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# K3 surfaces: moduli spaces and Hilbert schemes

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*Dedicated to the memory of Ferran Serrano*

Abstract

Let $X$ be an algebraic $K3$ surface. Fix an ample divisor $H$ on $X$, $L \in Pic(X)$ and $c_2 \in \mathbb{Z}$. Let $M_H(r; L, c_2)$ be the moduli space of rank $r$, $H$-stable vector bundles $E$ over $X$ with $\det(E) = L$ and $c_2(E) = c_2$. The goal of this paper is to determine invariants $(r; c_1, c_2)$ for which $M_H(r; L, c_2)$ is birational to some Hilbert scheme $Hilb^l(X)$.

## 1. Introduction

Let $X$ be an algebraic $K3$ surface defined over the complex number field; i.e, $X$ is an algebraic surface with the trivial canonical line bundle $K_X \simeq O_X$ and the vanishing irregularity $q(X) = 0$.

Fix an ample divisor $H$ on $X$. For a line bundle $L$ on $X$ and an integer $c_2 \in \mathbb{Z}$, let $M_H(r; L, c_2)$ be the moduli space of rank $r$, $H$-stable (in the sense of Mumford-Takemoto) vector bundles $E$ over $X$ with $\det(E) = L$ and $c_2(E) = c_2$. It is well known that for $c_2$ sufficiently large $M_H(r; L, c_2)$ is non-empty and irreducible. Moreover, $M_H(r; L, c_2)$ is smooth and has the expected dimension equal to $-\chi(End_0(E)) = 2rc_2 - (r-1)L^2 - (r^2-1)\chi(O_X) = 2rc_2 - (r-1)L^2 - 2(r^2-1)$.

In 1984, Mukai ([5]) proved that the moduli spaces of simple sheaves over $X$ has a symplectic structure. On the other hand, it is well known that the Hilbert schemes

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$Hilb^l(X)$ of 0-dimensional subschemes of $X$ with length $l$ have also a symplectic structure and it seems natural to look for a closer relation between Hilbert schemes $Hilb^l(X)$ and the moduli spaces $M_H(r; L, c_2)$. In [7], T. Nakashima proposes the following:

**Problem.** To determine for arbitrary $K3$ surfaces $X$, all invariants $(r; L, c_2)$ for which $M_H(r; L, c_2)$ are birational to some $Hilb^l(X)$.

For the rank 2 case, the first contribution to this problem is due to K. Zuo. He proved:

**Theorem** ([12; Theorem 1])

*Suppose $X$ is an algebraic $K3$ surface and $H$ is an ample line bundle on $X$. Let $M_H(2; 0, k(n))$ be the moduli space of $H$-stable rank 2 vector bundles $E$ over $X$ with $\det(E) = 0$, $c_2(E) = k(n) := n^2H^2 + 3$, $n \in \mathbb{N}^+$ and let $Hilb^{2k(n)-3}(X)$ be the Hilbert scheme of 0-dimensional subschemes of $X$ of length $2k(n) - 3$. Then there is a birational map*

$$\phi : M_H\big(2; 0, k(n)\big) \simeq Hilb^{2k(n)-3}(X)\,.$$

Later on T. Nakashima generalized Zuo's Theorem to the triples $(r; L, c_2) = (2; L, k(n))$ where $k(n) := (n^2 + n + \frac{1}{2})L^2 + 3$ and $L$ is an arbitrary ample line bundle ([6]). In the arbitrary rank case almost nothing is known. Very recently, T. Nakashima has proved:

**Theorem** ([7; Theorem 0.2]; see also [10])

*Let $S$ be a $K3$ surface with $(D, H)$ of degree one. If $c = \frac{D^2}{2} + r + 1$ and $c \geq h^0(D) + 1$ then $M_H(r; D, c)$ is birational to the Hilbert scheme $Hilb^c(S)$ of zero dimensional cycles of length $c$.*

We would like to stress that the hypothesis $(D, H)$ being of degree one is very "restrictive". The goal of this paper is to prove the following:

**Theorem A**

*Let $X$ be an algebraic $K3$ surface and $H$ an ample line bundle on $X$. Let $M_H(r; c_1, k(n))$ be the moduli space of $H$-stable rank $r$ vector bundles $E$ over $X$ with $\det(E) = c_1$, $c_2(E) = k(n) := \frac{c_1^2}{2} + \frac{r}{2}n^2H^2 + nc_1H + (r+1)$ and let $Hilb^{l(n)}(X)$ be the Hilbert scheme of 0-dimensional subschemes of $X$ of length $l(n)$. For $n >> 0$ there is a birational map:*

$$\phi : M_H\big(r; c_1, k(n)\big) \longrightarrow Hilb^{l(n)}(X)$$

*where $l(n) := k(n) + \frac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H$.*

Notice that when $r = 2$ we recover the results of K. Zuo and T. Nakashima.

