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On the asymptotic behavior of solutions for a class of second order nonlinear differential equations

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Abstract

We study asymptotic behavior of solutions for a class of second order nonlinear differential equations. Using Bihari's inequality, we obtain conditions under which all continuable solutions are asymptotic to $at+b$ as $t \to +\infty$, where a, b are real constants.

1. Introduction

In this paper, we are concerned with the nonlinear differential equation

$$u'' + f(t, u, u') = 0 \tag{1}$$

which is used in modeling of a large number of physical systems.

In the study of the asymptotic behavior of solutions of differential equation (1) the problem of establishing of conditions for the existence of solutions which approach those of equation $u'' = 0$, i.e., which behave like nontrivial linear functions $at + b$ as $t \to \infty$ is of great interest.

This problem has been treated recently by Constantin [4] and Dannan [6] for the equation (1), while the particular case of equation (1)

$$u'' + f(t, u) = 0 \tag{2}$$

was discussed by Cohen [3] and Tong [11]; see also Bellman [1] and Trench [12] for the linear case of the differential equation (2). All results cited above were obtained using the Gronwall-Bellman inequality [1] and its generalizations due to Bihari [2] and Dannan [6].

We mention also the recent results of Naito [7] and Philos and Purnaras [8] concerning the nonlinear equation

$$u'' + a(t)f(u) = 0$$

obtained by an averaging technique. For the detailed phase plane analysis of the asymptotic behavior of solutions for autonomous nonlinear differential equation

$$u'' + f(u, u') = 0$$

including as a particular case celebrated Liénard equation we refer the reader to the paper by Rogovchenko and Villari [10].

In this paper, we use the technique similar to that used by Constantin [4] to prove results applicable to another class of systems and we note that our proofs are simpler. We also give examples illustrating obtained theorems. In the sequel, it is assumed that all solutions of equation (1) are continuously extendable throughout the entire real axis. We note that the results in this paper pertain only to continuable solutions of equations (1) and (2).

2. Main results

Theorem 1

Suppose that the function $f(t, u, v)$ satisfies the following conditions:

(i) $f(t, u, v)$ *is continuous in* $D = \{(t, u, v) : t \in [t_0, \infty),\ u, v \in \mathrm{R}\}$, *where* $t_0 \geq 1$*;*

(ii) there exist continuous functions $h, g : \mathrm{R}_+ \to \mathrm{R}_+$ *such that*

$$| f(t, u, v) | \leq h(t)\ g\left(\frac{| u |}{t}\right)\ | v |,$$

where for $s > 0$ *the function* $g(s)$ *is positive and nondecreasing,*

$$\int_{t_0}^{\infty} h(s)\ ds < \infty,$$

and if we denote

$$G(x) = \int_{t_0}^{x} \frac{ds}{sg(s)},$$

then $G(+\infty) = +\infty$.

Then every solution $u(t)$ *of equation* (1) *is asymptotic to* $at+b$, *where* a, b *are real constants.*

Proof. It follows from (i) by standard existence theorems (see, for example, [5, Existence Theorem 3]) that equation (1) has solution $u(t)$ corresponding to the initial data $u(t_0) = c_1, \quad u'(t_0) = c_2$.

Two times integrating (1) from t_0 to t, we get for $t \geq t_0$

$$u'(t) = c_2 - \int_{t_0}^{t} f\big(s, u(s), u'(s)\big)\, ds, \tag{3}$$

$$u(t) = c_2(t - t_0) + c_1 - \int_{t_0}^{t} (t-s)\, f\big(s, u(s), u'(s)\big)\, ds. \tag{4}$$

It follows from (3) and (4) that for $t \geq t_0$

$$| u'(t) | \leq | c_2 | + \int_{t_0}^{t} | f\big(s, u(s), u'(s)\big) | \; ds,$$

$$| u(t) | \leq (| c_1 | + | c_2 |)t + t \int_{t_0}^{t} | f\big(s, u(s), u'(s)\big) | \; ds,$$

so applying (ii) we get for $t \geq t_0$

$$| u'(t) | \leq | c_2 | + \int_{t_0}^{t} h(s)\; g\left(\frac{| u(s) |}{s}\right) \; | u'(s) | \; ds, \tag{5}$$

$$\frac{| u(t) |}{t} \leq | c_1 | + | c_2 | + \int_{t_0}^{t} h(s)\; g\left(\frac{| u(s) |}{s}\right) \; | u'(s) | \; ds. \tag{6}$$

