The universal rank-$(n - 1)$ bundle on $G(1, n)$ restricted to subvarieties

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En record de Ferran Serrano, extraordinari
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Abstract

We classify those smooth $(n-1)$-folds in $G(1, \mathbb{P}^n)$ for which the restriction of the rank-$(n-1)$ universal bundle has more than $n+1$ independent sections. As an application, we classify also those $(n-1)$-folds for which that bundle splits.

It is a classical problem to study which projective varieties of small codimension are not linearly normal (i.e. isomorphically projected from higher projective spaces). The first observation is that any n-dimensional variety can be projected to $\mathbb{P}^{2n+1}$, but is expected to produce singular points when projected to $\mathbb{P}^{2n}$. Hence, n-dimensional varieties of codimension at most n are expected to be linearly normal. For $n = 2$, Severi proved (see his classical paper [6]) that the Veronese surface is the only smooth surface in $\mathbb{P}^5$ that can be isomorphically projected to $\mathbb{P}^4$. In general, projectability (or linear normality) is characterized by the dimension of the secant varieties. A recent thorough study of secant varieties can be found in [7], where a lot of projectability results are given.

More generally, one can study which varieties are isomorphically mapped under the projection from a Grassmannian $G(k, N)$ of k-planes in $\mathbb{P}^n$ to another $G(k, n)$ induced by linear projections from $\mathbb{P}^N$ to $\mathbb{P}^n$. In the particular case $k = 1$, we have

that $G(1, n)$ has even dimension $2(n - 1)$. Thus, it is natural to ask which $(n - 1)$-dimensional varieties in $G(1, n)$ are projected from a bigger $G(1, N)$. This problem was solved in [3] for $n = 3$, and then studied in [1] for any dimension. In the latter, we introduced an appropriate notion of secant varieties for Grassmannians of lines. However, the theory only worked if we added the hypothesis of "uncompressedness", i.e. that the union of the lines parameterized by our varieties has the expected dimension.

The situation in the other extreme case, when k is very big, is just the opposite, since the union of the corresponding k-planes can never have the expected dimension, except for varieties of big codimension). The motivation of this paper was to understand this new phenomenon of compressedness that makes the theory so different for a general k. To this purpose, we will study the maximal values of k. In particular, we will study which varieties of $G(k, n)$ are projected from higher Grassmannians for $k = n - 1, n - 2$. The case $k = n - 1$ is easy (see Proposition 2.1), so that we will study more in detail the case $k = n - 2$. Since $G(n - 2, n)$ has again even dimension $2(n - 2)$, we will also specially study its $(n - 1)$-dimensional subvarieties that come from a bigger $G(n - 2, N)$. Another reason to study this case is that to a variety in $G(n - 2, n)$ corresponds by duality another variety in $G(1, n)$. In fact we will write our results in terms of varieties in $G(1, n)$. Varieties of dimension $n - 1$ in $G(1, n)$ (particularly the cases $n = 3, 4$) have been thoroughly studied by the classical geometers one century ago, and this research has retaken interest nowadays. It should also be mentioned that projectability of the dual varieties can also be interpreted as some kind extendability of the original variety (see Remark 1.1).

To my knowledge, for $n \geq 4$, few things are known about the dual varieties of varieties in $G(1, n)$ (or more likely, the classical geometers knew but never wrote up). This is why we needed to devote most of the first section to not only recall classical examples of those $(n - 1)$-folds, but also to carefully describe their dual varieties in some case.

In section 2, we give a characterization (see Theorem 2.3) of which $(n - 1)$-folds of $G(1, n)$ verify that its dual variety in $G(n - 2, n)$ is projected from a higher Grassmannian of $(n - 2)$-planes. On the other hand, since projectability is related to the restriction of the rank-$(n - 1)$ universal bundle, we are able to characterize when such a restriction to an $(n - 1)$-fold splits (see Proposition 2.4). The same problem was solved in [3] for the restriction of the rank-two universal bundles of $G(1, 3)$ to surfaces (in fact we took our method from there). Such a classification can be considered as the very first step towards the difficult problem of studying the stability of the restriction of the universal bundles of Grassmannians to subvarieties.

