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The subbundles of decomposable vector bundles over an elliptic curve

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Dedicated to F. Serrano

Abstract

Let C be an elliptic curve and E, F polystable vector bundles on C such that no two among the indecomposable factors of $E \oplus F$ are isomorphic. Here we give a complete classification of such pairs (E, F) such that E is a subbundle of F.

In [1] M. Atiyah classified the vector bundles over an elliptic curve C. But still, there are several natural open questions on the structure of the subbundles of a fixed vector bundle on C. The aim of this paper is to give a reasonably complete answer to this question restricting slightly the vector bundles involved (see Corollary 0.2). A vector bundle F on a smooth projective curve is called polystable if it is the direct sum of stable vector bundles with the same slope $\mu(F)$. In particular a polystable vector bundle is semistable. The notion of polystability is very natural over an elliptic curve, because very few vector bundles over an elliptic curve are stable, while for all integers r, d with $r > 0$ there exist polystable vector bundles with rank r and degree d. All this paper is devoted to the proof of the following result.

Theorem 0.1

Fix integers x, y, a_i, $1 \le i \le x$, r_i, $1 \le i \le x$, b_j, $1 \le j \le y$, s_j, $1 \le j \le y$, with $x > 0$, $y > 0$, $r_i > 0$ for every i, $s_j > 0$ for every j and $a_i/r_i < b_j/s_j$ for every i and every j. Let C be an elliptic curve. Fix polystable vector bundles E_i, $1 \le i \le x$, and F_j, $1 \le j \le y$, with $\mathrm{rank}(E_i) = r_i$, $\deg(E_i) = a_i$, $\mathrm{rank}(F_j) = s_j$, $\deg(F_j) = b_j$. Set $E := \oplus_{1 \le i \le x} E_i$, $r := \sum_{1 \le i \le x} r_i = \mathrm{rank}(E)$, $F := \oplus_{1 \le j \le y} F_j$

and $s := \sum_{1 \le j \le y} s_j = \mathrm{rank}(F)$. *We assume that no two among the indecomposable factors of* E *are isomorphic and that no two among the indecomposable factors of* F *are isomorphic. Then:*

(a) If $r \le s$ *there is an injective map (as sheaves)* $f : E \to F$ *and the general* $f \in H^0(C, \mathrm{Hom}(E, F))$ *has this property;*

(b) If $r < s$ *there is an injective map* $f : E \to F$ *with Coker* (f) *locally free and the general* $f \in H^0(C, \mathrm{Hom}(E, F))$ *has this property;*

(c) If $r > s$ *there is a surjective map* $f : E \to F$ *and the general* $f \in H^0(C, \mathrm{Hom}(E, F))$ *has this property.*

A particular case of Theorem 0.1 (part (b) with $s = 1$) is the following result which was the main aim of this paper.

Corollary 0.2

Fix integers x, a_i, $1 \le i \le x$, r_i, $1 \le i \le x$, b *and* s *with* $x > 0$, $r_i > 0$ *for every* i, $s > r := \sum_{1 \le i \le x} r_i$ *and* $a_i/r_i < b/s$ *for every* i. *Let* C *be an elliptic curve. Fix polystable vector bundles* E_i, $1 \le i \le x$, *and* F *with* $\mathrm{rank}(E_i) = r_i$, $\deg(E_i) = a_i$, $\mathrm{rank}(F) = s$, $\deg(F) = b$. *Set* $E := \oplus_{1 \le i \le x} E_i$. *We assume that no two among the indecomposable factors of* E *are isomorphic and that no two among the indecomposable factors of* F *are isomorphic. Then* F *has a saturated subbundle isomorphic to* E.

The case $x = y = 1$ of Theorem 0.1 was proved in [2], Proposition 1.6. We will use this particular case for the proof of the general case. In [2] it was shown how to use results on polystable subbundles of polystable bundles on elliptic curves to obtain non trivial results on the same topic on bielliptic curves of genus > 1. We stress two features of the statements of 0.1 and 0.2: the results are independent from the isomorphism class of C and depend essentially only from the numerical data of a decomposition of the bundles into irreducible factors. For smooth curves of genus > 1 these features should not hold. In particular it is obvious that the Brill-Noether theory of the curve must play an important role, but that our problem requires much finer informations on the curve. Furthermore, we are able to work over an algebraically closed field of arbitrary characteristic. We stress the importance of the assumption "no two among the indecomposable factors of E and F are isomorphic". For instance $\mathcal{O}_C^{\oplus 2}$ has no degree 1 line bundle as quotient bundle, because no degree 1 line bundle on C is spanned; however, if $M \in \mathrm{Pic}^0(C)$ and M is not trivial, every degree 1 line bundle is a quotient of $\mathcal{O}_C \oplus M$ by Corollary 0.2. We hope to apply 0.1 and 0.2 for the corresponding problem on a bielliptic curve $\pi : X \to C$; we start with a polystable bundles E on C and we obtain bundles on X making elementary transformations of $\pi^*(E)$; a similar approach was used in [2].

