tag{3.23}$$

Applying Burkholder's inequality for Hilbert space valued martingales, Hölder's inequality, the above mentioned result by Bally and Pardoux and (3.22) we obtain, for $i=2,3,5$,

$$\sup_{0<\varepsilon\leq 1}\ \sup_{x,t}\ E\Big|\int_0^t\int_0^1 |M_i^\varepsilon(r,z;t,x)|^2 dr\,dz\Big|^{p/2} \leq C, \tag{3.24}$$

and, for $i=4,6$,

$$\begin{aligned}\sup_{x,t}\ E\Big|\int_0^t\int_0^1 |M_i^\varepsilon(r,z;t,x)|^2 dr\,dz\Big|^{p/2}&\\ \leq\ C\int_0^t \sup_x\ E\Big|\int_0^s\int_0^1 |D_{rz}\,X_1^\varepsilon(s,y)|^2 dr\,dz\Big|^{p/2} ds\,.&\end{aligned} \tag{3.25}$$

Hence (3.23)-(3.25) and Gronwall's Lemma ensure

$$\sup_{0\leq\varepsilon\leq 1}\ \sup_{x,t}\ E\Big|\int_0^t\int_0^1 |D_{rz}\,X_1^\varepsilon(t,x)|^2 dr\,dz\Big|^{p/2} \leq C\,, \tag{3.26}$$

for any $p\in(2,\infty)$.

For the Malliavin derivatives of higher order we use induction. □

We can now give the expansion of the density of the solution to the parabolic stochastic differential equation (1.2). In the next theorem $t\in(0,T]$, $x\in(0,1)$ are fixed and $p^\varepsilon_{t,x}(y)$ denotes the density of $X^\varepsilon(t,x)$ for $\varepsilon\in(0,1]$ at $y=\psi_{X_0}(t,x)$.

Theorem 3.5

Assume (H1), (H2). Set $\sigma^2 = E\big(|X_1^0(t,x)\,|^2\big)$. *Then*

$$p^\varepsilon_{t,x}(y) = \frac{1}{\varepsilon}\left\{\frac{1}{\sqrt{2\pi}\,\sigma} + \sum_{j=1}^N \varepsilon^j\,\frac{1}{j!}\,p_j + \varepsilon^{N+1}\,\tilde p^\varepsilon_{N+1}\right\}, \tag{3.27}$$

The coefficients p_j *are null for odd* j. *For even* $j\in\{2,\dots,N\}$

$$p_j = E\left\{\mathbb{1}_{\{X_1^0(t,x)>0\}}\,P_j\right\},$$

with

$$P_j = \sum_{k=1}^j\ \sum_{\substack{\beta_1+\ldots+\beta_k=j\\ \beta_1,\ldots,\beta_k\geq 1}} c_j(\beta_1,\dots,\beta_k)\ H_{k+1}\left(X_1^0(t,x), \prod_{\ell=1}^k \frac{1}{\beta_\ell+1}\,X^0_{\beta_{\ell+1}}(t,x)\right).$$

Moreover, $P_j\in\oplus_{n=0}^{3j+1}\mathcal{H}_n$, *where* $\mathcal{H}_n$ *denotes the* k*-th Wiener-Chaos, and* $\sup_{\varepsilon\in(0,1]}\big(|\tilde p^\varepsilon_{N+1}|\big)$ *is finite.*

Proof. The random variables $\hat{X}^\varepsilon(t,x)$ defined by (3.1) for $\varepsilon \in (0,1]$ and $\hat{X}^0(t,x) = X_1^0(t,x)$ (see (3.19)) satisfy the assumptions of Theorem 2.2. Therefore the expansion (3.27) follows from (2.7) and (3.2).

Consider the expansion (2.10) with $F^\varepsilon = \hat{X}^\varepsilon(t,x)$ and an even, smooth function f. Since the Wiener sheet W has a symmetric law, the random variables $f\big(\hat{X}^\varepsilon(t,x)\big)$ and $f\big(\hat{X}^{-\varepsilon}(t,x)\big)$ have the same law. Therefore the odd coefficients in the Taylor expansion are zero.

It is clear from the structure of the stochastic evolution equations defining $X_\beta^0(t,x)$, $\beta \in \mathbb{Z}^+$, (see (3.6) and (3.7)) that $X_\beta^0(t,x) \in \oplus_{n=0}^{\beta}\ \mathcal{H}_n$. Then, for $k \in \{1,\ldots,j\}$, $\beta_1,\cdots,\beta_k \geq 1$ with $\beta_1+\ldots+\beta_k = j$, $\prod_{\ell=1}^k \frac{1}{\beta_\ell+1}\ X^0_{\beta_\ell+1}(t,x) \in \oplus_{n=0}^{j+k}\mathcal{H}_n$. This yields $P_j \in \oplus_{n=0}^{3j+1}\mathcal{H}_n$. Indeed, if ϕ is a non-degenerate Gaussian random variable and ψ belongs to $\oplus_{n=0}^{m}\ \mathcal{H}_n,\ m \geq 0$, $H_k(\phi,\psi) \in \oplus_{n=0}^{m+k}\ \mathcal{H}_n$ (see Lemma 2.6 in [8]).

This completes the proof of the Theorem. □

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