In order to deal with the intermediate values of k, it would be needed to have a full knowledge of varieties containing more linear spaces than expected. As far as I know, these kind of results are known only when the dimension of the linear spaces is close to the maximum allowed. There are results by Rogora (see [4]) for varieties with a lot of lines, and for general linear spaces, there are classical results by B. Segre (see [5], or Lemma 2.2 for a particular case), which we used for the case $k = n-1$. I hope that a development of such a theory will help in the future to deal with the projectability theory for any value of k.

1. Preliminaries and examples

Notation. All the varieties we consider will be complex and integral and, unless otherwise specified, they are also assumed to be smooth. For a vector bundle E over a variety X, $H^i(X, E)$ means the i-th cohomology space of a vector bundle over a variety X, while $h^i(X, E)$ will denote the dimension of that space. We denote by $G(k, n)$ the Grassmannian of k-linear spaces in the complex projective space $\mathbb{P}^n$. More generally, if Λ is a linear subspace of $\mathbb{P}^n$, $G(k, \Lambda)$ will denote the set of k-linear subspaces of Λ. We denote by $\mathcal{Q}$ and $\mathcal{S}$ the universal vector bundles of respective ranks $k+1$ and $n-k$. More precisely, $\mathcal{Q}$ and $\mathcal{S}$ will be the vector bundles appearing in the universal exact sequence

$$0 \to \check{\mathcal{S}} \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \mathcal{Q} \to 0.$$

This means that, if Y is a subvariety of $G(k, n)$, then the restriction $\mathcal{Q}_{|Y}$ is the vector bundle that embeds Y in $G(k, n)$, while $\mathcal{S}_{|Y}$ embeds Y in the dual Grassmannian $G(n-k-1, n) = G(n-k-1, \mathbb{P}^{n*})$. The image of Y in $G(n-k-1, \mathbb{P}^{n*})$ will be denoted by Y^*, and will be called the *dual subvariety* of Y. For us, a *congruence* in $G(1, n)$ will mean a smooth irreducible subvariety of $G(1, n)$ of dimension $n-1$. By abuse of notation, a k-plane of a variety $Y \subset G(k, n)$ will just mean a k-plane in $\mathbb{P}^n$ corresponding to a point of Y.

Remark 1.1. Assume that a subvariety $Y \subset G(k, n)$ verifies that $h^0(Y, \mathcal{S}_{|Y}) \geq n+2$. This means that the dual $Y^* \subset G(n-k-1, n)$ is an isomorphic projection of a subvariety $Y' \subset G(n-k-1, n+1)$ (this projection being induced by a linear projection from $\mathbb{P}^{n+1*}$ to $\mathbb{P}^{n*}$) which is not contained in $G(n-k-1, H)$ for any hyperplane $H \subset \mathbb{P}^{n+1*}$. In this situation, we will say that Y^* is a *nontrivial projection* of Y', or that Y^* *is projected* from $G(n-k-1, n+1)$. Dually, this means that there exists a subvariety $\hat{Y} \subset G(k+1, n+1)$, parameterizing $(k+1)$-planes in $\mathbb{P}^{n+1}$ not all of them passing through a point, such that the intersection of those $(k+1)$-planes with a hyperplane $\mathbb{P}^n$ produces exactly our family Y of k-planes. Therefore, in this sense, $H^0(Y, \mathcal{S}_{|Y})$ measures how far Y can be extended.

We look now at different examples of congruences in $G(1, n)$. These will be the examples appearing in the statements of our main results in the next section.

