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Filtrations by complete ideals and applications

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To the memory of Ferran Serrano

Abstract

Infinitely near base points and Enriques' unloading procedure are used to construct filtrations by complete ideals of $\mathbb{C}\{x,y\}$. It follows a procedure for getting generators of the integral closure of an ideal.

Introduction

A geometric theory of infinitesimal base conditions (or virtual multiplicities) for plane curves was build by F. Enriques at the beginning of this century ([7], L. IV, chap. II). Its main results are a characterization of the infinitesimal base conditions that can be sharply satisfied by a curve (consistent conditions), and a procedure (unloading) giving the behavior of generic curves subjected to a non-consistent family of base conditions.

About twenty years later, O. Zariski developed an algebraic and (in his own words) parallel theory dealing with complete (i.e., integrally closed) ideals of two-dimensional local rings ([12] and also [13], II, app. 5) whose main result is a theorem of unique factorization of these ideals as a product of irreducible ones. Zariski's theory was partially extended to more general rings, mainly rings of rational surface singularities or regular rings of higher dimension by the work of Lipman ([9]), Cutkosky ([5], [6]) and Campillo, González-Sprinberg and Lejeune-Jalabert ([1]),

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while Enriques' theory seems to have been forgotten for many years. Updated versions of Enriques' theory, providing modern proofs, can be found in [3] and [4], its relation with Zariski's theory is explained in [8] and its extension to curves on a rational surface singularity has been obtained by Reguera ([11]).

Our main purpose in this note is to show how to use Enriques' theory, and in particular unloading, to get filtrations between pairs of embodied complete ideals. As an application we describe a procedure giving a system of generators of the integral closure of a zero-dimensional ideal of series in two variables.

We will use the results and keep into the hypothesis and conventions of the first half (Sections 0 to 6) of [3]. In particular the base field will be the complex one and we will deal with convergent power series. The formal and algebraic case can be dealt with similarly. We fix a point O on a smooth analytic surface S, we denote by $\mathcal{O}$ the local ring of S at O ($\mathcal{O} = \mathbb{C}\{x, y\}$ if x, y are local coordinates at O) and by m its maximal ideal. As in [4], we call *clusters* the non-empty finite sets of points equal or infinitely near to O such that with each point they contain all the precedent (by the ordering of the blow-ups) ones. A pair $\mathcal{K} = (\mathcal{K}, \nu)$, where K is a cluster and $\nu : K \longrightarrow \mathbb{Z}$ an arbitrary map, will be called a *weighted cluster*. Weighted clusters were called just clusters in [3]. A weighted cluster $\mathcal{K} = (\mathcal{K}, \nu)$ is called consistent if and only if it verifies the proximity inequalities: for any $p \in K$, the virtual multiplicity $\nu(p)$ of p is non-less than the sum of the virtual multiplicities of the points proximate to p in K.

1. A filtration

If $\mathcal{K} = (K, \nu)$ is a weighted cluster we denote by $H_{\mathcal{K}}$ the corresponding ideal of $\mathcal{O}$: the non-zero elements of $H_{\mathcal{K}}$ are all equations of all germs of curve at O going through $\mathcal{K}$. It is well known, and it is in fact the main link between Enriques' and Zariski's theories, that $H_{\mathcal{K}}$ is a complete ideal and all complete zero-dimensional ideals of $\mathcal{O}$ are of this form for a suitable $\mathcal{K}$ (see [12]).

Assume there are given two consistent weighted clusters $\mathcal{K} = (K, \nu)$ and $\mathcal{K}' = (K', \nu')$ and write $H = H_{\mathcal{K}}$, $H' = H_{\mathcal{K}'}$ for the corresponding ideals. Assume that $H \supset H'$, $H \neq H'$: we will show how to get finitely many consistent weighted clusters $\mathcal{K}_i = (K_i, \nu_i)$, $i = 0, \ldots, n$ so that $\mathcal{K}_0 = \mathcal{K}$, $\mathcal{K}_n = \mathcal{K}'$,

$$H_{\mathcal{K}_0} \supset H_{\mathcal{K}_1} \supset \ldots \supset H_{\mathcal{K}_n}$$

and $\dim H_{\mathcal{K}_{i-1}}/H_{\mathcal{K}_i} = 1$ for $i = 1, \ldots, n$. In the sequel a sequence of weighted clusters such as $\{\mathcal{K}_i\}_{i=0,\ldots,n}$ above will be called a *flag* of clusters with ends $\mathcal{K}_0$ and $\mathcal{K}_n$.

