Collect. Math. **49**, 2–3 (1998), 293–315

Boundedness for codimension two submanifolds of quadrics

Maria Lucia Fania and Giorgio Ottaviani*

Dipartimento di Matematica, Università de L'Aquila, Via Vetoio Loc. Coppito, 67100 L'Aquila, Italy
E-mail: fania@univaq.it, ottavian@univaq.it

In ricordo di Fernando Serrano

Abstract

It is proved that there are only finitely many families of codimension two subvarieties not of general type in Q_6.

1. Introduction

Ellingsrud and Peskine proved in [10] that smooth surfaces in $\mathbf{P}^4$ not of general type have bounded degree. In [6] this result has been extended to any non general type codimension two submanifolds in $\mathbf{P}^{n+2}$, of dimension $n \geq 2$.

In the same spirit Arrondo, Sols and De Cataldo proved in [1], [9] the following result

Theorem 1.1 (Arrondo, Sols, De Cataldo)

Let $X = X_n \subset Q_{n+2}$ be a smooth variety not of general type of dimension n embedded in the smooth quadric Q_{n+2} of dimension $n+2$. Let $n \geq 2$, $n \neq 4$. Then $deg(X)$ is bounded.

More precisely in [1] it is proved the case $n = 2$ while in [9] it is proved the case $n = 3$ and it is observed that the case $n \geq 5$ follows by an inequality on Chern classes along the lines of [13].

The aim of this paper is to drop the assumption $n \neq 4$ from the previous theorem. In fact we show the following

* Both authors were supported by MURST and by GNSAGA of CNR.

Theorem 1.2

Let $X = X_4 \subset Q_6$ be a smooth 4-dimensional variety not of general type. Then $\deg(X)$ *is bounded.*

As it is well known, see Proposition 2.7, the theorem implies that there are only finitely many families of codimension two subvarieties not of general type in Q_6.

The paper is structured as follows. In section 2 we fix our notation and give preliminary results that will be needed later on in the paper. The sections 3 and 4 are devoted to bounding the degree of a non general type 4-fold $X \subset Q_6$.

In the last section we consider the problem of boundedness of non general type 4-folds in $\mathbf{P}^7$. Among the log-special type 4-fold in $\mathbf{P}^7$ (i.e., the image of the adjunction mapping has dimension less than 4) there are still two hard cases to be considered, namely the quadric bundles over surfaces and the scrolls over threefolds. We show that quadric bundles over surfaces have bounded degree with the only exception of those which lie in a 5-fold of degree 8.

The same technique can be applied to a manifold X of dimension $n+1$ embedded in $\mathbf{P}^{2n+1}$ which is a quadric bundle over a surface. One can prove that there exists a function $F(n)$ such that $deg(X) \leq F(n)$ or X is contained in a variety of dimension $n+2$ and degree $[\frac{n(n+1)(4n^2+4n+1)}{6(4n-1)}]$, where $[x]$ is the greatest integer less than or equal to x.

2. Notations and preliminaries

Throughout this article, unless otherwise specified, X denotes a smooth connected projective 4-fold defined over the complex field $\mathbb{C}$, which is contained in Q_6. Its structure sheaf is denoted by $\mathcal{O}_X$. For any coherent sheaf $\mathcal{F}$ on X, $h^i(\mathcal{F})$ is the complex dimension of $H^i(X, \mathcal{F})$ and $\chi = \chi(\mathcal{O}_X) = \sum_i (-1)^i h^i(\mathcal{O}_X)$. The following notation is used:

X, smooth 4-fold in Q_6;
H class of hyperplane section of X, $H = \mathcal{O}_{\mathbf{P}^7}(1)_{|X}$;
K class of canonical bundle of X;
X^3 generic 3-fold section of X;
S generic surface section of X;
C generic curve section of X;
g genus of C;
c_i = Chern classes of X;
N the normal bundle of X in $Q_6, N_{X/Q_6}$.

Using the self-intersection formula for the embedding of $X \subset Q_6$

$$c_2(N) = \frac{1}{2} d H^2 \tag{1}$$

we get the following formulae for KH^3, K^2H^2, K^3H, K^4 as function of d, g, $\chi(\mathcal{O}_X)$, $\chi(\mathcal{O}_{X^3})$, $\chi(\mathcal{O}_S)$.

$$K \cdot H^3 = 2g - 2 - 3d \tag{2}$$

$$K^2 \cdot H^2 = 6\chi(\mathcal{O}_S) - 12g + 12 + \frac{13}{2}d + \frac{1}{4}d^2 \tag{3}$$

$$K^3 \cdot H = -24\chi(\mathcal{O}_{X^3}) - 48\chi(\mathcal{O}_S) + 48g - 48 + 3d - 3d^2 + d(g-1) \tag{4}$$

$$\begin{aligned} K^4 = {} & 120\chi(\mathcal{O}_X) + 216\chi(\mathcal{O}_{X^3}) + \chi(\mathcal{O}_S)\frac{9d+472}{2} \\ & + \frac{5d^3 + 1098d^2 - 16d(45g+434) - 6144(g-1)}{48}. \end{aligned} \tag{5}$$

Note that (2) follows from the adjuction formula.

To prove (3) we reason as follows. From the long exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{Q_6|X} \longrightarrow N \longrightarrow 0 \tag{6}$$

we get that

$$c_2(N) = 16H^2 + 6H \cdot K + K^2 - c_2\,.$$

Such equality along with the self-intersection formula (1) gives

$$c_2 = \Big(16 - \frac{1}{2}d\Big)H^2 + 6H \cdot K + K^2. \tag{7}$$

Hence by dotting (7) with H^2 we have

$$c_2 \cdot H^2 = \Big(16 - \frac{1}{2}d\Big)H^4 + 6H^3 \cdot K + H^2 \cdot K^2. \tag{8}$$

On the other hand

$$c_2 \cdot H^2 = 12\chi(\mathcal{O}_S) - K^2 \cdot H^2 - (12g - 12 - 11d)\,. \tag{9}$$

In fact using the following exact sequences

$$0 \longrightarrow T_{X^3} \longrightarrow T_{X|X^3} \longrightarrow H_{X^3} \longrightarrow 0$$

$$0 \longrightarrow T_S \longrightarrow T_{X^3|S} \longrightarrow H_S \longrightarrow 0$$

we get that

$$c_{2|X^3} = c_2(X^3) - K_{X^3} \cdot H_{X^3}\,. \tag{10}$$

Now (9) will follow if we dot (10) with H and if we use the following facts:

$$c_2(X^3) \cdot H_{X^3} = c_2(S) - K_S \cdot H_S = 12\chi(\mathcal{O}_S) - K_S^2 - (2g - 2 - d) \tag{11}$$

and

$$K_S^2 = K^2 \cdot H^2 + 8g - 8 - 8d\,. \tag{12}$$

Combining (8) and (9) we get (3).

In order to prove (4) we do the following. By dotting (7) with $H\cdot$ K we get

$$c_2 \cdot H \cdot K = \Big(16 - \frac{1}{2}d\Big)K \cdot H^3 + 6K^2 \cdot H^2 + K^3 \cdot H\,. \tag{13}$$

On the other hand if we dot (10) with K and we use the fact that $c_2(X^3) \cdot K_{X^3} = -24\chi(\mathcal{O}_{X^3})$ we have

$$c_2 \cdot H \cdot K = -24\chi(\mathcal{O}_{X^3}) - 12\chi(\mathcal{O}_S) + 8g - 8 - 6d\,. \tag{14}$$

Now (4) is gotten by putting together (13) and (14).