EXAMPLE 1.2: Let us consider on $Y = \mathbb{P}^r \times \mathbb{P}^{n-1-r}$ the vector bundle $\mathcal{O}_Y(1, 0) \oplus \mathcal{O}_Y(0, 1)$. This defines an embedding of Y as a congruence in $G(1, n)$ consisting of the set of lines joining two disjoint linear spaces of dimensions r and $n - 1 - r$. The dual of $\mathcal{S}_{|Y}$ is the kernel of the epimorphism

$$H^0(Y, \mathcal{O}_Y(1)) \otimes \mathcal{O}_Y \to \mathcal{O}_Y(1, 0) \otimes \mathcal{O}_Y(0, 1) \,.$$

Since $H^0(Y, \mathcal{O}_Y(1))$ splits as $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \oplus H^0(\mathbb{P}^{n-r-1}, \mathcal{O}_{\mathbb{P}^{n-r-1}}(1))$, it immediately follows that $\mathcal{S}_{|Y} \cong pr_1^*(T_{\mathbb{P}^r}(-1)) \oplus pr_2^*(T_{\mathbb{P}^{n-r-1}}(-1))$.

EXAMPLE 1.3: Assume that $Y \subset G(1, n)$ is contained in a $G' = G(1, H)$ for some hyperplane $H \subset \mathbb{P}^n$. If $\mathcal{Q}'$ and $\mathcal{S}'$ are the universal bundles on G', it holds that $\mathcal{Q}_{|G'} = \mathcal{Q}'$ and $\mathcal{S}_{|G'} \cong \mathcal{S}' \oplus \mathcal{O}_{G'}$. In particular, $\mathcal{S}_{|Y} \cong \mathcal{S}'_{|Y} \oplus \mathcal{O}_Y$. Reciprocally, if $\mathcal{S}_{|Y} \cong E \oplus \mathcal{O}_Y$ for some rank-$(n - 2)$ vector bundle E, then Y is contained in some $G(1, H)$. Indeed, such a decomposition implies that all the $(n-2)$-planes of the dual congruence Y^* pass through a fix point. Dualizing, all the lines of Y are contained in a hyperplane H.

EXAMPLE 1.4: Let Q be a smooth quadric in $\mathbb{P}^5$. It is well-known that Q has two families of planes, each of them parameterized by $\mathbb{P}^3$. The map from $\mathbb{P}^3$ to $G(2, \mathbb{P}^5)$ defining any of the two families is given by the vector bundle $\Omega_{\mathbb{P}^3}(2)$. The dual Grassmannian is another $G(2, 5)$ and it also holds that the dual of the set of planes in Q is again the set of planes in the dual quadric $Q^* \subset \mathbb{P}^{5*}$. If we take a general hyperplane $\mathbb{P}^4$ in $\mathbb{P}^5$, then we obtain a smooth three-dimensional quadric Q', and any line in Q' is the intersection of $\mathbb{P}^4$ with a unique plane in Q in each family. By Remark 1.1, the dual congruence $Y^* \subset G(2, 4)$ of the congruence $Y \subset G(1, 4)$ of lines in Q' is projected from a threefold $Y' \subset G(2, 5)$, and this Y' is dual to one of the two family of planes in Q. Hence, Y' itself consists of the set of planes contained in a smooth quadric.

Finally we recall now two examples from [2].

EXAMPLE 1.5: Let us consider the congruence Y of $G(1, n)$ studied in [2] 3.3. Geometrically, it is described in the following way. In $\mathbb{P}^n$ there are a plane Π' and a linear space Λ of dimension $n - 2$ meeting in one point P and there is a smooth conic $C \subset \Pi'$ passing through P. Then the congruence consists of the closure of the set of lines joining a point of $\Lambda \setminus \{P\}$ and a point of $C \setminus \{P\}$. In order to describe its dual congruence, we look at the following alternative description. We choose in C a

point P_0 different from P. The stereographic projections of C from P and P_0 define an isomorphism φ between the pencil A of hyperplanes containing Λ and the pencil of lines in Π' passing through P_0. Specifically, a hyperplane in the pencil A meets C in P plus another point Q, and we define the image of that hyperplane to be the line P_0Q in the second pencil. The congruence is then a scroll parameterized by A in the following way. For any hyperplane $H \in A$, the lines of the congruence associated with it are those obtained intersecting H with the set of planes containing the line $\varphi(H)$.