By adding points with virtual multiplicity zero to both clusters we may assume that $K = K'$. Since we assume $\mathcal{K} \neq \mathcal{K}'$ there is $p \in K$ so that $\nu(p) \neq \nu'(p)$. Assume we choose such p so that $\nu(q) = \nu'(q)$ for all q preceding p and fix any germ of curve ξ going sharply through $\mathcal{K}'$, this is with $e_q(\xi) = \nu'(q)$ for $q \in K'$ and no other singular points ([3], 3.6): since ξ goes through $\mathcal{K}$ too, necessarily $\nu(p) < \nu'(p)$. Then we choose any point p_1, not already in K and in the first neighborhood of p, and define a weighted cluster $\mathcal{Q}_1$ by taking all points in $\mathcal{K}$ with their own virtual multiplicities and furthermore the point p_1 taken with the virtual multiplicity one.

First of all, notice that $H_{\mathcal{Q}_1} \supset H_{\mathcal{K}'}$. Indeed, $\mathcal{K}'$ being consistent, it is enough to see that all germs going sharply through $\mathcal{K}'$ are going through $\mathcal{Q}_1$ and this is clear because any such germ ξ has effective multiplicity at p strictly bigger than $\nu(p)$, the virtual multiplicity of p in $\mathcal{Q}_1$, so that its virtual transform contains the exceptional divisor of blowing-up p and hence has multiplicity at least one at p_1.

On the other hand it is clear from our construction that $H_{\mathcal{K}} \supset H_{\mathcal{Q}_1}$ and even that $\dim H_{\mathcal{K}}/H_{\mathcal{Q}_1} \leq 1$, as a further point counted once adds a single linear equation to those defining $H_{\mathcal{K}}$ as a linear subspace of $\mathcal{O}$. Furthermore we have $H_{\mathcal{K}} \neq H_{\mathcal{Q}_1}$ since $\mathcal{K}$ is consistent and so there are germs of curve going sharply through $\mathcal{K}$ and going not through the point p_1. Thus $\dim H_{\mathcal{K}}/H_{\mathcal{Q}_1} = 1$

Lastly, since $\mathcal{Q}_1$ may be non-consistent, we perform successive unloadings from it, till getting an equivalent consistent weighted cluster we call $\mathcal{K}_1$ (updated versions of Enriques' unloading procedure may be found in [3], Section 4 and 5 and [4], Section 4.5). Since $H_{\mathcal{Q}_1} = H_{\mathcal{K}_1}$ all our conditions on $\mathcal{K}_1$ are fulfilled. If still $\mathcal{K}_1 \neq \mathcal{K}'$ we repeat the former procedure from $\mathcal{K}_1$ to get $\mathcal{K}_2$, and so on till getting $\mathcal{K}'$ after $\dim H_{\mathcal{K}}/H_{\mathcal{K}'}$ steps.

The reader may notice that in case of just a single consistent weighted cluster $\mathcal{K}$ being given, a similar and even easier procedure gives rise to an infinite sequence of consistent weighted clusters $\mathcal{K}_i$, $i \geq 0$ so that $\mathcal{K}_0 = \mathcal{K}$, $H_{\mathcal{K}_{i-1}} \supset H_{\mathcal{K}_i}$ and $\dim H_{\mathcal{K}_{i-1}}/H_{\mathcal{K}_i} = 1$ for $i > 0$.

2. Base points and integral closure

Let I be a zero-dimensional ideal of $\mathcal{O}$. The *cluster of base points of* I is a consistent weighted cluster $\mathcal{K}(I)$ defined in the following way. Call Γ the family of germs defined by the non-zero elements of I. Start by taking the point O with virtual multiplicity $\nu(O)$ equal to the minimal multiplicity at O of the germs in Γ. Then discard from Γ the germs with multiplicity bigger than $\nu(O)$ and call Γ_1 the family of the remaining ones. If these germs do not share any point in the first neighborhood of O, then our cluster is just O with virtual multiplicity $\nu(O)$. Otherwise we take all points the germs in Γ_1 share in the first neighborhood, each point p with virtual multiplicity equal to the minimum of the multiplicities at p of the germs in Γ_1. Again discard the germs whose multiplicities are not the minimal ones and look for the points the remaining germs share in the first neighborhoods of the former ones, and so on. Notice that at each step we are discarding germs whose equations are in a union of finitely many ideals strictly contained in I and therefore we keep germs enough to generate I with their equations. The procedure clearly ends after finitely many steps. Otherwise, because of Noether's formula ([10]) we would be able to find systems of generators of I any two of the germs they define having intersection multiplicity bigger than any preassigned number. This implies that any two elements of I share a common factor against the hypothesis of I to be zero-dimensional.