We now prove (5). From the sequence (6) we get that

$$24H^3 = c_3 + c_2 \cdot (6H + K) - \frac{1}{2}dK \cdot H^2 \tag{15}$$

$$22H^4 = c_4 + c_3 \cdot (6H + K) - \frac{1}{2}dH^2 \cdot c_2\,. \tag{16}$$

Thus

$$c_3 = (3d - 72)H^3 - (52 - d)H^2 \cdot K - 12H \cdot K^2 - K^3 \tag{17}$$

$$\begin{aligned} c_4 =&454d - 26d^2 + \frac{1}{4}d^3 + (384 - 12d)H^3 \cdot K + \Big(124 - \frac{3}{2}d\Big)H^2 \cdot K^2 \\ &+ 18H \cdot K^3 + K^4. \end{aligned} \tag{18}$$

By Riemann Roch theorem we know that

$$-720\chi(\mathcal{O}_X) = K^4 - 4K^2 \cdot c_2 - 3c_2^2 + K \cdot c_3 + c_4\,. \tag{19}$$

Combining (7), (19), (18), (17), (4), (3) and (2) we get (5).

Theorem 2.1 ([8], Theorem 5.1)

Let C be an irreducible reduced curve of arithmetic genus g and degree d in the projective space $\mathbf{P}^{n+1}$. Assume that C is not contained on any surface of degree $<$ s, with $d > \frac{2s}{n-1}\prod_{i=1}^{n-1}\sqrt[n-i]{n!s}$. Then

$$g-1 \leq \frac{d(d-1)}{2s}+\frac{d(s-2n+1)}{2(n-1)}+\frac{(d+s-1)(s-1)}{2s} + \frac{(d-1)(n-2)(s+n-2)}{2s(n-1)}+\frac{(s-1)^2}{2(n-1)} \ .$$

Proposition 2.2 ([1], Proposition 6.3)

Let C be a smooth curve of degree d and genus g in Q_3 that is not contained in any surface in Q_3 of degree strictly less than $2k$. Then

$$g-1 \leq \frac{d^2}{2k}+\frac{1}{2}(k-4)d \ .$$

Theorem 2.3 (Castelnuovo bound [11])

Let V be an irreducible nondegenerate variety of dimension k and degree d in $\mathbf{P}^n$. Put

$$M=\left[\frac{d-1}{n-k}\right] \quad \textit{and} \quad \varepsilon = d-1-M(n-k)$$

where $[x]$ is the greatest integer less than or equal to x. Then

$$p_g(V)=h^0(\tilde{V},\Omega^k) \leq \binom{M}{k+1}(n-k)+\binom{M}{k}\varepsilon$$

where $\widetilde{V}$ is a resolution of V (i.e., $\widetilde{V}$ is a smooth variety mapping holomorphically and birationally to V).

Proposition 2.4

Let X be a smooth 4-fold in Q_6. Then

$$\chi\Big(\mathcal{O}_X(t)\Big) = \frac{1}{24}dt^4+\frac{1}{12}(2-2g+3d)t^3+\frac{1}{24}\Big(12\chi(\mathcal{O}_S)-12g+12+11d\Big)t^2 + \frac{1}{12}\Big(12\chi(\mathcal{O}_{X^3})+6\chi(\mathcal{O}_S)-4g+4+3d\Big)t+\chi(\mathcal{O}_X).$$

Proof. By the Riemann-Roch theorem we have

$$\chi\Big(\mathcal{O}_X(t)\Big) = \frac{1}{24}H^4t^4+\frac{1}{12}(KH^3)t^3+\frac{1}{24}(H^2\cdot K^2+c_2\cdot H^2)t^2-\frac{1}{24}c_2\cdot H\cdot Kt+\chi(\mathcal{O}_X),$$

where $c_i = c_i(T_X)$. We now use (9) and (14) to get our claim. □

Proposition 2.5

Let $S \subset Q_4$ be a surface of degree d contained in an irreducible threefold of degree σ, with σ minimal. Then

$$p_g(S) = h^2(\mathcal{O}_S) \leq \frac{d^3}{24\sigma^2} + \frac{d^2(\sigma-4)}{8\sigma} + \frac{d(2\sigma^2 - 12\sigma + 23)}{12}.$$

Proof. Let C be the generic curve section of S. By [1], Proposition 6.4 for $d >> 0$ we have

$$g - 1 \leq \frac{d^2}{4\sigma} + \frac{1}{2}(\sigma - 3)d.$$

We let

$$G(t) = \chi\Big(\mathcal{O}_{Q_4}(t)\Big) = \binom{t+5}{5} - \binom{t+3}{5}$$

and

$$\widetilde{F}(t) = G(t) - G(t-\sigma) - G\Big(t - \frac{d}{2\sigma}\Big) + G\Big(t - \sigma - \frac{d}{2\sigma}\Big).$$

We set

$$P(t) = dt + \Big(-\frac{d^2}{4\sigma} + \frac{1}{2}(3-\sigma)d\Big)$$

and

$$F(t) = \begin{cases} \widetilde{F}(t) & \text{if} \quad t \leq -1 \\ \widetilde{F}(0) - 1 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \geq 1 \end{cases}$$

We have the following exact sequence

$$H^1\Big(\mathcal{O}_C(t)\Big) \longrightarrow H^2\Big(\mathcal{O}_S(t-1)\Big) \longrightarrow H^2\Big(\mathcal{O}_S(t)\Big) \longrightarrow 0$$

from which it follows that

$$\begin{aligned} &- h^2\Big(\mathcal{O}_S(t)\Big) + h^2\Big(\mathcal{O}_S(t-1)\Big) \leq h^1\Big(\mathcal{O}_C(t)\Big) \\ &= \left\{ \begin{array}{ll} -\chi\Big(\mathcal{O}_C(t)\Big) \leq -P(t) & \text{if} \quad t \leq -1 \\ -\chi(\mathcal{O}_C) + 1 \leq -P(0) + 1 & \text{if} \quad t = 0 \end{array} \right\} \\ &= F(t-1) - F(t), \quad \text{for} \quad t \leq 0\,. \end{aligned}$$

The same holds for $t \geq 1$ since $h^1\Big(\mathcal{O}_C(t)\Big) = 0$ and

$$F(t-1) - F(t) = \begin{cases} \geq 0 & \text{if} \quad t = 1 \\ \ \ 0 & \text{if} \quad t \geq 2\,. \end{cases}$$

From this it follows that $F(t) - h^2\Big(\mathcal{O}_S(t)\Big)$ is non increasing and since for t going to infinity it goes to zero it follows that $h^2\Big(\mathcal{O}_S(t)\Big) \leq F(t)$ for all t. Thus evaluating it at $t = 0$ we have

$$F(0) = \frac{d^3}{24\sigma^2} + \frac{d^2(\sigma-4)}{8\sigma} + \frac{d(2\sigma^2 - 12\sigma + 23)}{12}$$

and hence our claim follows. Note also that

$$F(t) - F(t-1) = \begin{cases} dt + \Big(-\dfrac{d^2}{4\sigma} + \dfrac{1}{2}(3-\sigma)d\Big) & \text{if} \quad t \leq -1 \\ \Big(-\dfrac{d^2}{4\sigma} + \dfrac{1}{2}(3-\sigma)d\Big) - 1 & \text{if} \quad t = 0\,. \end{cases}$$

On passing note that if d is a multiple of 2σ then $\widetilde{F}(t)$ is the Hilbert polynomial of the complete intersection $V_{\sigma,\frac{d}{2\sigma}}$ in Q_4 of hypersurfaces in $\mathbf{P}^5$ of degree σ and $\frac{d}{2\sigma}$ while $F(t)$ corresponds to $h^2\Big(\mathcal{O}_{V_{\sigma,\frac{d}{2\sigma}}}(t)\Big)$. $\square$

Proposition 2.6 ([9], Theorem 3.1)

Let $X^3 \subset V_\sigma \subset Q_5$. Then

$$-\chi(\mathcal{O}_{X^3}) \geq \frac{1}{192\sigma^3}d^4 + \text{l.t.} \quad \text{in } \sqrt{d}.$$

Proposition 2.7

For any fixed integer d_0 there are only finitely many irreducible components of the Hilbert scheme of 4-folds in Q_6 that contain 4-folds with $d \leq d_0$.

