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A look into the Severi varieties of curves in higher codimension

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To the memory of Fernando Serrano

Abstract

We show some non-emptiness results for the Severi varieties of nodal curves with fixed geometric genus in $\mathbf{P}^n$, $n > 2$. For $n = 3$, we also fix a vector bundle E of rank 2 and look at the variety $V_\delta(E)$ parameterizing sections of E whose 0-locus is nodal, with fixed geometric genus. We establish some basic facts about $V_\delta(E)$ and prove some (almost sharp) non-obstructedness results for these varieties.

Introduction

The study of spaces parameterizing singular curves with fixed geometric genus in a projective variety X (usually called the "Severi varieties" of X) actually receives a lot of attention, partially due to its connections with other fields of geometry and physics. The case in which X is a projective surface, gave recently rise to a huge amount of literature (let just recall [3] and [8] for $X = \mathbf{P}^2$, [5] for K3 surfaces and

We thank the organizers for this chance of contributing to the volume in honor of the memory of F. Serrano; we knew him in the early 80's, when he was a Ph.D. student in Boston and we spent several excellent days together in the warm atmosphere of the Math Department at Brandeis University.

[4], [10] for surfaces of general type). In higher dimension, enumerative results are known for some classes of varieties, which are relevant for applications; on the other hand, we feel some lack of systematic studies for what should be the next relevant case, from the point of view of algebraic geometry: curves in higher projective spaces.

The aim of this note is to propose two ways for attaching the argument; the first one is a classical approach, focused on the subsets of the Hilbert scheme formed by curves with fixed number of nodes (and no other singularities); the tangent space to these objects is a subsheaf N' of the normal bundle. When $h^1(N') = 0$ and the number of nodes is small, we are able to use standard smoothing results of [12] §5, to produce inductively examples of nodal curves in $\mathbf{P}^n$, obtaining some non-emptiness results for the corresponding Severi varieties. Remark 1.8 shows also that these varieties may have singularities.

In $\mathbf{P}^3$, we propose also a different approach. Curves in projective surfaces are organized in linear systems, hence they are parameterized by projective spaces, and one can look inside these spaces to produce results on the Severi varieties. Going to $\mathbf{P}^3$, one obtains a partially similar organization thinking to a curve as the 0-locus of a section of some reflexive sheaves F, so that $\mathbf{P}(H^0(F))$ somehow gives a projective space dominating a subvariety in which the curve moves; it seemed us natural to ask for an analogue of Severi varieties when we fix a reflexive sheaf and look at its sections. We are able to consider here only the case in which F is a vector bundle, for otherwise its singularities introduce some nasty correction terms in our results.

In section 2, we go through a local deformation theory for the Severi subvariety of $\mathbf{P}(H^0(E))$, $E =$ vector bundle of rank 2, parameterizing sections whose 0-loci have a fixed number of nodes (and no other singularities). In section 3 we prove some regularity results for the dimension of these objects, when we take a suitable twist $E(m)$ of E, with $m \gg 0$, and the number of nodes is small with respect to m. Non-emptiness results for these Severi varieties are given at the end of the paper.

We always work over an algebraically closed field of characteristic 0; we often indicate with $\mathcal{O}$ the structure sheaf of $\mathbf{P}^n$.

1. Curves in projective spaces

We present here some results holding for the Severi varieties of curves $C \subset \mathbf{P}^n$, when the number of nodes is "small" and the relative normal bundle has no H^1.

In this section, Y is an irreducible curve in $\mathbf{P}^n$, with arithmetic genus p_a, degree d and only δ nodes for singularities (a *nodal* curve). Let N_Y be its normal sheaf in

$\mathbf{P}^n$, which is locally free of rank $n-1$; let T^1_Y be the first cotangent sheaf of Y (see [11]) and call N'_Y the kernel of the natural surjection:

$$0 \to N'_Y \to N_Y \to T^1_Y \to 0$$

Then one has the following relations:

$$\deg(N_Y) = (n+1)d + 2(p_a(Y)-1)$$

$$\chi(N_Y) = (n+1)d - (n-3)(p_a(Y)-1)$$

$$\chi(N'_Y) = (n+1)d - (n-3)(p_a(Y)-1) - \delta$$

$$h^0(N'_Y) \leq h^0(N_Y) \leq h^0(N'_Y) + \delta$$

$$h^1(N_Y) \leq h^1(N'_Y) \leq h^1(N_Y) + \delta$$

Definition 1.1. Call $\mathcal{V}$ the irreducible component of the Hilbert scheme of $\mathbf{P}^n$ containing Y; let $\mathcal{V}_\delta$ be the subset of $\mathcal{V}$ formed by points which parameterizes *nodal* curves of geometric genus $p_a(Y) - \delta$.