Denote by $z(t)$ the right-hand side of inequality (6):

$$z(t) = | c_1 | + | c_2 | + \int_{t_0}^{t} h(s)\; g\left(\frac{| u(s) |}{s}\right) \; | u'(s) | \; ds,$$

then (5) and (6) imply that

$$| u'(t) | \leq z(t), \qquad \frac{| u(t) |}{t} \leq z(t). \tag{7}$$

The function $g(s)$ is nondecreasing for $s > 0$, and hence we get by (7)

$$g\left(\frac{\mid u(t) \mid}{t}\right) \leq g\big(z(t)\big),$$

so we conclude that for $t \geq t_0$

$$z(t) \leq 1+ \mid c_1 \mid + \mid c_2 \mid + \int_{t_0}^{t} h(s)\ g\big(z(s)\big)\ z(s)\ ds. \tag{8}$$

Applying Bihari's inequality [2] to (8), we get for $t \geq t_0$

$$z(t) \leq G^{-1}\left(G(1+ \mid c_1 \mid + \mid c_2 \mid) + \int_{t_0}^{t} h(s)\ ds\right),$$

where

$$G(w) = \int_{t_0}^{w} \frac{ds}{sg(s)}$$

and $G^{-1}(w)$ is the inverse function of $G(w)$ which is defined for $w \in \big(G(+0), +\infty\big)$, $G(+0) < 0$, and it is increasing.

Now put

$$K = G(1+ \mid c_1 \mid + \mid c_2 \mid) + \int_{t_0}^{\infty} h(s)\ ds < \infty.$$

Since $G^{-1}(w)$ is increasing, we get

$$z(t) \leq G^{-1}(K) < \infty,$$

so it follows from (7) that

$$\frac{\mid u(t) \mid}{t} \leq G^{-1}(K) \quad \text{and} \quad \mid u'(t) \mid \leq G^{-1}(K).$$

By (ii), we have

$$\int_{t_0}^{t} \mid f\big(s, u(s), u'(s)\big) \mid\ ds \leq \int_{t_0}^{t} h(s)\ g\left(\frac{\mid u(s) \mid}{s}\right)\ \mid u'(s) \mid\ ds$$

$$\leq \mid c_1 \mid + \mid c_2 \mid + \int_{t_0}^{t} h(s)\ g\left(\frac{\mid u(s) \mid}{s}\right)\ \mid u'(s) \mid\ ds = z(t) \leq G^{-1}(K),$$

therefore

$$\int_{t_0}^{\infty} \mid f\big(s, u(s), u'(s)\big) \mid\ ds$$

exists, as well as there exists an $a \in \mathrm{R}$ such that

$$\lim_{t\to\infty} u'(t) = a.$$

Further, in the same way as in [1, 3, 11], we can ensure that there exists a solution with the property

$$\lim_{t\to\infty} u'(t) \neq 0.$$

Finally, by the l'Hospital's rule, we conclude that

$$\lim_{t\to\infty} \frac{\mid u(t) \mid}{t} = \lim_{t\to\infty} u'(t) = a$$

and thus there exists a $b \in \mathrm{R}$ such that

$$\lim_{t\to\infty} \{u(t) - (at+b)\} = 0,$$

so the proof is now complete. □

Example 1: Consider the nonlinear differential equation

$$u'' - \frac{2t}{(t+1)^2(t+2)} \frac{u^3}{u^3+t^3} \ln(1+u^2) \frac{u'}{2+\cos(u')} = 0. \tag{9}$$

Applying Theorem 1, we deduce that for any solution $u(t)$ of equation (9) there exist real a, b such that $u(t) = at + b + o(t)$ as $t \to \infty$.

Theorem 2

Suppose that apart from assumption (i) of Theorem 1 the function $f(t,u,v)$ satisfies the following condition, (ii′) there exist continuous functions $h, g : \mathrm{R}_+ \to \mathrm{R}_+$ such that

$$\mid f(t,u,v) \mid \leq h(t) \frac{\mid u \mid}{t} g(\mid v \mid),$$

where for $s > 0$ the function $g(s)$ is positive and nondecreasing,

$$\int_{t_0}^{\infty} h(s)\, ds < \infty,$$

and if we denote

$$G(x) = \int_{t_0}^{x} \frac{ds}{sg(s)},$$

then $G(+\infty) = +\infty$.

Then every solution $u(t)$ of equation (1) is asymptotic to $at + b$, where a, b are real constants.

The proof of the theorem is analogous to that of Theorem 1 and thus it is omitted.