## 2. Generalities

In this section we collect some basic facts needed in the sequel.

Let $X$ be a smooth algebraic surface, $Z \subset X$ a 0-dimensional subscheme of length $l$ and $D \in Pic(X)$. Any $r-1$ linearly independent elements $e_1, \cdots, e_{r-1} \in Ext^1(I_Z(D), O_X)$ define an exact sequence:

$$0 \longrightarrow O_X^{r-1} \longrightarrow E \longrightarrow I_Z(D) \longrightarrow 0$$

where $E$ is a rank $r$ torsion free sheaf on $X$ with Chern classes $c_1(E) = D$ and $c_2(E) = l$.

Definition 2.1. Let $H$ be an ample divisor on a smooth algebraic surface $X$. For a torsion free sheaf $F$ on $X$ one sets

$$\mu_H(F) := \frac{c_1(F)H}{rk(F)}, \qquad P_F(m) := \frac{\chi(F \otimes O_X(mH))}{rk(F)}.$$

The sheaf $F$ is $H$-semistable (resp. $G$-semistable with respect to $H$) if

$$\mu_H(E) \leq \mu_H(F) \quad \big(P_E(m) \leq P_F(m) \quad \text{for} \quad m >> 0\big)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then $F$ is $H$-stable (resp. $G$-stable with respect to $H$).

One easily checks the implications:

$$H - stable \Rightarrow G - stable \Rightarrow G - semistable \Rightarrow H - semistable.$$

Let us recall the formulas for the Chern classes and the Euler-Poincaré characteristic for vector bundles on non-singular projective surfaces with canonical line bundle $K = K_X$.

**2.2.** Let $E$ be a rank r vector bundle on a non-singular projective variety of dimension n and let $L$ be a line bundle on $X$. Then,

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) c_1(L)^{k-i}.$$

**2.3.** Let $E$ be a rank r vector bundle on a non-singular projective surface. Let $c_1$ and $c_2$ be the Chern classes of $E$. Then,

$$\chi(E) = \sum_{i=0}^{2} (-1)^i \dim H^i(X, E) = r\big(1 + p_a(X)\big) + c_1(-K/2) + \big(c_1^2 - 2c_2\big)/2.$$

Given a line bundle $L$ on $X$, an integer $c_2$ and an ample line bundle $H$ on $X$, we will denote by $\overline{M}_H(r;L;c_2)$ the moduli space of rank $r$, torsion free sheaves $F$ on $X$, $G$-semistable with respect to $H$ with $c_1(F)=L$ and $c_2(F)=c_2$; and by $M_H(r;L;c_2)\subset\overline{M}_H(r;L;c_2)$ the open subset parameterizing rank $r$, $H$-stable vector bundles $F$ over $X$ with $c_1(F)=L$ and $c_2(F)=c_2$.

We will end this section reviewing a well known result on moduli spaces of rank $r$ torsion free sheaves on smooth algebraic surfaces that we will use later on.

**Theorem 2.4**

*Let $X$ be a smooth algebraic surface, $L$ an ample divisor on $X$ and $c_1\in Pic(X)$. For all $c_2>>0$, the moduli space $\overline{M}_L(r;c_1,c_2)$ of $G$-semistable with respect to $L$, rank $r$ torsion free sheaves on $X$ (resp. $M_L(r;c_1,c_2)$ of $L$-stable, rank $r$ vector bundles on $X$), is a generically smooth, irreducible projective (resp. quasi-projective) variety of the expected dimension $2rc_2-(r-1)c_1^2-(r^2-1)\chi(O_X)$.*

*Proof.* See [2], [8] and [9]. □

## 3. Main Construction

From now on, $X$ is assumed to be an algebraic $K3$-surface defined over the complex number field; i.e., $X$ is an algebraic surface with the trivial canonical line bundle $K_X\simeq O_X$ and the vanishing irregularity $q(X)=0$.