Proof. By Theorem 2.3, for $d \leq d_0$, there are finitely many possible values for g, $p_g(S), p_g(X^3), p_g(X)$ and hence for $\chi(\mathcal{O}_S), \chi(\mathcal{O}_{X^3}), \chi(\mathcal{O}_X)$ since $h^2(\mathcal{O}_{X^3}) \leq p_g(S)$ and $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_{X^3}) = h^1(\mathcal{O}_X) = 0$. Thus there are only finitely many possibilities for the Hilbert polynomial

$$\begin{aligned}\chi\Big(\mathcal{O}_X(t)\Big) =& \frac{1}{24}dt^4 + \frac{1}{12}(2 - 2g + 3d)t^3 + \frac{1}{24}\Big(12\chi(\mathcal{O}_S) - 12g + 12 + 11d\Big)t^2 \\ &+ \frac{1}{12}\Big(12\chi(\mathcal{O}_{X^3}) + 6\chi(\mathcal{O}_S) - 4g + 4 + 3d\Big)t + \chi(\mathcal{O}_X). \ \square\end{aligned}$$

3. 4-Folds on a hypersurface of fixed degree

Let X be a 4-fold of degree d in Q_6 contained in an integral hypersurface $V_\sigma \in |\mathcal{O}_{Q_6}(\sigma)|$.

Theorem 3.1

Let $X \subset V_\sigma \subset Q_6$ be as above. There is a polynomial $P_\sigma(t)$ of degree 10 in $\sqrt{d}$ with positive leading coefficient, such that

$$\chi(\mathcal{O}_X) \geq P_\sigma(\sqrt{d}).$$

Proof. Look at the following three exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^7}(t-2) \longrightarrow \mathcal{O}_{\mathbf{P}^7}(t) \longrightarrow \mathcal{O}_{Q_6}(t) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{Q_6}(t-\sigma) \longrightarrow \mathcal{O}_{Q_6}(t) \longrightarrow \mathcal{O}_V(t) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}_{X,V}(t) \longrightarrow \mathcal{O}_V(t) \longrightarrow \mathcal{O}_X(t) \longrightarrow 0\ .$$

We use the first one to compute $\chi(\mathcal{O}_{Q_6}(t))$ and the second one to compute

$$\begin{aligned}\chi\Big(\mathcal{O}_V(t)\Big) = {} & \frac{\sigma}{60}t^5 + \frac{(6\sigma-\sigma^2)}{24}t^4 + \frac{\sigma(\sigma^2-9\sigma+26)}{18}t^3 - \frac{\sigma(\sigma^3-12\sigma^2+52\sigma-96)}{24}t^2 \\ & + \frac{\sigma(3\sigma^4-45\sigma^3+260\sigma^2-720\sigma+949)}{180}t \\ & - \frac{\sigma(\sigma^5-18\sigma^4+130\sigma^3-480\sigma^2+949\sigma-942)}{360}\ .\end{aligned}$$

Now use Proposition 2.4, $\mu := \mu_\sigma = \frac{1}{2}d^2 + \sigma(\sigma-3)d - 2\sigma(g-1)$ and the third exact sequence to compute

$$\begin{aligned}\chi\Big(\mathcal{I}_{X,V}(t)\Big) = {} & \frac{\sigma}{60}t^5 + \frac{1}{24}\Big((6-\sigma)\sigma - d\Big)t^4 \\ & + \Big(\frac{3d^2 + 6d\sigma(\sigma-6) - 2\big(3\mu - 2\sigma^2(\sigma^2-9\sigma+26)\big)}{72\sigma}\Big)t^3 \\ & - \Big(\frac{12\chi(\mathcal{O}_S)\sigma - 2d^2 + d\sigma(19-4\sigma) + 4\mu + \sigma^2(\sigma^3-12\sigma^2+52\sigma-96)}{24\sigma}\Big)t^2 \\ & - \Big(\frac{180\chi(\mathcal{O}_S)\sigma + 360\chi(\mathcal{O}_{X^3})\sigma - 15d^2 + 30d\sigma(4-\sigma) + 30\mu}{360\sigma} \\ & - \frac{2\sigma^2(3\sigma^4-45\sigma^3+260\sigma^2-720\sigma+949)}{360\sigma}\Big)t \\ & - \frac{\sigma(\sigma^5-18\sigma^4+130\sigma^3-480\sigma^2+949\sigma-942)}{360} - \chi(\mathcal{O}_X) \\ =: {} & Q(t) - \chi(\mathcal{O}_X)\ .\end{aligned}$$

Thus $\chi(\mathcal{O}_X) = Q(t) - \chi(\mathcal{I}_{X,V}(t))$. Define

$$t_1 := \min\left\{t \in \mathbb{N} | \delta := 2\sigma t - d > 0, \frac{\delta^2}{2} - \mu - \delta\sigma(\sigma - 3) > 0\right\}.$$

Then

$$\frac{d}{2\sigma} \leq t_1 \leq \frac{d}{2\sigma} + \frac{\sqrt{2d}}{2} + \sigma \ .$$

By plugging t_1 we get that

$$\begin{aligned} Q(t_1) \geq & \frac{1}{60 \cdot 2^5 \sigma^4} d^5 - \frac{1}{24 \cdot 2^4 \sigma^4} d^5 - \frac{1}{24 \cdot 2^3 \sigma^4} d^5 \\ & - \frac{1}{2^3 \sigma^2} d^2 \chi(\mathcal{O}_S) - \frac{d}{2\sigma} \chi(\mathcal{O}_{X^3}) + \text{l.t.} \quad \text{in } \sqrt{d} \ . \end{aligned}$$

We now use Proposition 2.6 and Proposition 2.5 in the above inequality to get

$$Q(t_1) \geq \frac{d^5}{\sigma^4} \Big(\frac{1}{60 \cdot 2^5} \Big) + \text{l.t.} \quad \text{in } \sqrt{d}$$

and thus

$$\begin{aligned} \chi(\mathcal{O}_X) & \geq Q(t_1) - \chi\Big(\mathcal{I}_{X,V}(t_1)\Big) \\ & \geq -\chi\Big(\mathcal{I}_{X,V}(t_1)\Big) + \frac{d^5}{\sigma^4} (\frac{1}{60 \cdot 2^5}) + \text{l.t.} \quad \text{in } \sqrt{d} \ . \end{aligned} \tag{20}$$

Moreover

$$-\chi\Big(\mathcal{I}_{X,V}(t_1)\Big) \geq -h^0\Big(\mathcal{I}_{X,V}(t_1)\Big) - h^2\Big(\mathcal{I}_{X,V}(t_1)\Big) - h^4\Big(\mathcal{I}_{X,V}(t_1)\Big) \ .$$

