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Uniformly countably additive families of measures and group invariant measures[1]

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Abstract

The extension of finitely additive measures that are invariant under a group permutations or mappings has already been widely studied. We have dealt with this problem in [15] and [16] from the point of view of Hahn-Banach's theorem and von Neumann's measurable groups theory. In this paper we construct countably additive measures from a close point of view, different to that of Haar's Measure Theory.

We will use the following notations:

(i) Ω will be a non-empty set.

(ii) T will be a group of permutations, i.e., one-to-one mappings from Ω onto Ω.

(iii) Σ will be a σ-algebra of subsets of Ω such that $\tau E \in \Sigma$ for all $E \in \Sigma$ and $\tau \in T$.

(iv) μ will be a countably additive (c.a.) probability measure defined on Σ. We will write $\tau\mu$ to denote the measure define by $\tau\mu(E) = \mu(\tau^{-1}(E))$ for all $E \in \Sigma$ and $\tau \in T$, where $\tau E = \{\tau x\colon\ x \in E\}$.

A case to be considered is that in which Σ is the class of all the subsets of Ω. If Ω is a topological space, the cases in which Σ is the class of either Baire or Borel sets (or a class containing these) are interesting.

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Proposition 1

Let μ be a countably additive measure. The following statements are equivalent:

(i) The family $(\tau\mu\colon\ \tau \in T)$ is uniformly countably additive.
(ii) For any sequence $S \subseteq T$, the sequence $(\tau\mu\colon \tau \in S)$ is uniformly countably additive.
(iii) If (E_n) is a sequence of pairwise disjoint subsets $E_n \in \Sigma$, then $\lim_n \tau\mu(E_n) = 0$ uniformly in $\tau \in T$.
(iv) If (E_n) is a non-increasing sequence of subsets $E_n \in \Sigma$ such that $\lim_n E_n = \emptyset$, then $\lim_n \tau\mu(E_n) = 0$ uniformly in $\tau \in T$.
(v) If (E_n) is a non-increasing sequence of subsets $E_n \in \Sigma$, then $\lim_n \tau\mu(E_n)$ exists uniformly in $\tau \in T$.
(vi) If (E_n) is a non-decreasing sequence of subsets $E_n \in \Sigma$, then $\lim_n \tau\mu(E_n)$ exists uniformly in $\tau \in T$.

Proof. From [5, I. 1. Proposition 17] it follows that (i), (ii) and (iii) are equivalent. On the other hand we have that (i) $\Rightarrow$ (vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) as in [5, I. 1. Corollary 18]. $\square$

DEFINITION 1. We will say that the pair (μ, T) is *uniformly countably additive* whenever the family $(\tau\mu\colon\ \tau \in T)$ is uniformly countably additive.

If μ is a measure that is invariant with respect to the group T, then (μ, T) is uniformly countably additive. The same conclusion holds if T is a finite group.

Proposition 2

Let X be a Köthe space such that X is σ-order continuous on the probability space (Ω, Σ, μ) and let $U_\tau f(\omega) = \tau f(\omega) = f(\tau^{-1}\omega)$ for all $f \in X$ and $\tau \in T$. Suppose that $\mu(E) = 0$ implies $\mu(\tau E) = 0$ for all $\tau \in T$ and

$$\big|\|f\|\big| = \sup\big\{\, \|\tau f\|\colon\ \tau \in T\big\} < \infty$$

for all $f \in X$. Then $\big|\|\cdot\|\big|$ is an equivalent norm to $\|\cdot\|$ on X such that $\big|\|\tau f\|\big| = \big|\|f\|\big|$ for every $f \in X$ and every $\tau \in T$, and (μ, T) is uniformly countably additive.