It is a standard fact that $\mathcal{V}_\delta$ is a quasi projective subvariety of $\mathcal{V}$ and $h^0(N'_Y)$ is its tangent space at Y.

We are interested here in the case $h^1(N'_Y) = 0$, which clearly implies $h^1(N_Y) = 0$, hence also $\dim \mathcal{V} = (n+1)\deg(Y) + 2(p_a(Y)-1)$. Observe that this condition gives a linear upper bound on $p_a(Y)$ in terms of $\deg(Y)$, because it implies $\chi(N_Y) \geq 0$.

Remark 1.2. Let $Y \subset \mathbf{P}^n$ be a nodal curve, with δ nodes, and assume $h^1(N'_Y) = 0$; fix $t \leq \delta$ and fix a subset T formed by t nodes of Y. Then by [12] §1, near the point of $\mathcal{V}$ corresponding to Y there is a germ A of $\mathcal{V}_t$, of codimension t in $\mathcal{V}$, parameterizing the deformations of Y which are equisingular near T. Furthermore, if $Y - T$ is connected, then a general curve parameterized by A is irreducible.

Using the smoothing results of [12] Theorem 5.2 and Theorem 6.1, we obtain:

Theorem 1.3

Fix integers n, p_a, d, δ such that $n \geq 3$, $d \geq n+1$ and $0 \leq \delta \leq p_a \leq (n(d-n)-1)/(n-1)$. Then there exists an integral, non degenerate nodal curve $Y \subset \mathbf{P}^n$ of degree d, arithmetic genus p_a, with exactly δ nodes and with $h^1(N'_Y) = 0$.

Proof. First we will check the result for all pairs (d, p_a) with $p_a \geq 0$ and $d = p_a + n$. If $p_a = 0$, then just use the rational normal curve in $\mathbf{P}^n$; then we make induction on p_a. Assume $p_a > 0$ and choose a number δ with $0 \leq \delta \leq p_a$; set $t = \max\{\delta - 1, 0\}$. Let W be a (necessarily linearly normal) integral curve with $\deg(W) = p_a + n - 1$, $p_a(W) = p_a - 1$, with exactly t nodes and with $h^1(N'_W) = 0$. Let D be a general line intersecting W at exactly 2 points and s et $X := W \cup D$; by [12], Lemma 5.1, we have $h^1(N'_X) = 0$. Let T be the union of $\mathrm{Sing}(W)$ and one of the points of $W \cap D$ and apply the previous Remark to X and T: the case $d = p_a + n$ follows.

Now assume $d > p_a + n$ and fix an integer δ with $0 \leq \delta \leq p_a$. Start with a linearly normal irreducible nodal curve $W' \subset \mathbf{P}^n$ with $\deg(W') = p_a + n$, $p_a(W') = p_a$, with δ nodes and with $h^1(N'_{W'}) = 0$. Let D' be a general smooth rational curve of $\mathbf{P}^n$, intersecting W' in one smooth point, with $\deg(D') = d - p_a - n$ and set $X' := W' \cup D'$ and $T = \mathrm{Sing}(W')$. By [12] Lemma 5.1, we have $h^1(N'_{X'}) = 0$ and applying the previous remark again to X' and T, we find the required curve.

Finally, for the case $d < p_a + n$, put $a = nd - p_a(n-1) - n^2$ and argue as in [12] Theorem 6.2: the case degree $= a$ and genus $= a + n$ has been proved above and one can reduce to this case applying repeatedly [12] 5.5 iii. $\square$

Remark 1.4. By the proof of Theorem 1.3 , one sees that the component $\mathcal{V}$ of $\mathrm{Hilb}(\mathbf{P}^n)$ containing Y has the "expected number of Moduli" in the sense of [12].

Furthermore, for the set of triples (d, p_a, n) considered in 1.3, then $\mathcal{V}$ is the component of the Hilbert scheme defined in [2] for $n \geq 4$ and in for $n = 3$.

Now we will obtain in the same way some inductive results in which the allowed number of nodes is smaller than the arithmetic genus of the irreducible curve.