This immediately leads to the following description of the dual congruence. There is a line $L \subset \mathbb{P}^{n*}$ and an isomorphism φ from L to the pencil of $(n-2)$-planes contained in a hyperplane $H' \subset \mathbb{P}^{n*}$ and containing an $(n-3)$-plane $\Lambda' \subset H'$. Then, for any point $P \in L$, the $(n-2)$-planes of Y^* associated with it are those spanned by P and the set of $(n-3)$-planes contained in $\varphi(P)$. Our next task is to describe this in terms of an embedding induced by a rank-$(n-1)$ vector bundle on an $(n-1)$-fold.

A pencil of $(n-2)$-planes is a map from $\mathbb{P}^1$ to $G(n-2,n)$ induced by the vector bundle $\mathcal{O}_{\mathbb{P}^1}^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore, the set of $(n-3)$-planes in some $(n-2)$-plane of the above pencil is parameterized by the projective bundle $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ (i.e. the relative dual projective bundle to the previous one). Let $p : X \to \mathbb{P}^1$ be the structure morphism and denote by $\mathcal{O}_X(1)$ its tautological line bundle. Then the embedding of X in $G(n-3,n)$ that associates to each point of X the $(n-3)$-plane in $\mathbb{P}^n$ it represents is given by the vector bundle $T_{X/\mathbb{P}^1}(-1)$. Hence, Y^* is the image in $G(n-2,n)$ of a morphism induced by the rank-$(n-1)$ vector bundle $T_{X/\mathbb{P}^1}(-1) \oplus p^*\mathcal{O}_{\mathbb{P}^1}(1)$. Observe that this vector bundle has $n+2$ independent sections, so that it embeds X in $G(n-2,n+1)$, and Y^* is just a linear projection of it. In fact, the description in $G(n-2,n+1)$ coincides with the one given for Y^*, but now the hyperplane H' and the line L are disjoint.

EXAMPLE 1.6: Finally, we recall now from [2] 3.2 an example of another congruence $Y \subset G(1,n)$ that will be of interest for us. We will not describe it here completely as in the previous example, but we will just pay attention to the few facts that will be needed. For this congruence, there is a conic $C \subset \mathbb{P}^n$ such that, for each point $P \in C$, the set of lines of Y passing through it is the set of lines contained in a hyperplane $H_P \ni P$ and passing through it. Then, Y has a structure of a scroll over $C \cong \mathbb{P}^1$, and, under the Plücker embedding, it is in fact a rational normal scroll of degree $2n-2$.

One way of constructing a congruence of this type is by taking in $G(1,n)$ the dependency locus of three sections of the vector bundle $\mathcal{Q} \oplus \mathcal{S}$. It is not difficult to check –by looking at its invariants– that this dependency locus is indeed a congruence of the desired type. However, a more intuitive way of seeing the scroll structure in

this construction is the following. A section of $\mathcal{Q} \oplus \mathcal{S}$ vanishes on the set of lines contained in a hyperplane $H \subset \mathbb{P}^n$ and passing through a point $P \in \mathbb{P}^n$. If the section is general, $P \notin H$ and hence this locus is empty; but for a special section, P belongs to H and the zero locus of the section is a Schubert variety isomorphic to $\mathbb{P}^{n-2}$. The space of sections of $Q \oplus S$ is $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \oplus H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^*$, and the incidence subset corresponding to pairs (H, P) for which $P \in H$ is defined by a homogeneous quadratic form (in fact, this form is linear in each of the two direct summands). This means that a general net A of sections of $Q \oplus S$ will meet the incidence subset in a conic C. This gives the scroll structure over C. From that construction, it is easy to see that $h^0(Y, \mathcal{S}_{|Y}) = n + 2$, which implies that Y^* is projected from $G(n-2, n+1)$.