It will be enough to bound from above $h^2\big(\mathcal{I}_{X,V}(t_1)\big)$ since $h^0\big(\mathcal{I}_{X,V}(t_1)\big)$ and $h^4\big(\mathcal{I}_{X,V}(t_1)\big)$ have been bounded in ([9], Lemma 3.3). In order to bound $h^2\big(\mathcal{I}_{X,V}(t_1)\big)$ we consider the following exact sequences:

$$0 \longrightarrow \mathcal{O}_{Q_5}(-\sigma) \longrightarrow \mathcal{I}_{X^3,Q_5} \longrightarrow \mathcal{I}_{X^3,V^4} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-2) \longrightarrow \mathcal{I}_{X^3,\mathbf{P}^6} \longrightarrow \mathcal{I}_{X^3,Q_5} \longrightarrow 0 \ .$$

By [2], Theorem 3.12 (b)

$$H^1\Big(\mathcal{I}_{X^3,\mathbf{P}^6}(t)\Big) = 0 \quad \text{for} \quad t \geq 4d - 7 \ .$$

The latter along with the above sequences give

$$H^1\Big(\mathcal{I}_{X^3,V^4}(t)\Big) = 0 \quad \text{for} \quad t \geq 4d - 7 \ .$$

From

$$0 \longrightarrow \mathcal{I}_{X,V}(k-1) \longrightarrow \mathcal{I}_{X,V}(k) \longrightarrow \mathcal{I}_{X^3,V^4}(k) \longrightarrow 0$$

it follows that

$$H^2\Big(\mathcal{I}_{X,V}(t)\Big) = 0 \quad \text{for} \quad t \geq 4d - 8 \ .$$

Moreover by ([9], Lemma 3.3) we have that

$$h^2\Big(\mathcal{I}_{X,V}(t_1)\Big) \leq \sum_{k=t_1+1}^{4d-7} h^1\Big(\mathcal{I}_{X^3,V^4}(k)\Big) \leq (4d-7)Ad^{\frac{7}{2}} + \dots \leq 4Ad^{\frac{9}{2}} + \dots \ .$$

Hence

$$\chi(\mathcal{O}_X) \geq Q(t_1) - \chi\Big(\mathcal{I}_{X,V}(t_1)\Big) \geq Ad^5 + \text{l.t.} \quad \text{in } \sqrt{d}$$

which gives our claim. □

Corollary 3.2

Let $X \subset V_\sigma \subset Q_6$ be as above. Assume that X is not of general type. Then there exists d_0 such that $\deg(X) \leq d_0$.

Proof. Consider the following exact sequence

$$0 \longrightarrow K_X(-1) \longrightarrow K_X \longrightarrow K_{X^3}(-1) \longrightarrow 0 \ .$$

Since X is not of general type we have $h^0\big(K_X(-1)\big) = 0$ and thus

$$p_g(X) = h^0(K_X) \leq h^0\Big(K_{X^3}(-1)\Big) \leq h^0(K_{X^3}) = p_g(X^3) \ . \tag{21}$$

This along with Harris bound give

$$\begin{aligned} \chi(\mathcal{O}_X) &= 1 + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X) + p_g(X) \leq 1 + h^2(\mathcal{O}_X) + p_g(X^3) \\ &\leq 1 + p_g(S) + p_g(X^3) \leq \frac{1}{216}d^4 + \text{l.t.} \quad \text{in } \sqrt{d} \ . \end{aligned} \tag{22}$$

On the other hand by the proof of Theorem 3.1 we obtain that

$$\chi(\mathcal{O}_X) \geq \frac{1}{1920\sigma^4}d^5 + \text{l.t.} \quad \text{in } \sqrt{d} \ . \tag{23}$$

The boundedness of d will now follow from (22) and (23). Hence we get our claim. □

4. Boundedness

Proposition 4.1

Let X be a smooth 4-fold in Q_6. Denote $\chi(\mathcal{O}_S)$, $\chi(\mathcal{O}_{X^3})$, $\chi(\mathcal{O}_X)$, by s, x and v respectively. Then
a) $s \leq \frac{2}{3}\frac{(g-1)^2}{d} + \frac{5}{3}(g-1) - \frac{1}{24}d^2 + \frac{5}{12}d.$
b) $-24x(2g-2-3d) \leq 36s^2 + 3s(d^2 - 22d - 16(g-1)) + \frac{1}{16}(d^4 - 92d^3 + 4d^2(12g + 193) + 32d(1-g)(g+8) + 768(g-1)^2).$
If X is not of general type then
c) $v \leq 2s - x\,.$

Proof. By the generalized Hodge index theorem we know that

$$(K^2 \cdot H^2)H^4 \leq (K \cdot H^3)^2 \tag{24}$$

$$(K \cdot H^3)(K^3 \cdot H) \leq (K^2 \cdot H^2)^2. \tag{25}$$

We observe that (2), (3) and (24) give a) while (2), (3), (4) and (25) give b). In order to prove c) we note that (21) along with the Lefschetz theorem give $h^2(\mathcal{O}_X) \leq h^2(\mathcal{O}_S)$. Moreover being $h^1(\mathcal{O}_X) = 0$ it follows that $\chi(\mathcal{O}_X) \leq \chi(\mathcal{O}_S) + p_g(X^3) = 2\chi(\mathcal{O}_S) - \chi(\mathcal{O}_{X^3})$, which is our claim. $\Box$

Proposition 4.2

Let X be a smooth 4-fold in Q_6. Then
a) $24\chi(\mathcal{O}_S) \geq d^2 - 2d - 24(g-1).$
b) $240\chi(\mathcal{O}_{X^3}) \leq 120\chi(\mathcal{O}_X) + \frac{1}{2}(280 - 9d)\chi(\mathcal{O}_S) - \frac{1}{48}\big(d^3 - 6d^2 + 16d(12g-13) - 960(g-1)\big).$

Proof. Since $N(-1)$ is globally generated, the Segre classes satisfy:

$$s_2\Big(N(-1)\Big) \cdot H^2 \geq 0, \quad s_4\Big(N(-1)\Big) \geq 0\ .$$

Recall that

$$s_2 = c_1^2 - c_2$$
$$s_4 = c_1^4 + c_2^2 - 3c_1^2.$$

Moreover
$c_1\Big(N(-1)\Big) = K + 4H$

$c_2\Big(N(-1)\Big) = (\frac{1}{2}d-5)H^2 - H\cdot K$. Hence by (2), (3), (4), (5)

$$\begin{aligned} 0 \le s_2\Big(N(-1)\Big)\cdot H^2 &= K^2\cdot H^2 + 16d + 9K\cdot H^3 - \Big(\frac{1}{2}d-5\Big)H^4 \\ &= 6\chi(\mathcal{O}_S) + 6(g-1) + \frac{1}{2}d - \frac{1}{4}d^2 \end{aligned} \tag{26}$$

$$\begin{aligned} 0 \le s_4(N(-1)) &= (434-13d)K\cdot H^3 + \Big(136 - \frac{3}{2}d\Big)K^2\cdot H^2 + 19K^3\cdot H \\ &+ K^4 + 521d - 29d^2 + \frac{1}{4}d^3 = -240\chi(\mathcal{O}_{X^3}) + 120\chi(\mathcal{O}_X) \\ &+ \frac{280-9d}{2}\chi(\mathcal{O}_S) - \frac{d^3-6d^2+16d(12g-13)-960(g-1)}{48} . \square \end{aligned} \tag{27}$$

Theorem 4.3

There are only finitely many irreducible components of the Hilbert scheme of smooth 4-folds in Q_6 that are not of general type.

