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The irregularity of ruled surfaces in $\mathbf{P}^3$

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This paper contains part of the doctoral thesis written by the first author under the advice of the second. Fernando Serrano was one of the members of the Ph.D. committee. That was one of the last opportunities for the people of Complutense University to enjoy both his mathematical talent and his human quality. This paper is devoted to his memory.

Abstract

Let S be a ruled surface in $\mathbf{P}^3$ with no multiple generators. Let d and q be nonnegative integers. In this paper we determine which pairs (d, q) correspond to the degree and irregularity of a ruled surface, by considering these surfaces as curves in a smooth quadric hypersurface in $\mathbf{P}^5$.

1. Introduction

The study of ruled surfaces in projective space is a classical subject in algebraic geometry. In this paper, we contribute to it by determining which pairs (d, q) appear as the degree and irregularity of an integral ruled surface in $\mathbf{P}^3$, the complex three dimensional projective space.

Consider a curve C embedded in the Grassmann variety of lines in $\mathbf{P}^3$, $G = G(1, 3)$, that we identify with the quadric hypersurface $Q_4 \subset \mathbf{P}^5$ after Plücker embedding.

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Let us remember that on $G(1,3)$ we have the universal sequence

$$0 \longrightarrow E \longrightarrow \mathcal{O}_G^4 \longrightarrow E'^{\vee} \longrightarrow 0$$

which restricted to C gives

$$0 \longrightarrow E_C \longrightarrow \mathcal{O}_C^4 \longrightarrow E'^{\vee}_C \longrightarrow 0$$

In this paper, a ruled surface will be a surface which is the corresponding birational embedding of $S = \mathbf{P}(E'^{\vee}_C)$ in $\mathbf{P}^3$

$$\mathbf{P}(E'^{\vee}_C) \hookrightarrow \mathbf{P}^3 \times \mathcal{O}_C \longrightarrow \mathbf{P}^3$$

Observe that the irregularity of S is the genus of C, and the degree of S coincides with the degree of the curve $C \subset \mathbf{P}^5$.

The universal property of the Grassmann variety guarantees that, conversely, from a birational embedding of a ruled surface $S = \mathbf{P}(F)$ on a given curve C, we can recover the immersion of C in G in such a way that $E'^{\vee}_C = F$.

Let us just mention that the Chow groups of G are

$$A_3(G) = \eta_3 \mathbf{Z}$$

$$A_2(G) = \eta_2 \mathbf{Z} \oplus \eta'_2 \mathbf{Z}$$

$$A_1(G) = \eta_1 \mathbf{Z}$$

where η_3 is a special linear complex, i.e. parameterizing all lines in $\mathbf{P}^3$ meeting a given line; the cycle η_2 is an α-plane, that parameterizes all lines in $\mathbf{P}^3$ passing through a given point; the cycle η'_2 is a β-plane, that parameterizes all lines of $\mathbf{P}^3$ contained in a given plane; finally, the cycle η_1 is a line pencil that parameterizes all lines contained in a given plane and passing through a given point of it. Then we have that

$$\eta_3^2 = \eta_2 + \eta'_2$$

$$\eta_3 \cdot \eta_2 = \eta_3 \cdot \eta'_2 = \eta_1$$

$$\eta_3 \cdot \eta_1 = pt$$

$$\eta_2^2 = \eta'^2_2 = pt$$

$$\eta_2 \cdot \eta'_2 = 0$$

and the Chow class of a complex, i.e. a threefold of G, is given by its *degree* or intersection number with a generic line pencil; the Chow class of a surface of G (a congruence, in classical terms), is given by its *bidegree* (d_1, d_2), i.e. the intersection number d_1 with a generic α-plane and the intersection number d_2 with a generic β-plane. We can see that the degree of a ruled surface of $\mathbf{P}^3$, or curve in G, appears as the intersection number of the curve with a generic special linear complex.

Thus, the problem of determining the possible values of the degree and irregularity of a ruled surface is related with that of studying the genus of projective curves. As every curve in $\mathbf{P}^n$ can be isomorphically projected to $\mathbf{P}^3$, the pairs (d, g) giving the degree and genus of a smooth curve in $\mathbf{P}^n$ can be found among those that appear as the degree and genus of curves in $\mathbf{P}^3$, a problem which was completely solved by Gruson and Peskine ([6], [7]). But the search for the pairs realized by nondegenerate curves in higher dimensional projective spaces was still an interesting problem. Rathmann [12] and Pasarescu [11], independently, obtained the solution for $n = 4, 5$, and Ciliberto considered the case $n = 6$.