2. Projection of subvarieties of Grassmannians

We first study the easiest Grassmannians, i.e. those which are in fact a dual projective space. We will see that the fact that a subvariety of any dimension in $G(n-1, n)$ there is projected from a higher Grassmannian is impossible except for curves, and will study in that case what is the situation for congruences (i.e. $n = 2$). More precisely, we have the following

Proposition 2.1

  If a subvariety $Y \subset G(n - 1, n)$ is projected from $G(n - 1, n + 1)$, then Y is a curve. Moreover, if $n = 2$, then Y is a conic. Precisely, Y consists of the set of tangent lines to a conic, and it is projected from the subvariety of $G(1, 3)$ parameterizing the set of lines of one of the rulings of a smooth quadric in $\mathbb{P}^3$; this is not projected from $G(1, 4)$.

Proof. Let Y be a subvariety of $G(n - 1, n)$ that is projected from a subvariety of $G(n-1, n+1)$. Writing $\mathbb{P}^n = G(n-1, n)$, this means that $h^0(Y, T_{\mathbb{P}^n}(-1)_{|Y}) > n+1$. But from the dual of the Euler exact sequence restricted to Y,

$$0 \to \mathcal{O}_Y(-1) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^* \otimes \mathcal{O}_Y s \to T_{\mathbb{P}^n}(-1)_{|Y} \to 0,$$

it follows that $h^1(Y, \mathcal{O}_Y(-1)) \neq 0$. Then, by the Kodaira vanishing theorem, Y must be a curve. If, moreover, $n = 2$, Y is a plane curve of some degree d, and from the exact sequence

$$0 \to T_{\mathbb{P}^2}(-1-d) \to T_{\mathbb{P}^2}(-1) \to T_{\mathbb{P}^2}(-1)_{|Y} \to 0$$

it follows that $h^0(Y, T_{\mathbb{P}^2}(-1)_{|Y}) > 3$ if and only if $h^1(Y, T_{\mathbb{P}^2}(-1-d)) \neq 0$. But it is clear that the latter holds if and only if $d = 2$. Also, in this case $h^0(Y, T_{\mathbb{P}^2}(-1)_{|Y}) = 4$, so that Y comes only from $G(1,3)$. In fact, $T_{\mathbb{P}^2}(-1)_{|Y} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, so that the corresponding curve in $G(1,3)$ parameterizes the lines in one of the rulings of a smooth quadric. $\square$

Remark. Let us give a more geometric proof of the above result to illustrate in part what will be our strategy for the case of subvarieties in $G(n-2, n)$. Assume $Y \subset G(n-1, n)$ is projected from a variety $Y' \subset G(n-1, n+1)$. In particular, the union in $\mathbb{P}^{n+1}$ of all the $(n-1)$-planes parameterized by Y' must be a hypersurface X. But, if $\dim Y \geq 2$, a variety X with so many linear spaces is necessarily a hyperplane. Indeed, through a general point of X there will be at least a one-dimensional family of $(n-1)$-planes of Y'; hence, through two general points of X there passes an $(n-1)$-plane contained in X, and this implies that X is linear. Therefore, the projection is trivial. On the other hand, if Y is a curve and $n = 2$, we recall from Remark 1.1 that this means that Y^* is the plane section of a ruled surface in $\mathbb{P}^3$. But it is a well-known fact that a ruled surface of degree d in $\mathbf{P}^3$ has a singular curve which meets a general rule in a scheme of degree $d-2$, so that its hyperplane section can be smooth if and only if $d = 2$.

Hence, in general we need to study projective varieties with a lot of linear spaces. This has been done by B. Segre in [5]. There is a recent paper by Rogora (see [4]) studying projective varieties with too many lines. For the reader's convenience, we will proof here, by using the result of Rogora, the part we will need of the classical result of Segre.