Let us fix a line bundle $c_1$ and an ample divisor $H$ on $X$. Let $n_0$ be an integer such that for all $n\geq n_0$, $c_1+rnH$ is ample. Set:

$$k(n):=\frac{c_1^2}{2}+\frac{r}{2}n^2H^2+nc_1H+(r+1)\,;$$
$$l(n):=k(n)+\frac{r(r-1)}{2}n^2H^2+(r-1)nc_1H.$$

**Construction.** Let $\mathcal{F}$ be the irreducible family of rank $r$ torsion free sheaves $F$ on $X$, $G$-semistable with respect to $H$ with Chern classes $(c_1,k(n))$ given by a non-trivial extension

$$0\longrightarrow O_X^{r-1}\longrightarrow F(nH)\longrightarrow I_Z(c_1+rnH)\longrightarrow 0$$

where $Z$ is a 0-dimensional subscheme of $X$ of length $|Z|=c_2(F(nH))=c_2(F)+(r-1)nc_1(F)H+\frac{r(r-1)}{2}n^2H^2=k(n)+(r-1)nc_1H+\frac{r(r-1)}{2}n^2H^2=l(n)$ such that $h^0(I_Z(c_1+rnH))=0$.

**Claim:** For $n>>0$, $\mathcal{F}$ is non-empty.

*Proof of the claim.* We fix $c_2' \in \mathbb{Z}$ such that $M_H(r;c_1,c_2') \neq \emptyset$ ([11]). It is well known that there exists an integer $n_{c_2'} \in \mathbb{Z}$ such that for all $n \geq n_{c_2'}$ and for any $E \in M_H(r;c_1,c_2')$, $E(nH)$ is generated by its global sections and $\chi(E(nH)) \geq r-1$. We choose $(r-1)$ generic sections of $E(nH)$ and we get an exact sequence:

$$0 \longrightarrow O_X^{r-1} \longrightarrow E(nH) \longrightarrow I_{\tilde{Z}}(c_1+rnH) \longrightarrow 0$$

where $\tilde{Z}$ is a 0-dimensional subscheme of $X$ of length $|\tilde{Z}| = c_2(E(nH)) = c_2' + \frac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H$.

Moreover, there exists an integer $l_{c_2'} \in \mathbb{Z}$ such that for all $l \geq l_{c_2'}$, if we choose appropriately $l$ generic points $p_1, \cdots, p_l$ and a surjective map:

$$\alpha : E \longrightarrow \oplus_{j=1}^{l} \mathbb{C}_{p_j} ,$$

then $F$, the kernel of $\alpha$, is a rank $r$, torsion free sheaf, $G$-semistable with respect to $H$ sitting into an exact sequence:

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1+rnH) \longrightarrow 0$$

where $Z = \tilde{Z} \cup \{p_1, \cdots, p_l\}$. (See [9] for more details).

For $n >> 0$ we can assume $k(n) - c_2' \geq l_{c_2'}$, and $n \geq \max\{n_{c_2'}, n_0\}$. Define $l := k(n) - c_2' \geq l_{c_2'}$. As we have seen, there exists an exact sequence:

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1+rnH) \longrightarrow 0$$

where $Z$ is a 0 dimensional subscheme of $X$ of length

$$\begin{aligned} |Z| = |\tilde{Z}| + l = & \left(c_2' + \frac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H\right) + l \\ = & \ k(n) + \frac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H \end{aligned}$$

and $F$ is a rank $r$, torsion free sheaf, $G$-semistable with respect to $H$ with Chern classes $c_1(F) = c_1$ and $c_2(F) = k(n)$.

Since $c_1+rnH$ is ample, by Kodaira's vanishing Theorem $h^i(O_X(c_1+rnH)) = 0$ for $i > 0$; and applying Riemann-Roch's Theorem we get:

$$h^0\big(O_X(c_1+rnH)\big) = \chi\big(O_X(c_1+rnH)\big) = \frac{c_1^2}{2} + \frac{r^2}{2}n^2H^2 + rnc_1H + 2\,.$$

On the other hand,

$$\begin{aligned} |Z| = & \ k(n) + \frac{r(r-1)}{2} n^2 H^2 + (r-1) n c_1 H \\ = & \ \frac{c_1^2}{2} + \frac{r^2}{2} n^2 H^2 + r n c_1 H + (r+1)\,. \end{aligned}$$

Therefore, since $0 < r - 1$,

$$h^0\big(O_X(c_1 + rnH)\big) - |Z| = -(r-1) < 0 \tag{1}$$

and hence for $l >> 0$ and $l$ generic points,

$$h^0\big(I_Z(c_1 + rnH)\big) = 0\,.$$

Putting altogether we get $F \in \mathcal{F}$, which proves our claim.