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1. Proof of Theorem 0.1

All the section is devoted to the proof of Theorem 0.1. The case $x = y = 1$ was proved in [2], Proposition 1.6. Hence we will assume $x + y \geq 3$ and we may use induction on the integer $x + y$. For a fixed pair (x, y) we will use induction on the integer $r + s$; note that even the starting case $r + s = x + y$ is not trivial if $x + y \geq 3$ and hence its proof must be covered by the proof of the inductive step or made separately. We start with a remark which will be used several times for the proof of 0.1. We divide the rest of the proof of 0.1 into 4 steps.

Remark 1.1. Since $H^0(C, \mathrm{Hom}(E, F))$ is a rational variety, there is a dense open subset U of $H^0(C, \mathrm{Hom}(E, F))$ such that all vector bundles $Im(f)$ with $f \in U$ have the same rank and isomorphic determinant. Furthermore, if a general $f \in H^0(C, \mathrm{Hom}(E, F))$ has $Im(f)$ not saturated, even the degree and the determinant of its saturation is constant in a Zariski open non-empty subset U' of U. If for a general $f \in H^0(C, \mathrm{Hom}(E, F))$ the bundle $Im(f)$ is not semistable, say $Im(f) \cong \oplus_{1 \leq i \leq t} A_i$ with $t \geq 2$, A_i semistable and $\mu(A_i) \neq \mu(A_j)$ for $i \neq j$, then all the integers $\mathrm{rank}(A_i)$, $1 \leq i \leq t$, $\deg(A_i)$, $1 \leq i \leq t$, and the isomorphism classes of all line bundles $\det(A_i)$, $1 \leq i \leq t$, are constant in a Zariski open subset of $H^0(C, \mathrm{Hom}(E, F))$. The isomorphism classes of these determinantal line bundles are the same for every $f \in H^0(C, \mathrm{Hom}(E, F))$ such that the integers t, $\mathrm{rank}(A_i)$ and $\deg(A_i)$ are the generic ones.

Proof of Theorem 0.1

Step 1. Note that by semicontinuity the second part of each assertion (a), (b) and (c) follows from the first part.

Step 2. Here we will prove part (a). Here and in steps 3 and 4 we assume $a_i/r_i \leq a_1/r_1$ for every i, $1 \leq i \leq x$ and $b_j/s_j \geq b_1/s_1$ for every j, $1 \leq j \leq y$. Note that these inequalities are not an essential restriction, since they may be always satisfied just permuting the indices of the vector bundles E_i and F_j. In order to obtain a contradiction we assume that for a general $f \in H^0(C, \mathrm{Hom}(E, F))\, \rho :=$

rank$(Im(f)) < r$. Set $t := \deg(Im(f))$ for general f. Let $G(f)$ be an indecomposable factor of $Im(f)$ with maximal slope, say $Im(f) \cong G(f) \oplus H(f)$ with either $H(f) = \{0\}$ or $\mu(H(f)) \le \mu(G(f))$. An easy dimension count shows that for general f the bundle $Im(f)$ has no direct factor isomorphic to a direct factor of E_1. Hence $\mu(G(f)) > a_1/r_1$. If $x \ge 2$ by the inductive assumption on $x+y$ the map $f|E_1$ is an embedding. If $x = 1$ we have a surjection with non-trivial kernel $E_1 \to G(f)$ because $\mu(G(f)) > a_1/r_1$ ([2], Proposition 1.6). We claim that for a general $M \in \mathrm{Pic}^0(C)$ we have a surjection $E \to H(f) \oplus (G(f) \otimes M)$ and an embedding $H(f) \oplus (G(f) \otimes M) \to F$. Since the determinant of the semistable component of maximal slope of $H(f) \oplus (G(f) \otimes M)$ depends on M, the claim would give a contradiction by the last two assertions of Remark 1.1. By construction we have a surjection $E \to H(f)$. By part (c) for the integers $x' := x$ and $y' := 1$ for every $M \in \mathrm{Pic}^0(C)$ there is a surjection $E \to G(f) \otimes M$; here if $y \ge 2$ we apply the inductive assumption on the value of $x + y$; if $y = 1$ to obtain the same result we use, for fixed $x + y$, the inductive assumption on $r + s$, because $\mathrm{rank}(E) + \mathrm{rank}(G(f)) \le r + \rho < r + s$. Hence for every $M \in \mathrm{Pic}^0(C)$ there are maps $E \to H(f) \oplus (G(f) \otimes M)$ whose projection onto the factors $H(f)$ and $G(f) \otimes M$ are surjective. If $M \cong \mathcal{O}_C$ the general map $E \to H(f) \oplus (G(f) \otimes M)$ is surjective; the integer $h^0(C, \mathrm{Hom}(E, H(f) \oplus (G(f) \otimes M)))$ does not depend on $M \in \mathrm{Pic}^0(C)$; hence for general $M \in \mathrm{Pic}^0(C)$ there is a surjection $E \to H(f) \oplus (G(f) \otimes M)$. An easy dimension count shows that for general f there is no $M \in \mathrm{Pic}^0(C)$ such that $G(f) \otimes M$ is a direct factor of F. Hence $h^0(C, \mathrm{Hom}(G(f) \otimes M, F))$ does not depend on $M \in \mathrm{Pic}^0(C)$. Hence for general M there is an embedding $G(f) \otimes M \to F$. As above we obtain the existence of an embedding $H(f) \otimes (G(f) \otimes M)$ for general M, proving the claim and hence proving part (a).