Corollary 1 [1, Theorem 5, p. 114]

Consider the equation

$$u'' + a(t)u = 0 \tag{10}$$

where

$$\int^{\infty} t|a(t)| \, dt < \infty.$$

Then $\lim_{t\to\infty} u'$ *exists, and the general solution of equation (10) is asymptotic to* $d_0 + d_1 t$ *as* $t \to \infty$*, where* d_1 *may be zero, or* d_0 *may be zero, but not both simultaneously.*

Proof. The conclusion of corollary follows from Theorem 2 with

$$h(t) = t\,a(t), \quad g(s) \equiv 1. \ \square$$

EXAMPLE 2: Consider the nonlinear differential equation

$$u'' + \frac{1}{t^3+1}\,\frac{u}{2+\sin(u)}\ \ln\left(1+(u')^2\right)\ln\Big(\ln\left(1+(u')^2\right)\Big) = 0. \tag{11}$$

It follows from Theorem 2 that for any solution $u(t)$ of equation (11) there exist real a, b such that $u(t) = at + b + o(t)$ as $t \to \infty$.

Finally, arguing in the same way as in Theorem 1, we can also prove the following general result which with $g_2(s) \equiv s$ gives us exactly Theorem 1 and with $g_1(s) \equiv s$ gives us exactly Theorem 2.

Theorem 3

Suppose that apart from the assumption (i) of Theorem 1 the function $f(t,u,v)$ *satisfies the following condition, (ii$''$) there exist continuous functions* h, g_1, g_2 : $\mathrm{R}_+ \to \mathrm{R}_+$ *such that*

$$| f(t,u,v) | \le h(t)\ g_1\left(\frac{|\,u\,|}{t}\right)\ g_2(|\,v\,|),$$

where for $s > 0$ *the functions* $g_1(s)$ *and* $g_2(s)$ *are positive and nondecreasing,*

$$\int_{t_0}^{\infty} h(s)\ ds < \infty,$$

and if we denote

$$G(x) = \int_{t_0}^{x} \frac{ds}{g_1(s)g_2(s)},$$

then $G(+\infty) = +\infty$.

Then every solution $u(t)$ of equation (1) is asymptotic to $at + b$, where a, b are real constants.

Corollary 2 (cf. [1, Chapter 7, Theorems 1–8]).

Consider the well-known Emden-Fowler equation

$$u'' + t^{\sigma} u^{n} = 0 \tag{12}$$

with $0 < n < 1$ *and* $\sigma + 2 < 0$.

Then $\lim_{t\to\infty} u'$ *exists, and all solutions of equation (12) are asymptotic to* $d_0 + d_1 t$ *as* $t \to \infty$.

Proof. The conclusion of corollary follows from Theorem 3 with

$$h(t) = t^{\sigma+n}, \quad g_1(s) = s^{n}, \quad g_2(s) \equiv 1. \ \square$$

Example 3: Consider the nonlinear differential equation

$$u'' - a(t) \left(\frac{u^2}{u^2 + t^2}\right)^{k} \left(\frac{u'}{1 + \max(u', 0)}\right)^{1-k} = 0, \tag{13}$$

where

$$a(t) = \frac{2(3 - t^2)(2t^4 + 2t^2 + 1)^{k}(2t^4 + 5t^2 + 1)^{1-k}}{(t^2 + 1)^{3} t^{1+2k} (t^2 + 3)^{1-k}}$$

and $k \in (0, 1)$.

Then Theorem 3 implies for any solution $u(t)$ of equation (13) the existence of real numbers a, b such that $u(t) = at + b + o(t)$ as $t \to \infty$.

Observe that

$$u(t) = \frac{t^3}{t^2 + 1}$$

is the solution of equation (13) with the initial data

$$u(1) = 1/2, \qquad u'(1) = 1,$$

which is asymptotic to t as $t \to \infty$.

We stress that the results of Constantin [4, Theorems 1 and 2] can not be applied to equations (9) and (11), while equation (13) is of interest since it has the exact solution.

Remark. It is important to note that by Theorems 1–3 and Corollaries 1 and 2 *all* solutions of equations (1), (10) and (12) respectively are asymptotic to $at + b$ as $t \to \infty$, though the restrictions on the function $f(t, u, u')$ may seem to be artificial. It is possible to relax these assumptions for a certain rather wide class of functions (see [6]), though the price to be paid for it is the desired asymptotic behavior *only for a part* of solutions of equation (1) with initial data satisfying a certain estimate (we refer to [9] for the details).

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