Proof. First of all we will prove that $X_{\left|\|\cdot\|\right|}$ is a Banach space. With this aim, since $X_{\|\cdot\|}$ is a Banach space it will be enough to prove that $\{f\colon \left|\|f\|\right| \le r\}$ is a closed subset of $X_{\|\cdot\|}$. Indeed, let $\left|\|f_n\|\right| \le r$ for all $n \in \mathbb{N}$ and let $f \in X$ be such that $\|f_n - f\| \to 0$. Passing to a subsequence if necessary we may suppose $\lim_n f_n(\omega) = f(\omega)$ almost everywhere, and so $\lim_n f_n(\tau^{-1}\omega) = f(\tau^{-1}\omega)$ a.e. and $\lim_n |\tau f_n|(\omega) = |\tau f|(\omega)$ a.e. since $\mu(\tau E) = 0$ whenever $\mu(E) = 0$.

As X is σ-order continuous we have $X' = X^*$ [10, 1.b.17, p. 29], so that if $g \in X'$, $\|g\| \le 1$, Fatou's theorem implies

$$\begin{aligned}\int_\Omega |g|\,|\tau f|\,d\mu &\le \underline{\lim}_n \int_\Omega |g|\,|\tau f_n|\,d\mu \\ &\le \underline{\lim}_n \|\tau f_n\| \le r\end{aligned}$$

and therefore $\|\tau f\| \le r$ for all $\tau \in T$, and $\left|\|f\|\right| \le r$.

Now, since $X_{\left|\|\cdot\|\right|} \to X_{\|\cdot\|}$ is a continuous linear biyective mapping between two Banach spaces, from Banach's homomorphism theorem it follows that this mapping is an isomorphism and there exists $k > 0$ such that

$$\|f\| \le \left|\|f\|\right| \le k\|f\|$$

for all $f \in X$. If (E_n) is a non-increasing sequence in Σ with $\lim_n E_n = \emptyset$, since X is σ-order continuous we have $\lim_n \|\chi_{E_n}\| = 0$ and, as $\left|\|\chi_{E_n}\|\right| \le k\|\chi_{E_n}\|$, it follows that $\lim_n \|\tau\chi_{E_n}\| = 0$ and $\lim_n \tau\mu(E_n) = 0$ uniformly in $\tau \in T$. If $X = L^p(\mu)$ we have $\tau\mu(E) \le k^p\mu(E)$ for each $E \in \Sigma$ and each $\tau \in T$ $(1 \le p < \infty)$. □

Remark. If $\mu(E) = 0$ does not imply $\mu(\tau E) = 0$ for every $\tau \in T$, then $\left|\|\cdot\|\right|$ is not equivalent to $\|\cdot\|$.

Proposition 3

Let X be a reflexive Köthe. Then, provided $\left|\|f\|\right| = \sup\{\|\tau f\|\colon \tau \in T\} < \infty$, the limit

$$f^\tau = \lim_n \frac{1}{n}\sum_{k=1}^{n-1} \tau^k f$$

exists in the norm topology for all $f \in X$, and

$$\left|\|f^\tau\|\right| = \min\left\{\left|\|f^\tau - g\|\right|\colon g \in Y_\tau\right\},$$

where $Y_\tau = \overline{\left\{\tau h - h\colon h \in X\right\}}$. Moreover, if $\tau^* = U_\tau^*$ is the adjoint of the operator U_τ defined by $U_\tau f = \tau f$, we have $\tau^* g = g$ for all $g \in Z_\tau = Y\tau^\perp (\subseteq X^* = X')$.

Proof. The existence of the limit defining f^τ follows from [6, VIII.5.4] since X is a reflexive space. Let $f = \tau h - h$, $h \in X$. Then

$$\overline{\lim_n} \left| \left\| \frac{1}{n} \sum_{k=1}^{n-1} \tau^k f \right\| \right| = \overline{\lim_n} \frac{1}{n} \left| \|\tau^n h - h\| \right|$$
$$\leq 2 \overline{\lim_n} \frac{\left| \|h\| \right|}{n} = 0 .$$

Moreover, if $u_j = \tau h_j - h_j \to f$ it turns out that

$$\overline{\lim_n} \left| \left\| \frac{1}{n} \sum_{k=1}^{n-1} \tau^k f \right\| \right| \leq \overline{\lim_n} \left| \left\| \frac{1}{n} \sum_{k=1}^{n-1} \tau^k (f - u_j) \right\| \right|$$
$$+ \overline{\lim_n} \left| \left\| \frac{1}{n} \sum_{k=1}^{n-1} \tau^k u_j \right\| \right| \leq \left| \|f - u_j\| \right|$$

and therefore $f^\tau = 0$ for each $f \in Y_\tau$.