Then it is clear from the former construction that there is a system of generators of I so that the germs they define are going through $\mathcal{K}(I)$ with effective multiplicities equal to the virtual ones and share no point other than those in $\mathcal{K}$. $\mathcal{K}(I)$ is therefore consistent and furthermore we have:

Lemma 2.1

The integral closure of I is $H_{\mathcal{K}(I)}$.

Proof. We know the ideals $H_{\mathcal{K}}$, $\mathcal{K}$ any consistent cluster, to be all the integrally closed zero-dimensional ideals. So assume that such an ideal contains I: then all germs defined by non-zero elements in I, and in particular those going through $\mathcal{K}(I)$ with effective multiplicities equal to the virtual ones, go through $\mathcal{K}$. Since these germs share no point but those in $\mathcal{K}(I)$, any other germ going through $\mathcal{K}(I)$ with effective multiplicities equal to the virtual ones must go through $\mathcal{K}$ too: since $H_{\mathcal{K}(I)}$ may be generated by equations of germs going through $\mathcal{K}(I)$ with effective multiplicities equal to the virtual ones, it follows that $H_{\mathcal{K}(I)} \subset H_K$ and hence the claim. $\square$

3. Generators of the integral closure

Let I be, as before, a m-primary ideal of $\mathcal{O}$ and write $\bar{I} = H_{\mathcal{K}(I)}$ for its integral closure. As it is clear from Nakayama's lemma, elements $f_1, \dots f_n \in \bar{I}$ are a minimal system of generators of $\bar{I}$ if and only if their classes modulo $m\bar{I}$ are a basis of $\bar{I}/m\bar{I}$ as a $\mathbb{C}$-vector space. Write $\mathcal{K} = (K, \nu) = \mathcal{K}(I)$ and denote by $\mathcal{K}'$ the weighted cluster obtained from $\mathcal{K}$ just by increasing the multiplicity of O by one: $\mathcal{K}' = (K, \nu')$,

$\nu'(O) = \nu(O) + 1$ and $\nu'(p) = \nu(p)$ if $p \neq O$. Since $\mathcal{K}$ is consistent, $\mathcal{K}'$ is consistent too and it is easy to check that $H_{\mathcal{K}'} = m\bar{I}$. Indeed, clearly $\mathcal{K}'$ is the weighted cluster of base points of $m\bar{I}$, while $m\bar{I}$ is integrally closed as both m and $\bar{I}$ are so.

As explained in section 1 above, one may construct a flag of clusters $\{\mathcal{K}_i\}$ with ends $\mathcal{K}_0 = \mathcal{K}$, $\mathcal{K}_n = \mathcal{K}'$. Then

$$\bar{I} = H_{\mathcal{K}_0} \supset H_{\mathcal{K}_1} \supset \ldots \supset H_{\mathcal{K}_n} = m\bar{I}$$

and $\dim H_{\mathcal{K}_{i-1}}/H_{\mathcal{K}_i} = 1$ for $i = 1, \ldots, n$. Thus, in order to get a minimal system of generators of $\bar{I}$, it is enough to pick, for $i = 1, \ldots, n$, an equation f_i of a germ going through $\mathcal{K}_{i-1}$ but going not through $\mathcal{K}_i$: a suitable germ going through $\mathcal{K}_{i-1}$ with effective multiplicities equal to the virtual ones will do the job and the branches of such a germ are easily determined from $\mathcal{K}_{i-1}$. By the way note that, as already well known ([9]), $n = \nu(O) + 1$: it is enough to compute the codimensons of both $\bar{I}$ and $m\bar{I}$ from $\mathcal{K}$ and $\mathcal{K}'$ using the classical formula ([7], L. IV, Ch. II, 17, or also [3, 6.1]).