Our result, 2.1, is stated and proved in the following section after a research carried on with the aid of two main tools. One is the work of Rathmann and Pasarescu, and as they look for the curves that achieve the given degrees and genera on certain surfaces of low degree in $\mathbf{P}^4$ and $\mathbf{P}^5$, we need to use the classification of smooth congruences: our second tool. The classification was completed up to degree 5 by Hernández and Sols in [10], and to degree 8 by Arrondo and Sols (see [1], [2]). The degree nine case was classified by Verra [13], and Gross did the classification of smooth congruences of degree 10.

Finally, as a byproduct of our study, in the last section we try to answer the question *which are the possible values of the degree and genus of a nondegenerate curve lying on a smooth quadric hypersurface in* $\mathbf{P}^n$*,* $n = 3, 4, 5$?. The case $n = 3$ is well known, and we can give a complete answer for nondegenerate smooth curves in $Q_4 \subset \mathbf{P}^5$. Concerning curves in $Q_3 \subset \mathbf{P}^4$ we have a partial one. The study of the genus of curves in Q_3 is an active area of research, let us just cite the works [2], [4] and the paper by M.A. de Cataldo [3].

2. The degree and irregularity of ruled surfaces

Let us denote by $\pi(d, n)$ the Castelnuovo bound for the genus of smooth degree d curves in $\mathbf{P}^n$ ([8]).

The main result in this paper is the following.

Theorem 2.1

Let $S \subset \mathbf{P}^3$ be an integral, nondegenerate degree d ruled surface with no multiple rules, and irregularity q. Let $C \subset G(1,3) = Q_4 \subset \mathbf{P}^5$ be the smooth curve corresponding to S after Plücker embedding of the Grassmannian of lines in $\mathbf{P}^3$.

1. *If $d = 2$, then S is a quadric surface and $q = 0$.*
2. *When $3 \leq d \leq 6$.*
   - *If C spans a $\mathbf{P}^3$, then $q \leq \pi(d,3)$ and a pair (d,q) is realized as the degree and genus of a smooth curve in G if and only if there are integer numbers $a, b \geq 0$ such that $d = a + b$ and $q = ab - a - b - 1$ (C is a smooth curve of type (a,b) in $Q_2 \subset \mathbf{P}^3$).*
   - *If $0 \leq q \leq \frac{1}{6}d^2 - \frac{5}{6}d + 1$, $d \geq 4$, there exists a degree d smooth curve in G, that spans at least a $\mathbf{P}^4$ and has genus q.*
3. *When $d \geq 7$.*
   - *If C spans a $\mathbf{P}^3$, then $q \leq \pi(d,3)$. A pair (d,q) gives the degree and genus of a smooth curve in G if and only if there are integer numbers $a, b \geq 0$ such that $d = a+b$ and $q = ab-a-b-1$ (C is a smooth curve of type (a,b) in $Q_2 \subset \mathbf{P}^3$).*
   - *If C spans at least a $\mathbf{P}^4$, then $q \leq \frac{1}{6}d^2 - \frac{5}{6}d + 1$, $d \geq 4$ and*
     - *given any pair (d,q) with $q \leq \frac{1}{8}d^2 - \frac{1}{2}d + 1$, there exists such a smooth degree d curve of genus q in G*
     - *if $q > \frac{1}{8}d^2 - \frac{1}{2}d + 1$, then C spans precisely a $\mathbf{P}^4$ and lies on a cubic surface contained in G. A pair (d,q) is realized by a smooth curve if there exist integral numbers $b \geq 0, c \geq 1$ such that $d = b + 2c$ and $q = (b-1)(c-1) + \frac{1}{2}c(c-1)$*

Proof. As S is not a degenerate ruled surface, i.e. it is not a cone nor it is contained in a plane, the corresponding smooth curve $C \subset G$ will not be contained in an α-plane, nor in a β-plane.

There are four possibilities for the minimum dimension of a projective space containing C to be:

If C is a plane curve, the plane that contains it does not lie on G. Then it cuts a conic in G and C is precisely that conic. This is the degree 2 case, and S is a smooth quadric surface in $\mathbf{P}^3$. The irregularity is 0.