On the other hand, since $\tau f^\tau = f^\tau$, we have that $f^{\tau\tau} = f^\tau$ and

$$\left| \|f^\tau\| \right| = \left| \|(f^\tau - (\tau h - h))^\tau\| \right| \leq \left| \|f^\tau - (\tau h - h)\| \right|$$

so that

$$\left| \|f^\tau\| \right| = \min \left\{ \left| \|f^\tau - g\| \right| : g \in Y_\tau \right\} .$$

Finally, if $g \in Z_\tau = Y_\tau^\perp$, we have

$$\big(h, \tau^* g - g) = (\tau h - h,\, g) = 0$$

for all $h \in X$, and therefore $\tau^* g = g$ for all $g \in Z_\tau$. □

Proposition 4

Under the conditions of Proposition 3, if $\mu(f) = \int_\Omega f d\mu$ *the following statements are equivalent:*

(i) $\mu(\tau f) = \mu(f)$ *for all* $f \in X$.
(ii) $\chi_\Omega \in Z_\tau = Y_\tau^\perp$.
(iii) $\tau^* = \tau^{-1}$.

Proof. (i) $\Leftrightarrow$ (ii): It is enough to take into account that

$$\mu(\tau f) - \mu(f) = \big(\tau f - f,\ \chi_\Omega\big)$$

for all $f \in X$.

(i) $\Rightarrow$ (iii): Since $X^* = X'$ it follows that

$$(\tau f, g) = \int_\Omega (\tau f) g d\mu = \int_\Omega f\big(\tau^{-1} g\big) d\mu = \big(f, \tau^{-1} g\big)$$

for all $f, g \in X$ [10, 1.b.17 and 1.b.18].

(iii) $\Rightarrow$ (ii): As $\tau^* = \tau^{-1}$ we have

$$\big(\tau f - f, \chi_\Omega\big) = \big(f, \tau^{-1}\chi_\Omega - \chi_\Omega\big) = 0$$

for all $f \in X$ and therefore $\chi_\Omega \in Y_\tau^\perp = Z_\tau$. □

It should be noted that Propositions 3 and 4 involve a generalization of the Mean Ergodic Theorem [6, VIII].

Proposition 5

Let $\mathcal{M}^1(\Omega)$ be the linear space of all finite Radon measures on a Hausdorff topological space Ω, and let $\mathcal{M}^1_+(\Omega)$ be the corresponding cone of positive measures. Then, if $A = \{\tau\mu\colon\ \tau \in T\} \subseteq \mathcal{M}^1_+(\Omega)$, the following statement are equivalent:

(i) (μ, T) is uniformly countably additive.

(ii) For every $\varepsilon > 0$ and universally bounded Lusin- measurable function f on Ω there exists a compact subset $K \subseteq \Omega$ such that the restriction $f|_K$ is continuous and $\bar{\mu}(\Omega\backslash K) < \varepsilon$ for every $\overline{\mu} \in A$.

(iii) A is uniformly innerly regular, i.e., for every Borel subset $B \subseteq \Omega$ and for every $\varepsilon > 0$ there exists a compact subset $K \subseteq B$ such that $\overline{\mu}(B\backslash K) < \varepsilon$ for every $\overline{\mu} \in A$.

Proof. It follows from [9, Theorem 15]. The equivalence between (i) and (iii) when Ω is a compact space was proved in [5, VI.2, Lemma 13]. □

Proposition 6

Let Ω be a Stonean space and suppose that $\tau f \in C(\Omega)$ for all $f \in C(\Omega)$ and $\tau \in T$. If for every sequence $S \subseteq T$ the closed subspace $X_S \subseteq X = l_\infty(S)$ spanned by $\{(\mu(\tau f))_{\tau\in S}\colon f \in C(\Omega)\}$ contains no isomorphic copy of l_∞, then (μ, T) is uniformly countably additive.