Proof. Let X be a smooth 4-fold in Q_6 that is not of general type. By Proposition 2.7 it is enough to bound $d = \deg X$. We will do so by considering separately the cases $2g-2-3d \le 0$ and $2g-2-3d > 0$.

Assume that $2g-2-3d \le 0$, i.e. $g-1 \le \frac{3}{2}d$. Using Proposition 4.1 along with a) in Proposition 4.2 we get

$$0 \le -\frac{1}{2}d^2 + 6d .$$

Hence d is bounded in this case.

Assume now that $2g-2-3d > 0$. Using b) in Proposition 4.2 along with c) and b) in Proposition 4.1 we get

$$\begin{aligned} 0 \le &\frac{540}{2g-2-3d}s^2 + \frac{117d^2 - 6d(3g+707) + 80(g-1)}{2(2g-2-3d)}s + \frac{d^4}{2g-2-3d} \\ &+ \frac{-d^3(g+2078) + 6d^2(229g+2842) + 304d(1-g)(3g+23) + 1824(g-1)^2}{24(2g-2-3d)} . \end{aligned}$$

Solving the above inequality with respect to s we see that either

$$s \ge \frac{b+\sqrt{L}}{2160} \ge \frac{b}{2160} \tag{28}$$

or

$$s \le \frac{b-\sqrt{L}}{2160} \tag{29}$$

where $b = -117d^2 + 6d(3g + 707) - 80(g - 1)$ and

$L = 5049d^4 - 36d^3(107g + 6793) + 36d^2(9g^2 - 8978g + 328809) + 960d(g - 1)(339g + 1915) - 6560000(g - 1)^2$.

If (28) holds then combining it with Proposition 4.1 we get

$$0 \leq \frac{d^2}{80} - \frac{d(3g + 557)}{360} + \frac{2(g - 1)^2}{3d} - \frac{46}{27}(g - 1). \tag{30}$$

If (29) holds then such inequality along with a) in Proposition 4.2 gives

$$0 \leq \frac{7}{864}d^4 - \frac{d^3(5g + 2203)}{6480} + \frac{d^2(595 - 263g)}{3240} + \frac{d(1 - g)(87g - 5749)}{1620} + \frac{7}{3}(g-1)^2. \tag{31}$$

Fix a positive integer k and let $d > 2k^2$. Assume that X does not lie on any hypersurface of Q_6 of degree strictly less than $2k$. Then by a well known theorem of Roth C does not lie on any hypersurface of Q_6 of degree strictly less than $2k$. Hence by Proposition 2.2 the genus of C satisfies

$$g - 1 \leq \frac{d^2}{2k} + \frac{1}{2}(k - 4)d\ . \tag{32}$$

Rewriting (30) in the following way

$$0 \leq (g - 1)\Big(\frac{2}{3d}(g - 1) - \frac{d}{120} + \frac{46}{27}\Big) - \frac{7}{4}d + \frac{d^2}{80}$$

and using (32) we get

$$(g - 1) \leq \frac{3k}{2(k - 40)}d + \text{l.t. in } d\ . \tag{33}$$

In the case (31) a similar reasoning yields

$$(g - 1) \leq \frac{21}{2}d + \text{l.t. in } d\ . \tag{34}$$

The following inequality, gotten by combining a) in Proposition 4.1 and a) in Proposition 4.2 will be needed:

$$0 \leq -\frac{1}{12}d^2 + \frac{1}{2}d + (g - 1)\Big(\frac{2}{3d}(g - 1) + \frac{8}{3}\Big)\ . \tag{35}$$

Plugging (33) and (32) in (35) gives

$$0 \leq d^2\Big(\frac{1}{2(k-40)} - \frac{1}{12}\Big) + \text{l.t. in } d\ . \tag{36}$$

Similarly, plugging (34) and (32) in (35) gives

$$0 \leq d^2\Big(\frac{7}{2k} - \frac{1}{12}\Big) + \text{l.t. in } d\ . \tag{37}$$

The coefficient of d^2, both in (36) and (37) is negative for k=47. Hence d is bounded from above if X is not in a hypersurface of degree strictly less than $2 \cdot 47$. If X is not of general type and is contained in a hypersurface of degree less than or equal to $2 \cdot 47$ then by Corollary 3.2 there exists d_0 such that $\deg(X) \leq d_0$. Hence the theorem is proved. □

5. Quadric bundles over surfaces in $\mathbf{P}^7$

Throughout this section X will denote a smooth 4-fold of degree d in $\mathbf{P}^7$ which is a quadric bundle over a surface. We will show that either its degree d is bounded or X is contained in a 5-fold of degree 8.

For 4-folds in $\mathbf{P}^7$ by the self-intersection formula we have:

$$c_3(N_{X/\mathbf{P}^7}) = dH^3.$$

5.1 From the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbf{P}^7|X} \longrightarrow N_{X/\mathbf{P}^7} \longrightarrow 0$$

we get that

$$\begin{aligned} c_3(X) &= (56-d)H^3 - 28H^2 \cdot c_1(X) + 8H \cdot \Big(c_1^2(X) - c_2(X)\Big) - c_1^3(X) \\ &\quad + 2c_1(X)c_2(X) \\ c_4(X) &= 70d - c_1(X)dH^3 - c_2(X)\Big(28H^2 - 8c_1(X)H + c_1^2(X) - c_2(X)\Big) \\ &\quad - c_3(X)\Big(8H - c_1(X)\Big). \end{aligned}$$

DEFINITION 5.2. A 4-fold X is called a geometric quadric bundle if there exists a morphism $p : X \longrightarrow B$ onto a normal surface B such that every fibre $p^{-1}(b)$ is isomorphic to a quadric. A 4-fold X is a quadric bundle in the adjunction theoretic sense if there exists a morphism $p : X \longrightarrow B$ onto a normal surface B and an ample Cartier divisor L on B such $p^*L = K + 2H$.

The following proposition relates the two notions.

Proposition 5.1

Let X be a quadric bundle in the adjunction theoretic sense. Then X is a geometric quadric bundle. Moreover the base B is smooth.

Proof. By ([3], Theorem 2.3) we know that p is equidimensional, being $\dim X = 4$. Moreover by ([5], Theorem 8.2) the base B is smooth. □

From now on our notations and arguments follow closely the ones in [7].

5.3 Notations

Let $p : X \longrightarrow B$ be a geometric quadric bundle in $\mathbf{P}^7$. We have a natural morphism $f : B \longrightarrow Gr(\mathbf{P}^3, \mathbf{P}^7)$.

Let S be a generic surface section of X. Then $p : S \longrightarrow B$ is finite 2:1. Let $2R \subset B$ be the *ramification divisor* of $p : S \longrightarrow B$.

We set $p_*\mathcal{O}_X(1) =: E$, a rank 4 vector bundle over B. We have $E = f^*(U^\vee)$, where U is the universal bundle of $Gr(\mathbf{P}^3, \mathbf{P}^7)$, in particular $\det\ E = f^*(\overset{2}{\wedge}\ U^\vee)$ is ample. Note that $W := P(E)$ is a $\mathbf{P}^3$-bundle in the natural incidence variety $\mathbf{P}^7 \times Gr(\mathbf{P}^3, \mathbf{P}^7)$ whose projection π into $\mathbf{P}^7$ is the variety V given by the union of all the 3-planes containing the quadrics of X.