Lemma 2.2

Let X be a (not necessarily smooth) hypersurface of $\mathbb{P}^{n+1}$ which is the union of an irreducible $(n-1)$-dimensional family Y' of $(n-2)$-planes. If $n \geq 4$, then either X is a hyperplane, or a quadric or the $(n-2)$-planes are distributed in a one-dimensional family of $(n-1)$-planes.

Proof. By Rogora's result, it suffices to show that X contains at least a $(2n-3)$-dimensional family of lines. We look first at the incidence variety

$$I = \{(\Lambda, L) \in G(n-2, n+1) \times G(1, n+1) \mid \Lambda \in Y', \ L \subset \Lambda\}.$$

The projection of I over Y' has fibers isomorphic to $G(1, n-2)$. Therefore, I is irreducible of dimension $3n - 7$. Since we want to prove that the image of I under the second projection has dimension at least $2n - 3$, it is enough to show that

a general line contained in a general $(n-2)$-plane of Y' is contained in at most an $(n-4)$-dimensional family of $(n-2)$-planes of Y'.

I claim first that two general $(n-2)$-planes of Y' do not meet along an $(n-3)$-plane. Otherwise, either all the $(n-2)$-planes of Y' are in the same $(n-1)$-plane, or they all contain the same $(n-3)$-plane A. In the latter case, we observe that the set of $(n-2)$-planes of $\mathbb{P}^{n+1}$ containing A has dimension three; and since $n \geq 4$, Y' must consist of the whole set of those planes. In either case, the union of the $(n-2)$-planes of Y' is not a hypersurface of $\mathbf{P}^{n+1}$, so that we get a contradiction.

We take now a general $(n-2)$-plane Λ of Y'. Because of the above claim, the intersection of Λ with the rest of $(n-2)$-planes of Y' will consist only of a family Y_1' of dimension $r \leq n-2$ of $(n-3)$-planes and a family Y_2' of dimension $n-1$ of $(n-4)$-planes. When we take a general line $L \subset \Lambda$, the dimension of the $(n-3)$-planes of Y_1' containing L is then $r-2$, while the dimension of the the $(n-4)$-planes of Y_2' containing L is $n-5$. Hence the dimension of $(n-2)$-planes of Y' containing L is at most $n-4$, as wanted. $\square$

Theorem 2.3

Assume $s \geq n-1$, $n \geq 4$ and let $Y \subset G(1,n)$ be a smooth variety of dimension s such that its dual $Y^* \subset G(n-2,n)$ is a nontrivial projection from $G(n-2,n+1)$. Then $s = n-1$ and Y is one of the following:
a) The congruence of lines in a smooth quadric of $\mathbb{P}^4$.
b) The congruence of Example 1.5.
c) The congruence of Example 1.6.

Moreover, in all three cases, Y^* is projected from $G(n-2,n+1)$, but not from $G(n-2,n+2)$.

Proof. Let $Y' \subset G(n-2,n+1)$ be a smooth variety of dimension s that can be projected isomorphically to $Y^* \subset G(n-2,n)$ and let $X \subset \mathbb{P}^{n+1^*}$ be the union of all the $(n-2)$-planes of Y'. Since Y' is projectable, this implies in particular that X is not the whole $\mathbb{P}^{n+1}$, so that it has dimension at most n and is not a linear space. Clearly, X has dimension exactly n, since otherwise two general points of it would lie in a linear space contained in X, and X would be linear. By Lemma 2.2, either X is quadric or the $(n-2)$-planes are distributed in a one-dimensional family of $(n-1)$-planes (for the last case, we apply the lemma for all subvarieties of Y' of dimension $n-1$, and observe that X, not being linear, cannot contain a two-dimensional family of $(n-1)$-planes).