Lemma 1.5

Fix integers n, x, y, m and δ, with $n \geq 3$, $x \geq n + 1$ and $0 \leq \delta \leq mn$. Assume the existence of a smooth connected non degenerate linearly normal curve $W \subset \mathbf{P}^n$ with $\deg(W) = x$, $p_a(W) = y$ and $h^1(N_W) = 0$. Then there exists an integral non degenerate nodal curve $Y \subset \mathbf{P}^n$ with $\deg(Y) = x + m(n-1)$, $p_a(Y) = y + mn$, $h^1(N'_Y) = 0$ and with exactly δ nodes. If furthermore W is linearly normal, then we may take also Y linearly normal.

Proof. Take m general hyperplanes H_i, $i = 1, \ldots, m$ and m smooth rational curves $C_i \subset H_i$, with $\deg(C_i) = n - 1$, C_i spanning H_i and intersecting W exactly at $n + 1$ points and quasi transversally. Set $X := W \cup (\bigcup C_i)$; then apply the procedure of [12] Theorem 5.2 to X. $\square$

Corollary 1.6

Fix integers n, x, y, m *and* δ, *with* $n \geq 4$, $m \geq 0$, $x \geq n+2$, $y \geq 2$, $(y-2)(n-2) \leq (x-n-2)(n-1)$ *and* $0 \leq \delta \leq mn$. *Then there exists an integral, non degenerate, linearly normal curve* $Y \subset \mathbf{P}^n$ *with* $\deg(Y) = x + m(n-1)$, $p_a(Y) = y + mn$, $h^1(N'_Y) = 0$ *and with exactly* δ *nodes.*

Proof. Apply Lemma 1.5 to the smooth curve W which is the general member of the component $W(x, y; n)$ of $\mathrm{Hilb}(\mathbf{P}^n)$ constructed in [2], §1. □

The previous corollary also works in $\mathbf{P}^3$, just using the component of $\mathrm{Hilb}(\mathbf{P}^3)$ constructed in [1]. However in this case, we can produce a more precise result, because we know all the pairs (d, p_a) which are the degree and genus of smooth curves $W \subset \mathbf{P}^3$, with $h^1(N_W) = 0$ (see [6] or [7]).

We use, in the next statement, the function $C_K : \mathbf{N} \to \mathbf{R}$ defined in [7].

Corollary 1.7

Fix integers x, y, m *and* δ, *with* $x \geq 5$, $m \geq 0$, $0 \leq y \leq C_K(x)$ *and* $0 \leq \delta \leq 3m$. *Then there exists an irreducible nodal curve* $Y \subset \mathbf{P}^3$ *with* $\deg(Y) = x + 2m$, $p_a(Y) = y + 3m$, $h^1(N'_Y) = 0$ *and with exactly* δ *nodes.*

Remark 1.8. Of course, there are cases in which $h^1(N_Y) \neq 0$ but nevertheless $\mathcal{V}_\delta$ is non-empty (and somehow it has the "expected dimension"). This happens, for instance, for complete intersection curves in $\mathbf{P}^3$, of type $5, d$, with $d \geq 5$, when $\delta \leq 5(n-1)^2/4$ (see [4] Proposition 4.7).

In $\mathbf{P}^3$, one can also produce examples of singular Severi varieties. In fact, if $\mathcal{V}$ is the Hilbert scheme of integral complete intersections of type $5, d$, $d > 5$ and $\delta = 5(n-1)^2/4$, then we have an obvious forgetful map $\mathcal{V} \to |\mathcal{O}_{\mathbf{P}^3}(5)|$, in which the fiber of $\mathcal{V}_\delta$ over a general quintic surface S is the Severi variety of curves in $|\mathcal{O}_S(d)|$ with δ nodes. Since this last variety is singular by [4] Proposition 4.5 and we are working in char. 0, then $\mathcal{V}_\delta$ cannot be smooth.

2. Rank 2 bundles: local theory

In this section, let E be a rank 2 bundle on $\mathbf{P}^3$, with Chern classes c_1, c_2 and let s be a general element of $H^0(E)$; call $Y := (s)_0$ the 0-locus of s and assume that Y is an irreducible curve, with only δ nodes for singularities (i.e. a nodal curve). We recall the usual exact sequence

$$0 \to \mathcal{O} \to E \to \mathcal{I}_Y(c_1) \to 0 \tag{1}$$

Y has degree $d = c_2$; the arithmetic and geometric genus of Y are given by:

$$p_a(Y) = (c_1 - 4)\frac{c_2}{2}$$

$$4g = p_a(Y) - \delta\,.$$

DEFINITION 2.1. Call $\mathcal{V}_\delta(E)$ the subset of $\mathbf{P}\big(H^0(E)\big)$ formed by those sections whose 0-locus is a *nodal* curve of geometric genus $p_a(Y) - \delta$.