Moreover $\pi^{-1}(X) = \widetilde{X}$ is smooth and isomorphic to X. We denote the natural projection of W onto B also by p and by H the divisor on W corresponding to $\mathcal{O}_W(1)$. Hence $\widetilde{X} = 2H - p^*L$ for some divisor L on B.

The divisor $D \subset B$ corresponding to points whose fibres are singular quadrics, is called the *discriminant divisor*. Moreover $D = c_1(E) - c_1(L \otimes E^\vee) = 2c_1(E) - 4L$. In fact $\widetilde{X}$ determines a section of $S^2E \otimes L^\vee$, hence a morphism $\phi : L \otimes E^\vee \longrightarrow E$. D is given by the equation $\det\ \phi = 0$. Thus our claim.

In order to bound d we need several preliminary computations.

Proposition 5.2

$$c_1(W) = 4H - p^*c_1(E) + p^*c_1(B)$$

$$c_2(W) = 6H^2 + H \cdot \Big(4p^*c_1(B) - 3p^*c_1(E)\Big) + p^*c_2(B) - p^*c_1(E) \cdot p^*c_1(B) + p^*c_2(E)$$

$$c_3(W) = 4H^3 + H^2 \cdot \Big(6p^*c_1(B) - 3p^*c_1(E)\Big) + H \cdot \Big(2p^*c_2(E) + 4p^*c_2(B) - 3p^*c_1(E) \cdot p^*c_1(B)\Big)$$

$$c_4(W) = 4H^3 \cdot p^*c_1(B) + H^2 \cdot \Big(6p^*c_2(B) - 3p^*c_1(E) \cdot p^*c_1(B)\Big)$$

$$H^4 - H^3 \cdot p^*c_1(E) + H^2 \cdot p^*c_2(E) = 0\ .$$

Proof. Consider the sequence

$$0 \longrightarrow \mathcal{O}_W \longrightarrow p^*E^\vee \otimes \mathcal{O}_W(1) \longrightarrow T_W \longrightarrow p^*T_B \longrightarrow 0 \ .$$

The Chern polynomial of $p^*E^\vee \otimes \mathcal{O}_W(1)$ is

$1+c_1(p^*E^\vee\otimes\mathcal{O}_W(1))t+c_2(p^*E^\vee\otimes\mathcal{O}_W(1))t^2+c_3(p^*E^\vee\otimes\mathcal{O}_W(1))t^3 = 1+[4H-p^*c_1(E)]t+[6H^2-3p^*c_1(E)\cdot H+p^*c_2(E)]t^2+[4H^3-3p^*c_1(E)\cdot H^2+2p^*c_2(E)\cdot H]t^3$.

On the other hand $ch(T_W) = ch(p^*E^\vee \otimes \mathcal{O}_W(1)) \cdot ch(p^*T_B)$ hence we get that $1+c_1(W)t+c_2(W)t^2+c_3(W)t^3+c_4(W)t^4+c_5(W)t^5 = \{1+[4H-p^*c_1(E)]t+[6H^2-3p^*c_1(E)\cdot H+p^*c_2(E)]t^2+[4H^3-3p^*c_1(E)\cdot H^2+2p^*c_2(E)\cdot H]t^3\}\cdot\{1+p^*c_1(B)t+p^*c_2(B)t^2\}$. Expanding the right hand side we get the first four equations. The last one is the Wu-Chern equation on $W = P(E)$, that is, $c_4(p^*E^\vee \otimes \mathcal{O}_W(1)) = 0$. □

Lemma 5.3

$c_1(E) = 2R - \frac{D}{2}, \quad L = R - \frac{D}{2}.$

Proof. We identify X and $\widetilde{X}$. We have $K_S = p^*(K_B + R)$, hence by the adjunction formula $K_X = -2H + p^*R + p^*K_B$. From Proposition 5.2 $K_W = -4H + p^*c_1(E) - p^*c_1(B)$. Putting this together with the adjunction formula $K_X = K_{W|X} + 2H - p^*L$ gives $-p^*L + p^*c_1(E) = p^*R$, that is $c_1(E) = L + R$. Substituting this in $c_1(E) = \frac{D+4L}{2}$ we get $L = R - \frac{D}{2}$ and hence $c_1(E) = 2R - \frac{D}{2}$. □

Proposition 5.4

$$\begin{aligned}
c_1(X) &= 2H - p^*K_B - p^*R \\
c_2(X) &= 2H^2 + H \cdot \Big(- p^*c_1(E) + 2p^*c_1(B)\Big) + p^*R^2 - p^*c_1(E)\cdot p^*R \\
&\quad - p^*c_1(B)\cdot p^*R + p^*c_2(B) + p^*c_2(E) \\
c_3(X) &= H^2 \cdot \Big(2p^*c_1(B) + p^*c_1(E) - 2p^*R\Big) + H \cdot \Big(2p^*c_2(B) - p^*c_1(B)\cdot p^*c_1(E) \\
&\quad - 2p^*R^2 + 3p^*R\cdot p^*c_1(E) - p^*c_1^2(E)\Big) \\
c_4(X) &= H^3 \cdot \Big(- 2p^*c_1(E) + 4p^*R\Big) + H^2\cdot\Big(2p^*c_2(B) + p^*c_1(B)\cdot p^*c_1(E) + 6p^*R^2 \\
&\quad + 3p^*c_1^2(E) - 9p^*R\cdot p^*c_1(E) - 2p^*R\cdot p^*c_1(B)\Big).
\end{aligned}$$

Proof. The following sequence

$$0 \longrightarrow T_X \longrightarrow T_{W|X} \longrightarrow \mathcal{O}_X\Big(2H + p^*\big(R - c_1(E)\big)\Big) \longrightarrow 0$$

along with Proposition 5.2 gives the proof. □

Lemma 5.5

Let Z, Z' be arbitrary divisors on B. Then
i) $H^2 \cdot p^*Z \cdot p^*Z' = 2Z \cdot Z'$
ii) $H^3 \cdot p^*Z = \Big(c_1(E) + R\Big) \cdot Z = \Big(3R - \frac{D}{2}\Big) \cdot Z.$

Proof. i) follows from the fact that the fibres of X over B are quadrics. As for ii) note that $H^3 \cdot p^*Z$ is equal to the intersection product in W

$$H^3 \cdot p^*Z \cdot \Big(2H + p^*R - p^*c_1(E)\Big).$$

Intersecting the Wu-Chern equation with p^*Z we get that

$$H^4 \cdot p^*Z - H^3 \cdot p^*Z \cdot p^*c_1(E) = 0.$$

Hence
$H^3 \cdot p^*Z \cdot \big(2H + p^*R - p^*c_1(E)\big) = 2H^4 \cdot p^*Z + H^3 \cdot p^*Z \cdot p^*R - H^3 \cdot p^*Z \cdot p^*c_1(E) = 2H^3 \cdot p^*Z \cdot p^*c_1(E) + H^3 \cdot p^*R \cdot p^*Z - H^3 \cdot p^*Z \cdot p^*c_1(E) = \big(c_1(E) + R\big) \cdot Z = \big(3R - \frac{D}{2}\big) \cdot Z$. □

Proposition 5.6

The surface $f(B)$ in $Gr(\mathbf{P}^3, \mathbf{P}^7)$ has bidegree $(\delta, c_2(E))$ and it holds

$$\delta = \deg V = \frac{d - R \cdot c_1(E) + c_1^2(E)}{2}, \quad c_2(E) = \frac{-d + R \cdot c_1(E) + c_1^2(E)}{2} \ .$$