Let us illustrate the procedure for getting generators of the integral closure by means of an example. Take $I = (x^5y - y^3, x^8 + 2x^5y)$: its cluster of base points $\mathcal{K}$ consists of the origin O and the points p_1, p_2 in its first and second neighborhoods on the x-axis, with respective virtual multiplicities 3, 3 and 2, plus the point p_3 in the first neighborhood of p_2 and proximate to p_1 with virtual multiplicity 1. Enriques diagrams of the clusters $\mathcal{K}$, $\mathcal{K}'$ as well as those of the intermediate weighted clusters giving rise to the filtration described above are shown in next figure (Enriques diagrams are explained in [7], Book 4, ch. 1, and also in [2], Section 3 and [4], Section 3.9). It easily turns out that by taking $f_1 = x^5y - y^3$, $f_2 = x(y^2 - x^5)(y - x^2)$, $f_3 = xy(y - x^2)^2$, and $f_4 = xy^2(y - x^2)$, the germ of $f_i = 0$ goes through $\mathcal{K}_{i-1}$ but not through $\mathcal{K}_i$. Hence f_1, f_2, f_3, f_4 generate the integral closure of I.

4. Gaps for non-complete ideals

Fix an m-primary ideal I and let $\mathcal{K} = (K, \nu)$ be its weighted cluster of base points and $\bar{I} = H_{\mathcal{K}}$ its integral closure. There are of course complete ideals contained in mI, namely suitable powers of the maximal ideal m. Anyway, Noether's $Af + B\varphi$ theorem (Northcott [10]) allows us to determine such an ideal from the cluster of base points of I: just take a new weighted cluster $\mathcal{K}' = (K', \nu')$ which has the same set of points as $\mathcal{K}$, $K' = K$, and multiplicities $\nu'(p) = 2\nu(p) - 1$ for $p \neq O$ and $\nu'(O) = 2\nu(O) + 1$. Then the weighted cluster of base points of mI, say $\mathcal{J}$ has the same points as $\mathcal{K}$ and also the same multiplicities, but for the point O whose virtual