It is a standard fact that $\mathcal{V}_\delta(E)$ is a quasi projective subvariety of $\mathbf{P}\big(H^0(E)\big)$.

We want to examine here the tangent space to $\mathcal{V}_\delta(E)$ at s.

Since, of course, the tangent space to $\mathbf{P}\big(H^0(E)\big)$ at s is isomorphic to $H^0(E)/(s)$, the question is to find which sections $s' \in H^0(E)$ represent infinitesimal deformations of s whose 0-locus still maintains nodes around the nodes of Y.

To do that, we may work locally around each node, in some open subset where E trivializes. So we fix a node P of Y and a suitable neighborhood U of P and we assume that Y is defined by the two equations f, g around P.

DEFINITION 2.2. Every section $s' \in H^0(E)$ defines, by sequence (1), a surface of degree c_1 containing Y; this surface is exactly the 0-locus of the wedge product $s \wedge s'$ and it will be denoted by $F(s')$. If s' is defined by the pair (f', g') in the neighborhood U, then in U the surface $F(s')$ has equation $fg' - f'g$.

Conversely, since the map $H^0(E) \to H^0(I_Y(c_1))$ surjects, every surface of degree c_1 containing Y is obtained in this way.

Proposition 2.3

Use the previous notation and identify $H^0(E)/(s)$ with the tangent space to $\mathbf{P}(H^0(E))$ at s. Then the tangent space to $\mathcal{V}_\delta(E)$ at s is given by those sections $s' \in H^0(E)$ for which the surface $F(s') = (s \wedge s')_0$ is singular at the nodes of $Y = (s)_0$.

Proof. Since by assumptions, the infinitesimal deformation of s defined by s' induces an equisingular deformation of Y, then it corresponds to an element of the normal bundle N_Y of Y, which goes to 0 in the cotangent map

$$T_{\mathbf{P}^3|Y} \to N_Y \to T^1.$$

Now N_Y the restriction $E \otimes \mathcal{O}_Y$, so we need just to prove that, locally around any node P of Y, the section s' goes to 0 in the composition $E \to E \otimes \mathcal{O}_Y \to \mathcal{O}_P \subset T^1$ if and only if the jacobian vector of $F(s')$ vanishes at P.

The question being local, we may take local coordinates x_1, $i = 1,2,3$ at P and trivialize E so that s, s' corresponds to pairs (f,g) and (f',g') of polynomials. With these notations, the map $T_{\mathbf{P}^3|Y} \to N_Y$ sends $\partial/\partial x_i$ to the pair $\big((\partial f/\partial x_i)_P, (\partial g/\partial x_i)_P\big)$ and these pairs are dependent, by assumptions; it follows that s' goes to 0 around P if and only if $\big(f'(P), g'(P)\big)$ linearly depends on each vector $\big((\partial f/\partial x_i)_P, (\partial g/\partial x_i)_P\big)$, i.e. if and only if, for all $i = 1,2,3$, we get:

$$g'(P)\Big(\frac{\partial f}{\partial x_i}\Big)_P - f'(P)\Big(\frac{\partial g}{\partial x_i}\Big)_P = 0$$

and since $f(P) = g(P) = 0$, this last is exactly $\Big(\frac{\partial(g'f - f'g)}{\partial x_i}\Big)_P$, the partial derivative of $F(s')$ along x_i at P. $\square$

Remark 2.4. Observe that, in fact, if the curve $C \subset \mathbf{P}^3$ is local complete intersection in a point $P \in C$ and P is a planar singularity of C, then, "heuristically", among the surfaces containing C, the condition of being singular in C has codimension 1.

To see this, let F, G be local generators for the ideal of C at P. Since P is a planar singularity, F and G are tangent at P, so after replacing F, G with some linear combination, we may assume F smooth and G singular at P; it follows that a surface $AF + BG$ is singular at P if and only if $A(P) = 0$.