If C is not a plane curve but lies on a three dimensional projective space, then this linear space cuts on the Grassmanian a two dimensional smooth quadric surface or a cone (depending on whether the $\mathbf{P}^3 \subset \mathbf{P}^5$ is general with respect to $G = Q_4 \subset \mathbf{P}^5$ or not: remember that C is integral and not a plane curve, so that the intersection cannot be the union of an α-plane and a β-plane). It is well known ([9], IV.6.4.1) that the values of the degree and genus for curves in the cone are also achieved by curves in the smooth quadric surface, Q_2. The latter has $\mathbf{Z} \oplus \mathbf{Z}$ as its Picard group, the generators being one rule from each of the two families of lines ruling out this surface. A curve in Q_2 of type (a, b) has degree $d = a + b$ and genus $(a-1)(b-1)$, and there are smooth curves in every class.

Finally, the curve C can span a $\mathbf{P}^4$ or a $\mathbf{P}^5$. Is in these cases when the results of Rathmann [12] and Pasarescu [11] give the key to decide which values the degree and genus of $C \subset G = Q_4$ can achieve.

To make the argument more transparent, let us recall the results proven for projective curves (we will present here the two theorems containing them as enunciated by Rathmann, the results in [11] are the same).

Theorem 1. Let C be a smooth irreducible curve of degree $d > 0$ and genus g in $\mathbf{P}^4$.

- *If C is nondegenerate, then $0 \leq g \leq \frac{1}{6}d^2 - \frac{5}{6}d + 1$.*
- *If C lies on a nondegenerate cubic surface, then there are integers b, c such that $d = b + 2c$, $g = (b-1)(c-1) + \frac{1}{2}c(c-1)$. For every $b \geq 0, c \geq 1$ or $(b, c) = (-1, 1)$ ó $(0, 0)$, such a curve exists and is nondegenerate for $d \geq 4$.*
- *If C is nondegenerate and not contained in a surface of degree at most three, then $0 \leq g \leq \frac{1}{8}d^2 - \frac{1}{2}d + 1$. (For $d < 7$ the bound given in the first point is stronger).*
- *For $4 \leq d \leq 6$, g satisfying the inequality of the first point and for $d \geq 7$, g satisfying the inequality of the third one, there exists a nondegenerate smooth irreducible curve in $\mathbf{P}^4$ having these values as invariants.*

Theorem 2. Let C be a smooth irreducible curve of degree d and genus g in $\mathbf{P}^5$.

- *If C is nondegenerate, then $0 \leq g \leq \frac{1}{8}(d-3)^2$.*
- *If C lies on a nondegenerate quartic surface, then either there are integers a, b such that $d = 2a + b$, $g = (a-1)(b-1)$ (for $(a, b) = (0, 1)$ or $(1, 0)$ or $a > 0, b > 0$ such a curve exists) or there is an integer $a > 0$ such that $d = 2a$, $g = \frac{1}{2}(a-1)(a-2)$. These curves are nondegenerate for $d \geq 5$.*
- *If C is nondegenerate and not contained in a surface of degree at most 4 then $0 \leq g \leq \frac{1}{10}d^2 - \frac{1}{2}d + 1$. (For $d < 9$ the bound given in the first point is stronger).*
- *For $5 \leq d \leq 8$, g satisfying the inequality given in the first point and $d \geq 9$, g satisfying the inequality of the third there exists a nondegenerate smooth irreducible curve in $\mathbf{P}^5$ having these values as invariants.*

The idea behind the proof of these theorems is that of finding curves with the invariants that we want by realizing them as curves on certain known surfaces. The surfaces that Rathmann and Pasarescu use to prove their results are, in general, the same. Our idea is to verify whether the surfaces lie on the quadric hypersurface in $\mathbf{P}^5$.

Let us observe that in both theorems, the bounds for the genus are well known (see, for example [8]). The ones appearing in the first point of each theorem are the Castelnuovo bounds $\pi(d,4)$ and $\pi(d,5)$, respectively.