In the first case, we first observe that an n-dimensional quadric that is a cone over a smooth r-dimensional quadric (and hence it has a vertex of dimension $n-r-1$)

contains linear spaces of dimension at most $n - \frac{r}{2}$. This implies that $r \leq 4$. If $r = 4$, then the family of $(n-2)$-planes contained in the quadric has dimension three, so that it has to be $s = 3$ and $n = 4$. Hence, by dimensional reasons, Y' must coincide with one of the two families of planes in a smooth quadric of $\mathbb{P}^5$. This is then example 1.4, so that Y is as stated in a). If $r = 3$, the family of $(n-2)$-planes contained in the quadric has dimension three, so that again $s = 3$ and $n = 4$, and it holds now that all the planes contained in the quadric are in a one-dimensional family of three-dimensional linear spaces. Finally, if $r \leq 2$, then n can take any value, but it always happens that the $(n-2)$-planes contained in a quadric are in a one-dimensional family of linear spaces of dimension $n - 1$. Rather than studying separately these last two subcases, we will consider them as a particular case of the following.

So we can assume from now on that the $(n-2)$-planes of Y' are distributed in a one-dimensional family of $(n-1)$-planes contained in X. We first observe that this property will be preserved under projection, i.e. the $(n-2)$-planes of Y^* are distributed in a one-dimensional family of hyperplanes of $\mathbb{P}^{n*}$. Dualizing, there is a curve C in $\mathbb{P}^n$ such that all the lines of Y meet C. If $s \geq n$, by dimensional reasons it has to be $s = n$ and Y will consist of the set of lines meeting C. But then Y cannot be smooth, since the bisecants to C provide singular points of Y. Hence, $s = n - 1$ and we can use the classification given in [2] of congruences such that all of their lines meet a given curve. However, instead of checking one by one all the cases in order to find out for which of them their dual is actually projected from $G(n-2, n+1)$, we will use a more conceptual method.

We know from [2] that C must be a smooth plane curve –in fact, either a line, or a conic or a smooth plane cubic. In the dual space, we have then a one-dimensional family of hyperplanes, all of them containing a fixed $(n-3)$-plane Λ. If we take a plane Π not meeting Λ, the intersection of the family of hyperplanes with Π produces a smooth one-dimensional family of lines in Π. The fact that the one-dimensional family of hyperplanes is projected from $\mathbb{P}^{n+1}$ is equivalent to the fact that this curve $C' \subset G(1, \Pi)$ is projected from $G(1,3)$. But, by Proposition 2.1, this is only possible if C' (and hence also C) is a smooth conic. It also holds that C' is not projected from $G(1,4)$, so that Y^* is not projected from $G(n-2, n+2)$. Looking again at the classification in [2], the only congruences in $G(1,n)$ for which C is a conic are those of Examples 1.5 and 1.6. We also checked that, for those examples, Y^* can be indeed projected from $G(n-2, n+1)$. $\square$

Remark. The same classification when $n = 3$ was done in [3], where three more cases are found. Even for one of them –the congruence of bisecants to a twisted cubic– it holds that its dual comes not only from $G(1,4)$, but also from $G(1,5)$.

Proposition 2.4

  Assume $n \geq 4$ and let Y be a congruence in $G(1, n)$. Then the vector bundle $\mathcal{S}_{|Y}$ splits if and only if Y is one of the following:
  a) A congruence contained in some $Y \subset G(1, H)$, for a hyperplane $H \subset \mathbb{P}^n$.
  b) The congruence of lines joining the points of two disjoint linear spaces Λ_1 and Λ_2 of dimensions l_1, l_2 with $l_1 + l_2 = n - 1$.
  c) The congruence in Example 1.5.