Step 3. Here we will prove part (c) for $x \ge 2$ and hence taking duals we will prove part (b) for $y \ge 2$. By induction on $x + y$ we may assume that all rational numbers a_i/r_i, $1 \le i \le x$, and b_j/s_j, $1 \le j \le s$, are different. By induction on $r + s$ we may assume $r - r_i \le s$ for every i. First we assume that $(b_j - 1)/s_j \le a_1/r_1$ for every j and that $r_1 < s$. We claim the existence an embedding of $g : E_1 \to F$. The claim is the case $(x', y', r', s') = (1, y, r_1, s)$ of part (b). We may assume the claim because $x' + y' \le x + y$ and $r_1 + s < r + s$. Fix such an embedding g and set $G_1 := g(E_1)$. Hence we obtain an exact sequence

$$0 \to G_1 \to F \to F/G_1 \to 0\,. \tag{1}$$

Set $A := \oplus_{2 \le i \le x} E_i$. We have $H^1(C, \mathrm{Hom}(A, E_1)) = 0$ because $a_i/r_i < a_1/r_1$ for $2 \le i \le x$. By the inductive assumption we have a surjection $A \to F/G_1$. Since

$H^1(C, \mathrm{Hom}(A, G_1)) = 0$ we may lift any surjection $A \to M/G_1$ to a map $A \to M$ which, together with g, induces a surjection $E = E_1 \oplus A \to F$. Now assume that $(b_j - 1)/s_j \leq a_1/r_1$ for every j and that $r_1 \geq s$, i.e. that $r_1 = s$ and that $r - r_i \leq s$ for every i. We may assume $x = 2$ and $r_2 \leq r_1 = s$. By part (a) there is an inclusion $u : E_1 \to F$. Since $h^0(C, \mathrm{Hom}(E_2, F)) > h^0(C, \mathrm{Hom}(E_2, E_1))$, there is a map $v : E_2 \to F$ with $v(E_2)$ not contained in $u(E_1)$. Since $(b_j - 1)/s_j \leq a_1/r_1$ for every j, the map $(u, v) : E \to F$ is surjective. Now assume the existence of an integer j with $1 \leq j \leq y$ and such that $(b_j - 1)/s_j > a_1/r_1$. Set $B(j) := \oplus_{1 \leq k \leq y, k \neq j} F_k$. Take a polystable bundle D with $\deg(D) = b_j - 1$, $\mathrm{rank}(D) = s_j$ and such that for general $L \in \mathrm{Pic}^0(C)$ the bundle $B(L) := B(j) \oplus (D \otimes L)$ has no two isomorphic indecomposable factors. By induction on $\deg(F)$ for fixed E we obtain a surjection from E to $B(L)$ for general L; the starting case of the induction is the case "$(b_j - 1)/s_j \leq a_1/r_1$ for every j" proved before. By part (a) for $x = y = 1$ we obtain the existence of an inclusion $D \otimes L \to F_j$ for general L and hence the existence of an inclusion $B(L) \to F$ for general L. Note that $\deg(B(L)) = \deg(F) - 1$ and $\mathrm{rank}(B(L)) = \mathrm{rank}(F)$. Hence if a general $f \in H^0(C, \mathrm{Hom}(E, F))$ is not surjective, we have $\deg(B(L)) = \deg(Im(f))$ and $\mathrm{rank}(B(L)) = \mathrm{rank}(Im(f))$. Hence we conclude by Remark 1.1.

Step 4. To finish the proof of 0.1 it remains to prove only part (c) for $x = 1$. By [2], Proposition 1.6, we may assume $y \geq 2$. Taking duals, it is sufficient to prove part (b) with $x \geq 2$ and $y = 1$. The proof of step 3 works just changing a few times the word "surjection" with the words "embedding with locally free cokernel" and taking $(a_i + 1)/r_i$ instead of $(b_j - 1)/s_j$. $\square$

References

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