Those appearing in the second, are the "nearly Castelnuovo" $\pi_1(d,4)$ and $\pi_1(d,5)$, where $\pi_1(d,r)$ is defined as follows given d and $r \geq 4$

$$\text{if} \quad m_1 = \left[\frac{d-1}{r}\right], \qquad \varepsilon_1 = d - m_1 r - 1$$

$$\mu_1 = \begin{cases} 1 & \text{if } \varepsilon_1 = r-1 \\ 0 & \text{if } \varepsilon_1 \neq r-1 \end{cases}$$

then

$$\pi_1(d,r) = \binom{m_1}{2} r + m_1(\varepsilon_1 + 1) + \mu_1 \,.$$

In [8] (Theorem 3.15) Harris proves that if a nondegenarate smooth irreducible degree d curve in $\mathbf{P}^r$ is such its genus g satisfies $\pi(d,r) \geq g > \pi_1(d,r)$ then the curve must be contained in a degree $r-1$ surface.

But smooth degree $r-1$ surfaces in $\mathbf{P}^r$ are surfaces of minimum degree and they are the rational normal scrolls, if $r \neq 5$. In the $r=5$ case they can be either rational normal scrolls or Veronese surfaces.

In order to prove the theorem for curves in $\mathbf{P}^5$, Rathmann studies:

1. Curves lying on a nondegenerate quartic surface in $\mathbf{P}^5$. This is a rational normal scroll or the Veronese surface. But we can see in [10, 1, 2] that both are surfaces lying on G. To be precise, they are smooth nondegenerate congruences of bidegree $(2,2)$ and $(1,3)$, respectively.
2. Curves that do not lie on a surface of degree at most 4. Then Rathmann considers curves on low degree surfaces: the Del Pezzo and Bordiga surfaces in $\mathbf{P}^5$. For every pair of integers (d,g) such that $d \geq 20$ and

$$\sqrt{2d+40}(d+30) - \frac{23}{2}d - 189 < g \leq \frac{1}{10}d^2 - \frac{1}{2}d + 1$$

there exists a nondegenerate smooth irreducible curve of degree d and genus g on the Del Pezzo surface in $\mathbf{P}^5$. This surface can be defined as $\mathbf{P}^2$ after blowing up four points in general position, and embedding the resulting surface by means of the complete linear system of the cubics passing through them. The surface so obtained lies on G: it is the nondegenerate congruence of bidegree $(2,3)$ in the list of [10, 1, 2].
If $0 \leq g \leq (d-5)$ or

$$0 \leq g \leq \frac{1}{14}d^2 - \frac{3}{14}d + \frac{9}{56}$$

there exists a nondegenerate smooth irreducible curve of degree d and genus g on the Bordiga surface in $\mathbf{P}^5$. This surface can be realized after blowing up 9 points in general position in $\mathbf{P}^2$ and embedding the resulting surface by the complete linear system of the quartic curves passing through them; these degree 7 surface is the nondegenerate smooth congruence in G of bidegree $(3,4)$ (see [1, 2] again).

Observe that we can conclude this way that every pair (d,g) which gives the degree and genus of a smooth nondegenerate curve in $\mathbf{P}^5$, can also be obtained as the pair of invariants of a smooth nondegenerate curve lying on the quadric hypersurface $G = Q_4$.

Concerning the problem for curves in $\mathbf{P}^4$, we can observe that Rathmann considers:

1. Curves lying on a nondegenerate cubic surface in $\mathbf{P}^4$. Such a surface is a rational normal scroll which can be realized as a degenerate congruence of bidegree $(1,2)$ (see [10, 1, 2]) by blowing up a point in $\mathbf{P}^2$ and embedding the surface so obtained by the complete linear system of all the conics passing through the point.
2. Curves that are not contained on a surface of degree at most 3. The existence of smooth irreducible curves with (d,g) as invariants is proven by finding those curves on smooth surfaces of low degree in $\mathbf{P}^4$. Rathmann considers the degree 4 Del Pezzo surface and the Bordiga surface, which is of degree 6. The Del Pezzo surface in $\mathbf{P}^4$ is isomorphic to the surface obtained blowing up $\mathbf{P}^2$ in 5 points in general position and embedding the resulting surface by means of the complete linear system of the cubics passing through them. The classification in [10, 1,2] shows that this surface is a smooth degenerate congruence of bidegree $(2,2)$. Rathmann proves that for $d \geq 5$ and g such that
$$(d+12)\sqrt{d+9} - \frac{11}{2}d - 35 \leq g \leq \frac{1}{8}d^2 - \frac{1}{2}d + 1$$

there exists a smooth nondegenerate curve in $\mathbf{P}^4$ of degree d and genus g on the Del Pezzo surface, thus on $G \cap \mathbf{P}^4$.