Proof. Assume $\mathcal{S}_{|Y} = E_1 \oplus E_2$, where each E_i has rank e_i (hence, $e_1 + e_2 = n - 1$). Since $\mathcal{S}$ is generated by its sections, so are E_1 and E_2. This implies that $h^0(Y, E_1) \geq e_1$ and $h^0(Y, E_2) \geq e_2$. Moreover, if some of the equalities hold, then we have respectively $E_1 \cong \mathcal{O}_Y^{\oplus e_1}$ or $E_2 \cong \mathcal{O}_Y^{\oplus e_2}$. But the fact that $\mathcal{S}_{|Y}$ contains the trivial line bundle as a summand is equivalent (see Example 1.2) to the fact that Y is contained in some $G(1, H)$, for a hyperplane $H \subset \mathbb{P}^n$, which is case a) in the statement. So we can assume $h^0(Y, E_1) \geq e_1 + 1$ and $h^0(Y, E_2) \geq e_2 + 1$.

  If $h^0(Y, E_i) = e_i + 1$, then it follows that in the dual $\mathbb{P}^{n*}$ there is a space of dimension e_i meeting all the $(n - 2)$-planes of Y^* in dimension $e_i - 1$. Dually, there is an $(n - e_i - 1)$-plane in $\mathbb{P}^n$ meeting all the lines of Y. Therefore, if both equalities hold, Y must consist of all the lines of $\mathbb{P}^n$ meeting two disjoint linear spaces of dimensions $(n - e_1 - 1)$ and $(n - e_1 - 1)$. This is the congruence of part b) in the statement, and we checked in Example 1.2 that $\mathcal{S}_{|Y}$ splits.

  So we can assume that, for some i, $h^0(Y, E_i) \geq e_i + 2$. But then $h^0(Y, \mathcal{S}_{|Y}) \geq n + 2$. Hence Y belongs to one of the three types of congruences in the statement of Theorem 2.3. If Y is of the first type, i.e. the congruence of lines in a three-dimensional quadric, as we remarked in Example 1.4, $\mathcal{S}_{|Y}$ is isomorphic to $\Omega_{\mathbb{P}^3}(2)$, hence indecomposable. If Y belongs to the second type, i.e. it is as in Example 1.5, we checked there that $\mathcal{S}_{|Y}$ splits.

  As for the case of the last type of congruences, the one in example 1.6, Y is a rational normal scroll of degree $2n - 2$, so it spans a linear space of dimension $3n - 4$. This means that, as a projective subvariety under the Plücker embedding of $G(1, n)$, it is contained exactly in $\binom{n-2}{2}$ linearly independent hyperplanes. The fibers of the scroll consist of the set of lines contained in a hyperplane $H \subset \mathbb{P}^n$ and passing through a point $p \in H$. Hence, $\mathcal{S}_{|Y}$ restricted to a fiber $F \cong \mathbb{P}^{n-2}$ of the scroll decomposes as $\mathcal{O}_F \oplus \Omega_F(2)$, and the second summand is indecomposable. Therefore, if $\mathcal{S}_{|Y}$ splits as $E_1 \oplus E_2$, it has to be $e_1 = 1$ and $e_2 = n - 2$ (or viceversa). Moreover, since $h^0(Y, \mathcal{S}_{|Y}) = n + 2$, either $h^0(Y, E_1) = 3$ and $h^0(Y, E_2) = n - 1$ or $h^0(Y, E_1) = 2$ and $h^0(Y, E) = n$. Hence, there are two linear subspaces $A_1, A_2 \subset \mathbb{P}^n$ of respective dimensions r and $n - r$ (with $r = 1$ or 2) meeting only at one point,

such that Y is contained in the Schubert varieties of lines meeting A_1 and A_2. This implies that Y is contained in $\binom{r}{2} + \binom{n-r}{2}$ independent hyperplanes, which is a contradiction, since this number is bigger than $\binom{n-2}{2}$ for any value of $r = 1, 2$. Therefore, $\mathcal{S}_{|Y}$ does not split for this last type of congruences. $\square$

Remark. When $n = 3$, there is one more case for which $\mathcal{S}_{|Y}$ splits (see [3]). Again, this new case is the congruence of bisecants to a twisted cubic.

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