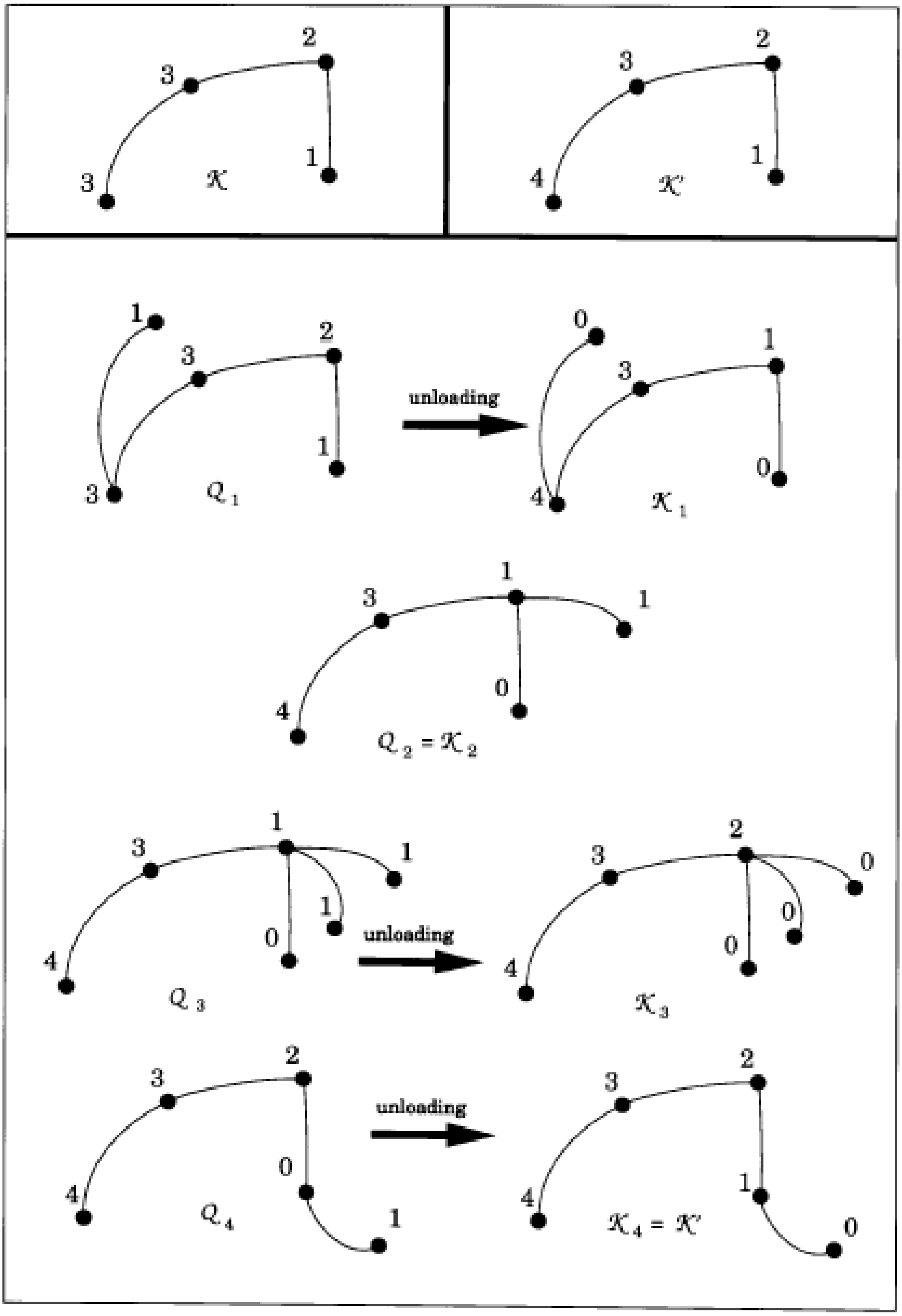

Figure. Getting generators for the integral closure of $I = (x^5y - y^3, x^8 + 2x^5y)$.

multiplicity is increased by one. Then if one takes any two elements $g_1, g_2 \in mI$ so that both germs $g_1 = 0$, $g_2 = 0$ go through $\mathcal{J}$ with effective multiplicities equal to the virtual ones and share no points outside of $\mathcal{J}$, Noether's $Af + B\varphi$ just guarantees that all germs going through $\mathcal{K}'$ have its equations in (g_1, g_2) and so, in particular, $H' = H_{\mathcal{K}'} \subset mI$ as wanted.

Now, let us fix a flag of clusters $\{\mathcal{K}_i\}_{i=0,\dots,n}$ with ends $\mathcal{K}$ and $\mathcal{K}'$. Write $H_i = H_{\mathcal{K}_i}$, $i = 0, \dots, n$. We have:

$$H = H_0 \supset H_1 \supset \ldots \supset H_n = H'$$

and $\dim H_{i-1}/H_i = 1$ for $i = 1, \dots, n$. By taking the traces on I, $I_i = H_i \cap I$ we get a sequence of embodied ideals

$$I = I_0 \supset I_1 \supset \ldots \supset I_n = H'$$

and $\dim I_{i-1}/I_i \leq 1$ for $i = 1, \dots, n$. Once the flag of clusters is fixed, we say that $\mathcal{K}_i$ is a *gap* for the ideal I if and only if $I_i = I_{i+1}$.

Notice that the flag of clusters may be chosen at once for all ideals I with the same cluster of base points (or the same integral closure): the gaps describe, in a certain sense, how far is the ideal I from its integral closure. One may view the clusters $\mathcal{K}_i$ as a sequence of singularities, each a first order specialization of the former one, the first of them being that of generic elements in the linear system of germs defined by I. Then the gaps are the singularities in the sequence that cannot be realized by an element of I.

As it is clear the number of gaps of I, say $\alpha(I)$, does not depend on the flag of clusters but only on the ideal I itself, as it is related to the codimension of I by the formula

$$\alpha(I) = \dim \mathcal{O}/I - \dim \mathcal{O}/\bar{I} = \dim \mathcal{O}/I - \sum_{p \in K} \frac{\nu(p)(\nu(p)+1)}{2}.$$

Obviously, in particular the ideal is integrally closed if and only if it has no gaps.

If for each index i corresponding to a non-gap $\mathcal{K}_i$ we take $f_i \in I_i - I_{i+1}$, $\{f_i\}$ is a system of generators of I (their classes generate I/H' and hence also I/mI) that in most senses behaves like an standard basis for the ideal I. Indeed, one may choose a second flag of clusters, $\mathcal{K}_{-\delta}, \dots, \mathcal{K}_0$ with ends the empty cluster and $\mathcal{K} = \mathcal{K}_0$, giving thus rise to a filtration by complete ideals

$$\mathcal{O} = H_{-\delta} \supset \ldots \supset H_0 = \bar{I},$$

and then elements $g_i \in H_i - H_{i+1}$ for $i = -\delta, \dots, -1$ and also for $i = 0, \dots, n$ and $\mathcal{K}_i$ a gap of I. The reader may easily see that any given element $g \in \mathcal{O}$ may be uniquely written in the form $g = \sum \lambda_i g_i + f$, $\lambda_i \in \mathbb{C}$ and $f \in I$.

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