Let us go back to the proof of our theorem. We were left with the case in which the curve $C \subset G \subset \mathbf{P}^5$ spanned a $\mathbf{P}^4$ or $\mathbf{P}^5$. But now we know that if C is of degree d and spans a $\mathbf{P}^4$ then its genus (which we again denote as q, being the irregularity of the ruled surface S) satisfies

$$q \leq \frac{1}{6}d^2 - \frac{5}{6}d + 1$$

and if it spans a $\mathbf{P}^5$, it is

$$q \leq \frac{1}{8}(d-3)^2.$$

So, if C does not lie in a $\mathbf{P}^3$ it holds that

$$q \leq \frac{1}{6}d^2 - \frac{5}{6}d + 1.$$

In order to prove that if $q \leq \frac{1}{8}d^2 - \frac{1}{2}d + 1$ then there is a smooth irreducible curve in G with (d, q) as its invariants, it is enough to recall that when

$$q \leq \frac{1}{10}d^2 - \frac{1}{2}d + 1$$

we can find such curves on the Bordiga and Del Pezzo surfaces in $\mathbf{P}^5$, which are smooth congruences on G.

We also know that if $d \geq 5$ and

$$(d+12)\sqrt{(d+9)} - \frac{11}{2}d - 35 \leq q \leq \frac{1}{8}d^2 - \frac{1}{2}d + 1$$

we can find smooth irreducible curves spanning a $\mathbf{P}^4$ and lying on the Del Pezzo surface in $\mathbf{P}^4$, so that we can find them in G. But as

$$(d+12)\sqrt{(d+9)} - \frac{11}{2}d - 35 \leq \frac{1}{10}d^2 - \frac{1}{2}d + 1$$

for $d \geq 9$ and the integral part (which is what is meaningful in bounding an integer number as the genus) of these two expressions is the same for $d = 7, 8$, we conclude the proof of our theorem in the case $d \geq 7$.

The case $3 \leq d \leq 6$ follows after observing that the existence of curves verifying the statement can be tested in the rational normal scroll, that lies on G. $\square$

3. The degree and genus of curves lying on quadric hypersurfaces

Let us consider the following problem: *which are the possible values of the degree and genus for curves lying on a quadric hypersurface Q_i, $i = 2, 3, 4$?*. Here, $Q_i \subset \mathbf{P}^{i+1}$ and we consider curves which are nondegenerate (i.e., do not lie in $\mathbf{P}^i$).

As a result of our research, we can give a complete answer to this problem in the cases $i = 2, 4$.

The case $i = 3$ is not completed at the moment. The point being that for $d \geq 7$ and g such that

$$0 \leq g \leq \frac{1}{12}d^2 - \frac{1}{6}d - \frac{11}{12}$$

Rathmann finds curves having the pair (d, g) as their invariants on the Bordiga surface in $\mathbf{P}^4$. This is a rational surface isomorphic to $\mathbf{P}^2$ blown up in ten points in general position and embedded in $\mathbf{P}^4$ by the complete linear system of the quartics passing through them. But *this surface does not lie on G*, as can be seen in the classification [1, 2]: the only smooth degenerate congruence of degree 6 is the one of bidegree $(3, 3)$, which is a $K3$ surface.

In a slightly wider range than the previously considered, Pasarescu finds the curves on a sextic surface in $\mathbf{P}^4$ with a double line (see [11]).

Gruson and Peskine, when studying the degree and genus problem in $\mathbf{P}^3$, consider also a quartic with a double line that, as in the case of Pasarescu's sextic, can be obtained by blowing up nine points in $\mathbf{P}^2$ that verify certain conditions and embedding the resulting rational surface afterwards.

In trying to imitate that approach, a smooth degenerate congruence that is a rational surface obtained after blowing up at least nine points in $\mathbf{P}^2$ embedded in G should be considered. But there are not such congruences according to the classification of congruences of low degree, as it is by now (for smooth congruences of degree less or equal than 5 see [10], up to degree 8 [1], degree 9 in [13] and 10 in [5]). This is, then, a problem which is still open.

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