**Lemma 3.1**

*With the above notation*

$$\dim \mathcal{F} = 2l(n)$$

*Proof.* By definition,

$$\begin{aligned} \dim \mathcal{F} = & \ 2|Z| + \dim Gr\big(r-1, Ext^1(I_Z(c_1 + rnH), O_X)\big) \\ & - \ \dim Gr\big(r-1, H^0(F(nH))\big) \end{aligned}$$

where $Gr(s, V)$ is the Grassmanian variety of $s$-dimensional subspaces of $V$ and $\dim Gr(s, V) = s \cdot \dim V - s^2$.

Consider the exact cohomology sequence:

$$0 \longrightarrow H^0 O_X^{r-1} \longrightarrow H^0 F(nH) \longrightarrow H^0 I_Z(c_1 + rnH) \longrightarrow \cdots$$

associated to the exact sequence:

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1 + rnH) \longrightarrow 0\,.$$

Since $h^0 I_Z(c_1 + rnH) = 0$, we obtain:

$$h^0 F(nH) = h^0 O_X^{r-1} = r - 1\,.$$

On the other hand, the exact cohomology sequence:

$$0 \longrightarrow H^0 I_Z(c_1 + rnH) \longrightarrow H^0 O_X(c_1 + rnH) \longrightarrow H^0 O_Z(c_1 + rnH) \longrightarrow$$

$$\longrightarrow H^1 I_Z(c_1 + rnH) \longrightarrow H^1 O_X(c_1 + rnH) \longrightarrow \cdots$$

associated to the exact sequence:

$$0 \longrightarrow I_Z(c_1 + rnH) \longrightarrow O_X(c_1 + rnH) \longrightarrow O_Z(c_1 + rnH) \longrightarrow 0\,,$$

together with the fact that $c_1 + rnH$ is ample and hence $h^i O_X(c_1 + rnH) = 0$ for $i > 0$, gives us:

$$\dim Ext^1\big(I_Z(c_1 + rnH)\big), O_X) = h^1 I_Z(c_1 + rnH) = |Z| - h^0 O_X(c_1 + rnH) = r - 1$$

where the last equality follows from (1). Putting altogether we conclude:

$$\begin{aligned}\dim \mathcal{F} = \;& 2l(n) + (r-1)\dim Ext^1\big(I_Z(c_1 + rnH), O_X\big) - (r-1)^2 \\ & - \big((r-1)h^0(F(nH)\big) - (r-1)^2) = 2l(n)\end{aligned}$$

which proves the lemma. $\square$

*Remark 3.2.* It follows from the definition of $l(n)$, $k(n)$ and Lemma 3.1 that for $n >> 0$,

$$\begin{aligned}\dim \mathcal{F} = \;& \dim Hilb^{l(n)}(X) = 2l(n) = 2rk(n) - (r-1)c_1^2 - 2(r^2 - 1) \\ = \;& \dim \overline{M}_H\big(r; c_1, k(n)\big).\end{aligned}$$

## 4. The birational correspondence to the Hilbert Scheme

The goal of this section is to prove **Theorem A**. We keep the notation introduced in section 3.

**Theorem 4.1**

*Any torsion free sheaf $F \in \mathcal{F}$ is simple.*

*Proof.* Applying the functor $\mathrm{Hom}(F(nH),\cdot)$ to the exact sequence:

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1 + rnH) \longrightarrow 0 \tag{2}$$

we get the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}\big(F(nH), O_X^{r-1}\big) &\longrightarrow \mathrm{Hom}\big(F(nH), F(nH)\big) \\ &\longrightarrow \mathrm{Hom}\big(F(nH), I_Z(c_1 + rnH)\big) \longrightarrow \cdots. \end{aligned}$$

Since $n >> 0$, by Serre's duality we have:

$$\dim \mathrm{Hom}(F(nH), O_X^{r-1}) = (r-1)h^2 F(nH) = 0\,.$$

Therefore, it is sufficient to see that $\dim \mathrm{Hom}(F(nH), I_Z(c_1 + rnH)) = 1$. To this end, we consider the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(I_Z(c_1 + rnH), I_Z(c_1 + rnH)) &\longrightarrow \mathrm{Hom}\big(F(nH), I_Z(c_1 + rnH)\big) \\ &\longrightarrow \mathrm{Hom}\big(O_X^{r-1}, I_Z(c_1 + rnH)\big) \longrightarrow \cdots \end{aligned}$$

obtained applying the functor $\mathrm{Hom}(\cdot, I_Z(c_1+rnH))$ to the exact sequence (2). Since $F \in \mathcal{F}$, $h^0 I_Z(c_1 + rnH) = 0$ and we get:

$$\dim \mathrm{Hom}\big(F(nH), I_Z(c_1 + rnH)\big) = \dim \mathrm{Hom}\big(I_Z(c_1 + rnH), I_Z(c_1 + rnH)\big) = 1$$

which proves the Lemma. □

For $n >> 0$, we have two natural rational morphisms:

$$\begin{aligned} \pi : \mathcal{F} &\longrightarrow Hilb^{l(n)}(X)\,; \\ e : \mathcal{F} &\longrightarrow \overline{M}_H\big(r; c_1, k(n)\big)\,. \end{aligned}$$