Proof. We intersect the Wu-Chern equation in Proposition 5.2 with H and we get

$$\delta - H^4 \cdot c_1(E) + H^3 \cdot c_2(E) = 0.$$

Now cut the equation $X = 2H + p^*R - p^*c_1(E)$ with H^4 and we obtain

$$d = 2\delta + H^3 \cdot p^*R \cdot p^*c_1(E) - H^3 \cdot p^*c_1^2(E).$$

From these two equalities and Lemma 5.5 we get our claim. □

Proposition 5.7

$$\begin{aligned}
c_1(X) &= 2H + p^*c_1(B) - p^*R\\
c_2(X) &= 2H^2 + H\cdot\Big(2p^*c_1(B) - 2p^*R + \frac{1}{2}p^*D\Big) - p^*R^2 + p^*c_2(E) - p^*c_1(B)\cdot p^*R\\
&\qquad + p^*c_2(B) + \frac{1}{2}p^*D\cdot p^*R\\
c_3(X) &= H^2\cdot\Big(-\frac{1}{2}p^*D + 2p^*c_1(B)\Big) + H\cdot\Big(\frac{1}{2}p^*c_1(B)\cdot p^*D - 2p^*c_1(B)\cdot p^*R\\
&\qquad + 2p^*c_2(B) - \frac{1}{4}p^*D^2 + \frac{1}{2}p^*D\cdot p^*R\Big)\\
c_4(X) &= H^3\cdot p^*D + H^2\cdot\Big(-\frac{1}{2}p^*c_1(B)\cdot p^*D + 2p^*c_2(B) + \frac{3}{4}p^*D^2 - \frac{3}{2}p^*D\cdot p^*R\Big).
\end{aligned}$$

Proof. Using Proposition 5.3-Proposition 5.6 and easy computations give the formulas for $c_2(X)$, $c_3(X)$, $c_4(X)$. □

Proposition 5.8. *Set*

$$P(d) = \frac{1}{9d^3 - 50d^2 - 10949d + 169120}, \quad x = K_B^2, \quad y = D\cdot R\ .$$

The following hold:

$$\begin{aligned}
R^2 &= -\frac{1}{2}P(d)\Big(-192864x + 2842d^3 + 332024d - 70224y - 53900d^2 - 15yd^2\\
&\qquad - 49d^4 + 4974yd - 3yd^3 + 9016dx\Big)\\
D^2 &= P(d)\Big(-882d^4 + 54yd^3 + 54684d^3 - 870yd^2 - 1100736d^2 + 190512dx\\
&\qquad - 39792yd + 7112448d - 3035648x + 709632y\Big)\\
c_2(B) &= \frac{1}{16}P(d)\Big(9d^5 - 328d^4 + 144d^3x + 3yd^3 + 3036d^3 - 173yd^2 - 37416d^2\\
&\qquad + 712d^2x + 2608yd - 102048dx + 728896d + 675584x - 8576y\Big)\\
K_B\cdot R &= -\frac{1}{8}P(d)\Big(-175392dx + 1722d^4 + 696696d^2 - 22686yd - 52332d^3\\
&\qquad - 11yd^3 - 21d^5 + 879yd^2 + 3864d^2x + 1983744x - 3415104d + 190784y\Big)\\
K_B\cdot D &= -\frac{1}{4}P(d)\Big(-63d^5 + 5292d^4 - 164556d^3 - 33yd^3 + 2237760d^2 + 13608d^2x\\
&\qquad + 2703yd^2 - 11176704d - 71880yd - 516208dx + 624384y + 4770304x\Big).
\end{aligned}$$

Proof. The idea will be to solve a linear system of five equations in the unknowns R^2, D^2, $c_2(B)$, $K_B \cdot R$, $K_B \cdot D$, with coefficients rational functions of d. In fact from 5.1 we have

$$c_3(X) = (56-d)H^3 - 28H^2 \cdot c_1(X) + 8H \cdot (c_1^2(X) - c_2(X)) - c_1^3(X) + 2c_1(X)c_2(X).$$

Substituting the values of $c_1(X)$, $c_2(X)$, $c_3(X)$ of Proposition 5.7 we get

$$\begin{aligned}(d-16)H^3 &+ \left(-12p^*R + \frac{3p^*D}{2} + 14p^*c_1(B)\right) \cdot H^2 + \Big(- 6p^*c_1(B)^2 - 10p^*R^2 \\ &+ 6p^*c_2(B) + 4p^*c_2(E) - \frac{p^*D^2}{4} + 6p^*c_1(B) \cdot p^*R \\ &- \frac{p^*c_1(B) \cdot p^*D}{2} + \frac{7p^*R \cdot p^*D}{2}\Big) \cdot H = 0 \ .\end{aligned}$$

Cutting respectively with H, p^*R, p^*K_B, p^*D we get four equations. For instance if we cut with p^*D we obtain:

$$(d-16)H^3 \cdot p^*D + H^2 \cdot \left(-12p^*R \cdot p^*(D) + \frac{3}{2}\, p^*D \cdot p^*D + 14p^*c_1(B) \cdot p^*D\right) = 0 \ .$$

Using now lemma 5.3 and lemma 5.5 we get

$$(d-16)\left(3R - \frac{D}{2}\right) \cdot D - 24R \cdot D + 3D \cdot D + 28c_1(B) \cdot D = 0$$

and simplifying

$$-72R \cdot D + 11D^2 + 3dR \cdot D - \frac{dD^2}{2} - 28K_B \cdot D = 0 \ .$$

The fifth equation is gotten by substituting the values of Proposition 5.7 in the second formula of 5.1:

$$\begin{aligned}c_4(X) = 70d - c_1(X)dH^3 - c_2(X)&\Big(28H^2 - 8c_1(X)H + c_1^2(X) - c_2(X)\Big) \\ &- c_3(X)\Big(8H - c_1(X)\Big) \ .\end{aligned}$$

Solving such system we get the claim. □

Proposition 5.9

$$c_2(E) = \frac{1}{4}P(d)\Big((41160d - 360640)\, x + \big(-95d^2 + 5005d - 69440\big)\, y - 165d^4 + 10390d^3 - 205070d^2 + 1225840d\Big)$$
$$g - 1 = \frac{1}{8}P(d)\Big(\big(1008d^2 - 49112d + 566720\big)\, x + \big(223d^2 - 9857d + 109120\big)\, y + 393d^4 - 21032d^3 + 353440d^2 - 1781360d\Big).$$

Proof. Using Proposition 5.6 and Lemma 5.3 we get

$$c_2(E) = \frac{1}{2}\left(6R^2 - \frac{5D \cdot R}{2} + \frac{D^2}{4} - d\right).$$

Moreover from the adjunction formula and Lemma 5.5 ii) we obtain

$$g - 1 = \frac{1}{2}d + \frac{1}{2}H^3(p^*K_B + p^*R) = \frac{1}{2}d + \frac{1}{2}(p^*K_B + p^*R)\Big(3R - \frac{D}{2}\Big)$$
$$= \frac{1}{2}d + \frac{3}{2}K_B \cdot R - \frac{1}{4}K_B \cdot D + \frac{3}{2}R^2 - \frac{1}{4}D \cdot R\,.$$

Substituting the values of Proposition 5.8 and simplifying we get the assertions. □

We need a Roth type result.

Proposition 5.10

Let X be a codimension 3 integral subvariety of $\mathbf{P}^n$ of degree d. If the generic section $C = X \cap \mathbf{P}^4$ with a linear $\mathbf{P}^4$ is contained in a surface $S_\sigma \subset \mathbf{P}^4$ of degree σ with $\sigma^2 \leq d$ then X itself is contained in a codimension 2 subvariety $V_\sigma \subset \mathbf{P}^n$ of degree σ.