Corollary 2.5

Let E be a rank 2 bundle and let $s \in H^0(E)$ be a global section such that the 0-locus $Y = (s)_0$ is a nodal curve, with nodes at $P_1, \dots, P_\delta$. Then the dimension of the tangent space to $\mathcal{V}_\delta(E)$ at s is $\geq h^0(E) - \delta$ and equality holds if and only the conditions of singularity imposed by the nodes $P_1, \dots, P_\delta$ to the surfaces of degree c_1 through Y are independent.

In general, we do not expect $\mathcal{V}_\delta(E)$ to be irreducible. However, for any component, we give the following:

Definition 2.6. Call $h^0(E) - \delta$ the *expected dimension* of the components of $\mathcal{V}_\delta(E)$.

We say that a component of $\mathcal{V}_\delta(E)$ is *regular* if its dimension equals the expected one.

Remark 2.7. When $h^0(E) \geq 4\delta$ and $\mathcal{V}_\delta(E)$ is non empty, then all the components of $\mathcal{V}_\delta$ have dimension greater or equal than the expected one.

Indeed, fix one point $P \in \mathbf{P}^3$ and look at the conditions imposed to a section $s \in H^0(E)$ to have the 0-locus Y singular at P and 2-codimensional; since the question is local, s can be replaced by a pair of polynomials (f_1, f_2) around P, so that the conditions are translated in having $f_1(P) = f_2(P) = 0$ and the Jacobian matrix of rank < 2 at P: a total of 4 conditions at most.

Now, if $\mathcal{V}_\delta$ is non empty and s is one of its elements, by $h^0(E) \geq 4\delta$, we can move arbitrarily the nodes of $Y = (s)_0$ and still get sections s' whose 0-loci are singular at these points; by semicontinuity, $s' \in \mathcal{V}_\delta$, hence the codimension of $\mathcal{V}_\delta$ in $H^0(E)$ is at most δ.

3. Rank 2 bundles: some regularity results

In this section we use the previous local analysis to prove some regularity results for the nodal Severi varieties of the rank 2 bundle $E(m)$, for $m \gg 0$, when the number δ of nodes is small with respect to m. Next we present some examples which measures the sharpness of our result.

Since we are going to see what happens for $m \gg 0$, we shall assume that $E(-1)$ is generated by global sections; this can always be achieved, indeed, twisting by some fixed k.

Proposition 3.1

Let E be a vector bundle on $\mathbf{P}^3$, with Chern classes c_1, c_2 and assume that $E(-1)$ is generated by global sections. Then for all $m \gg 0$ and for $\delta \leq m+1$, the Severi variety $\mathcal{V}_\delta\big(E(m)\big)$ is either empty or smooth, of the (expected) dimension $h^0\big(E(m)\big) - 1 - \delta$.

Proof. Assume that $s \in \mathcal{V}_\delta\big(E(m)\big)$ and call Y the 0-locus of s. By our assumptions on E and by the exact sequence $0 \to \mathcal{O} \to E(m) \to \mathcal{I}_Y(c_1+2m) \to 0$ we see that $\mathcal{I}_Y(c_1+m-1)$ is generated by global sections, so that Y is contained in some smooth surface X of degree c_1+m, since it has just planar singularities.

By our assumptions on δ, surfaces X' of degree m separate the δ nodes of Y, so that, for each node P, one can find a surface $X \cup X'$ of degree $c_1 + 2m = c_1\big(E(m)\big)$ which passes through Y and is singular at its nodes, except for P; it follows that the nodes of Y impose exactly δ conditions of singularity to the surfaces of degree $c_1\big(E(m)\big)$ through Y, thus by Corollary 2.5, the tangent space of $\mathcal{V}_\delta\big(E(m)\big)$ at s has dimension $\dim \mathbf{P}\big(H^0(E(m)\big) - \delta$ and the claim follows by Remark 2.7, since for $m \gg 0$ one easily checks by Riemann-Roch that $h^0\big(E(m)\big) \geq 4(m+1)$. $\square$

One cannot hope for a much stronger result; indeed, even in the case of splitting bundles, as soon as δ becomes slightly bigger that m, $\mathcal{V}_\delta\big(E(m)\big)$ acquires singular points:

Example 3.2: Consider $E = \mathcal{O}(1) \oplus \mathcal{O}(4)$ and fix $\delta = m + 4$; then $\mathcal{V}_\delta\big(E(m)\big)$ has some singular point.