The fiber $\pi^{-1}(Z)$ over $Z \in Hilb^{l(n)}(X)$ is identified with a non-empty open subset of the Grassmanian variety

$$Gr\big(r-1, Ext^1(I_Z(c_1 + rnH), O_X)\big)$$

and the fiber $e^{-1}(F)$ over $F \in \overline{M}_H(r; c_1, k(n))$ is canonically isomorphic to a non empty Zariski open subset of

$$Gr\big(r-1, H^0(F(nH))\big)\,.$$

Notice that for the dimension computations of section 3, we have:

$$\dim(\pi^{-1}(Z)) = \dim(e^{-1}(F)) = 0$$

for all generic $Z \in Hilb^{l(n)}(X)$ and for all generic $F \in \overline{M}_H(r; c_1, k(n))$ respectively.

Let us see that $e$ is an injection. Assume that there are two non trivial extensions:

$$0 \longrightarrow O_X^{r-1} \xrightarrow{\alpha_1} F(nH) \xrightarrow{\alpha_2} I_Z(c_1 + rnH) \longrightarrow 0\,;$$
$$0 \longrightarrow O_X^{r-1} \xrightarrow{\beta_1} F(nH) \xrightarrow{\beta_2} I_{Z'}(c_1 + rnH) \longrightarrow 0$$

where $Z$ and $Z'$ are 0-dimensional subschemes of $X$ of length $l(n)$.

From the fact that $h^0 I_Z(c_1 + rnH) = h^0 I_{Z'}(c_1 + rnH) = 0$ we get:

$$\dim \operatorname{Hom}\big(O_X^{r-1}, I_Z(c_1 + rnH)\big) = \dim \operatorname{Hom}\big(O_X^{r-1}, I_{Z'}(c_1 + rnH)\big) = 0\,.$$

Thus, $\beta_2 \circ \alpha_1 = \alpha_2 \circ \beta_1 = 0$. So, there exists $\gamma \in Aut(F(nH)) \simeq \mathbb{C}$ (Lemma 4.1) such that $\beta_2 = \gamma \circ \alpha_2$. Therefore, $Z \simeq Z'$ and hence, $e$ is an injection.

Since $h^0 F(nH) = r - 1$, $\pi$ is also an injection and by Remark 3.2

$$\dim \mathcal{F} = \dim Hilb^{l(n)}(X) = \dim \overline{M}_H\big(r; c_1, k(n)\big)\,.$$

Hence, $e$ and $\pi$ are birational maps. Composing, we get a birational map:

$$e\pi^{-1} = \psi : \overline{M}_H\big(r; c_1, k(n)\big) \longrightarrow Hilb^{l(n)}(X)\,.$$

Moreover, since $M_H(r; c_1, k(n))$ is an open dense subset of $\overline{M}_H(r; c_1, k(n))$, restricting $\psi$ to $M_H(r; c_1, k(n))$ we obtain the birational morphism claimed in the Theorem A.

*Remark 4.2.* The pullback of the symplectic structure on $Hilb^{l(n)}(X)$ via the birational map $\phi$ of Theorem A, gives a symplectic structure on $M_H(r; c_1, k(n))$. This symplectic structure coincides with the symplectic structure given by Mukai ([5]).

*Remark 4.3.* In ([4]) we describe explicitly the birational map $\phi\colon M_H(r; c_1, k(n)) \longrightarrow Hilb^{l(n)}(X)$ and, as application, we check that the Hodge numbers of the moduli space $\overline{M}_H(r; c_1, k(n))$ and the Hilbert scheme $Hilb^{l(n)}(X)$ coincide. Furthermore, since the Hodge numbers of $Hilb^{l(n)}(X)$ can be expressed in terms of the Hodge numbers $h^{p,q}(X)$ of $X$ (see [3]; [1]), we deduce that the Hodge numbers of $\overline{M}_H(r; c_1, k(n))$ can be computed in terms of $h^{p,q}(X)$.

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