Proof. We first check that the generic section $S = X \cap \mathbf{P}^5$ with a linear $\mathbf{P}^5$ is contained in a 3-fold $Y_\sigma \subset \mathbf{P}^5$ of degree σ. By the assumptions, the surface S_σ is unique. On the contrary, suppose there are two such surfaces S'_σ and S''_σ. There exists a linear projection π from a point $p \in \mathbf{P}^4$ on a hyperplane $\mathbf{P}^3$ such that $\pi(S'_\sigma)$ and $\pi(S''_\sigma)$ are two irreducible distinct surfaces of degree σ containing $\pi(C)$ against Bezout theorem. We get a rational map from $(\mathbf{P}^5)^\vee$ in the Hilbert scheme of degree σ surfaces of $\mathbf{P}^4$. It follows that the closure of the union of all surfaces S_σ is the 3-fold Y_σ we looked for.

By iterating this process we get the thesis. □

Proposition 5.11

Let X be a quadric bundle in $\mathbf{P}^7$. Then $d \leq 2963$ or X is contained in a 5-fold of degree 8.

Proof. We consider the possible values of x and y compatible with the following three inequalities:

$D \cdot R \geq 0$, (Lemma 4.15 in [7])
$c_2(E) \geq 0$,
$c_1(E) \cdot D \geq 0$.

Using Lemma 5.3, Proposition 5.8, Proposition 5.9 and easy computations we have that the three above inequalities correspond respectively to the following

$$y \geq 0$$

$$(41160d - 360640)\,x + \left(-95d^2 + 5005d - 69440\right) y - 165d^4 + 10390d^3 - 205070d^2 + 1225840d \geq 0$$

$$-\,(190512d - 3035648)\,x - \left(18d^3 - 670d^2 + 4004d + 33152\right) y + 882d^4 - 54684d^3 + 1100736d^2 - 7112448d \geq 0\,.$$

These inequalities for $d \geq 26$ bound the inside of a triangle whose vertices are:

$$A_d = \left(\frac{d\left(33d^3 - 2078d^2 + 41014d - 245168\right)}{8232d - 72128}, 0\right)$$

$$B_d = \left(\frac{9d\left(d^3 - 62d^2 + 1248d - 8064\right)}{1944d - 30976}, 0\right)$$

$$C_d = \left(\frac{\left(33d^3 - 2192d^2 + 46900d - 316064\right) d}{8232d - 131712}, \frac{2d\,(69d - 872)}{21d - 336}\right).$$

The minimum and the maximum of $g-1$ considered as a function in x and y with d fixed (see Proposition 5.9) have to be attained in one of the vertices. Substituting the coordinates of A_d, B_d, C_d in the expression of $g-1$ we get respectively

$$g - 1 = \frac{\left(33d^2 - 293d - 552\right) d}{588d - 5152}, \frac{\left(63d^2 - 1320d + 5192\right) d}{972d - 15488}, \frac{\left(33d^2 - 407d - 1008\right) d}{588d - 9408}\,.$$

It is a straightforward computation to see that for $d \geq 65$ we have

$$\frac{\left(33d^2 - 293d - 552\right) d}{588d - 5152} \leq g - 1 \leq \frac{\left(63d^2 - 1320d + 5192\right) d}{972d - 15488}\,. \tag{38}$$

We now distinguish two cases. Suppose first that the curve section C is not contained in any surface of degree 8. Then from Theorem 2.1 and (38) we have

$$\frac{\left(33d^2 - 293d - 552\right) d}{588d - 5152} \leq \frac{d^2}{18} + \frac{5}{3}d + \frac{347}{18}$$

that is $3d^3 - 8881d^2 - 29706d + 893872 \leq 0$, which implies $d \leq 2963$.

If C is contained in an octic, then from Proposition 5.10. it follows that also X is contained in an octic. $\square$

Remark 5.12. We remark that a similar reasoning gives an analogous result for manifolds X of dimension $n+1$ and degree d embedded in $\mathbf{P}^{2n+1}$ which is a quadric bundle over a surface. More precisely we have the following

Proposition 5.13

Let X be a manifold of dimension $n+1$ and degree d embedded in $\mathbf{P}^{2n+1}$ which is a quadric bundle over a surface. Then there exists a function $F(n)$ such that $d \leq F(n)$ or X is contained in a variety of dimension $n+2$ and degree $[\frac{n(n+1)(4n^2+4n+1)}{6(4n-1)}]$, where $[x]$ is the greatest integer less than or equal to x.

Proof. Similarly as in proposition 5.11. we get that

$$\frac{3(4n-1)}{n(n+1)(4n^2+4n+1)}d^2 + O(d) \leq g - 1 \leq \frac{3(2n+1)}{(n+1)(2n^3+2n^2+2n+3)}d^2 + O(d) \ .$$

The details are left to the reader. We now use Theorem 2.1 and Proposition 5.10 to see that there exists a function $F(n)$ such that $d \leq F(n)$ or X is contained in a variety of dimension $n+2$ and degree $[\frac{n(n+1)(4n^2+4n+1)}{6(4n-1)}]$. $\square$

References

1. E. Arrondo and I. Sols, On congruences of lines in the projective space, *Mém. Soc. Math. France* **50**, (1992).
2. D. Bayer and D. Mumford, What Can Be Computed in Algebraic Geometry?, Computational Algebraic Geometry and Commutative Algebra (Cortona 1991), *Symposia Math.* **34**, C.U.P. (1993), 1–48.
3. M. Beltrametti and A.J. Sommese, New properties of special varieties arising from adjunction theory, *J. Math. Soc. Japan* **43** (1991), 381–412.

4. M. Beltrametti and A.J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, volume 16 of Expositions in Mathematics, De Gruyter, 1995.
5. G.M. Besana, On the geometry of conic bundles arising in adjunction theory, *Math. Nachr.* **160** (1993), 223–251.
6. R. Braun, G. Ottaviani, M. Schneider and F.O. Schreyer, *Boundedness of nongeneral type 3-folds in* $\mathbf{P}^5$, in Complex Analysis and Geometry, Ed. V. Ancona and A. Silva, 311–338. Plenum Press, New York 1993.
7. R. Braun, G. Ottaviani, M. Schneider and F.O. Schreyer, Classification of Conic Bundles in $\mathbf{P}^5$, *Ann. Sc. Norm. Sup. Pisa* **23**(1) (1996), 69–97.
8. C. Ciliberto, L. Chiantini and V. Di Gennaro, The genus of projective curves, *Duke Math. J.* **70**(2) (1993), 229–245.
9. M.M.A. de Cataldo, A finiteness theorem for low-codimension nonsingular subvarieties of quadrics, *Trans. Amer. Math. Soc.* **349**(6) (1997), 2359–2370.
10. G. Ellingsrud and C. Peskine, Sur les surfaces lisses de $\mathbf{P}^4$, *Invent. Math.* **95** (1989), 1–11.
11. J. Harris, A bound on the geometric genus of projective varieties, *Ann. Sc. Norm. Sup. Pisa*, **8**(4) (1981), 35–68.
12. G. Ottaviani, On 3-folds in $\mathbf{P}^5$ which are scrolls, *Ann. Sc. Norm. Sup. Pisa* **19**(4) (1992), 451–471.
13. M. Schneider, Boundedness of low-codimensional submanifolds of projective space, *Int. J. Math.* **3** (1992), 397–399.