Indeed choose points $P_1, \dots, P_{m+2}$ on a line L and take 2 surfaces F and G, passing through the P_i's, such that F, of degree $m+1$, has ordinary double points at the P_i (so it contains L) and no other singularities while G, of degree $m+4$, is smooth and transversal to F and L. The existence of these surfaces F and G follows by standard generalization arguments, starting for instance with reducible objects.

The the curve $Y = G \cap F$ has ordinary nodes at the P_i and no other singularities, so the section $s = (F, G) \in H^0\big(E(m)\big)$ belongs to $\mathcal{V}_\delta\big(E(m)\big)$; on the other hand the tangent space to $\mathcal{V}_\delta\big(E(m)\big)$ at s is given by surfaces of degree $2m + 5 = c_1\big(E(m)\big)$, passing through Y and singular at the points P_i. Forget now P_1 and consider the surfaces of degree $2m + 5$ through Y, which are singular at $P_2, \dots, P_\delta$. Since $2(\delta - 1) > 2m + 5$, they must clearly contain L and since L and the two tangents of Y at P_1 span $\mathbf{P}^3$, they are all singular at P_1; it follows that $P_1, \dots, P_\delta$ do not impose independent singularity conditions to the surfaces of degree $2m + 5$ containing Y: the conditions imposed are 1 less than the expected number.

When the points P_i are moved so that they are no longer aligned, then they impose independent singularity conditions to the surfaces of degree $2m+5$ through Y; since we have $h^0\big(E(m)\big) > 4\delta$, by Remark 2.7 and by an easy dimension count, one shows that the subset of $\mathcal{V}_\delta\big(E(m)\big)$ corresponding to curves with aligned nodes cannot fill any component of $\mathcal{V}_\delta$. It follows that the previous section s correspond in fact to an obstructed point of $\mathcal{V}_\delta$.

With a similar procedure, one can extend Example 3.2 to some other rank 2 bundles E.

Using reduction to open subsets and a Bertini argument, then one can prove the non emptiness of $\mathcal{V}_\delta$, in the range of Theorem 3.1.

Proposition 3.3

For $m \gg 0$ and for $\delta \leq m$ then $\mathcal{V}_\delta E(m)$ is non empty.

Proof. Let us start by observing that since $E(-1)$ is generated by global sections, then there exists a finite open cover of $\mathbf{P}^3$ such that for any U of the cover, one has two sections of $H^0\big(\mathbf{P}^3, E(-1)\big)$ which give an isomorphism $E_U(-1) \to \mathcal{O}_U^2$; it follows that for any pair (F, G) of surfaces of degree $m+1$ there is a global section $s \in H^0\big(\mathbf{P}^3, E(m)\big)$ with $(s)_0 = F \cap G$ over U.

Take a general set T of $\delta \leq m$ points in the intersection of all the open sets of the previous cover and let Q be any point in $\mathbf{P}^3 - T$; take an element U of the cover, which contains Q; since surfaces of degree $m + 1$ separate the points and the tangent planes at $T \cup \{Q\}$, then in the set of pairs of surfaces (F, G) which have singular intersection at T, generically $F \cap G$ has only nodes at T and the subset of pairs with singular intersection also at Q has codimension 4.

For general T in the symmetric product of $\mathbf{P}^3$, let $V \subset H^0\big(E(m)\big)$ be the set of sections whose 0-locus is singular at T; for what we said above, the subset $\{(s, Q) \in V \times (\mathbf{P}^3 - T)$: the 0-locus of s is singular at $Q\}$ has a component with 4-codimensional fibers over $\mathbf{P}^3 - T$, whose general point maps to $s \in V$ such that $(s)_0$ has nodes at T. It follows by a standard dimension count that there are sections $s \in V$ whose 0-locus has nodes at T and no other singularities. $\square$

Remark 3.4. Our method also produces results for rank 2 bundles on higher dimensional projective space (observe that they are relevant even for splitting bundles) or for rank two bundles over other smooth threefolds. Both generalizations are not explored here.

Similarly, one could apply the method to reflexive sheaves F on $\mathbf{P}^3$, aiming for results on general space curves; however, in this case one has to put the singular points of F into the picture, which seem to alter our statements in many nasty ways.

Remark 3.5. In any event, observe that one cannot simply apply our results from section 1 to study the 0-loci Y of rank 2 bundles, for in general we may have $H^1(N_Y) \neq 0$.

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