Proof. Since Ω is a Stonean space and X_S does not contain any copy of l_∞, the bounded linear operator $T\colon C(\Omega) \to X_S$ defined by $T(f) = (\mu(\tau f))_{\tau\in S}$ is weakly compact (by virtue of [5, VI.2, Theorem 10 (Rosenthal)]). So, From [5, VI.2, Theorem 5] it follows that the measure $G(E) = (\mu(T\chi_E))_{\tau\in S} = (\mu(\tau E))_{\tau\in S}$ is countably additive on the σ-algebra of Borel sets, and therefore (μ, T) is uniformly countably additive. □

It should be noted that if μ is a measure which is invariant under T, then every space X_S is isomorphic to $\mathbb{R}$ and therefore X_S does not contain any isomorphic copy of l_∞.

Proposition 7

If the pair (μ, T) is uniformly countably additive, then there exists a probability measure $\lambda\colon \Sigma \to \mathbb{R}$ such that

$$\lim_{\lambda(E)\to 0} \tau\mu(E) = 0$$

uniformly in $\tau \in T$ and

$$0 \le \lambda(E) \le \sup_{\tau\in T} \tau\mu(E)\,.$$

Therefore, $\lambda(E) \to 0$ if and only if $\sup_{\tau\in T} \tau\mu(E) \to 0$. Moreover, there exists a sequence $(\tau_n) \subseteq T$ such that

$$\lambda(E) = \sum_{n\in\mathbb{N}} a_n \tau_n \mu(E)$$

for all $E \in \Sigma$, where $a = (a_n)_{n\in\mathbb{N}} \in l_1$ verifies $\|a\|_1 = 1$ and $a_n \ge 0$.

Proof. Since μ is a countably additive probability measure, this result is a consequence of [5, I.2 Theorem 4 and its proof and Corollary 5]. □

Proposition 8

Suppose that the pair (μ, T) is uniformly countably additive. Then, for every $\sigma \in T$ there exists a countably additive measure μ_σ such that $\mu_\sigma(\sigma E) = \mu_\sigma(E)$ for all $E \in \Sigma$ and $0 \le \mu_\sigma(E) \le \sup_{\tau\in T} \mu(\tau E)$, $\mu_\sigma(\Omega) = 1$.

Proof. Let "Lim_n" be a Banach's generalized limit on l_∞ [12, 7.2]. Then

$$\mu_\sigma\colon\ \ \mu_\sigma(E) = \mathrm{Lim}_n\, \mu\big(\sigma^n E\big) \qquad (E \in \Sigma)$$

is a finitely additive measure, invariant under σ, that is $\mu_\sigma(\sigma E) = \mu_\sigma(E)$, with $\mu_\sigma(\Omega) = \mu(\Omega) = 1$. Let (E_n) be a sequence in Σ such that $E_n \searrow \emptyset$. Then $\lambda(E_n) \to 0$ for the measure λ of Proposition 7, and therefore $\lim_n \mu(\tau E_n) = 0$ uniformly in $\tau \in T$, which implies $\lim_n \mu_\sigma(E_n) = 0$ and μ_σ is a countably additive measure. Finally, it is clear that $0 \le \mu_\sigma(E) \le \sup_{\tau\in T}\, \mu(\tau E)$ for all $E \in \Sigma$. $\square$

Proposition 9

Suppose that (μ, T) is uniformly countably additive. Then, if T is a commutative group, there exists a c.a. probability measure $\nu\colon \Sigma \to \mathbb{R}$ such that ν is invariant under T and $0 \le \nu(E) \le \sup_{\tau\in T}\, \mu(\tau E)$.

Proof. We will use transfinite induction. Suppose $(\mu_\rho)_{\rho<\alpha}$ is a well ordered family of probability measures such that $0 \le \mu_\rho(E) \le \sup_{\tau\in T} \tau\mu(E)$ and let $(T_\rho)_{\rho<\alpha}$ be an increasing family of subsets of T such that $T_{\rho+1}\backslash T_\rho$ is a singleton for $\rho+1 < \alpha$, and $T_\rho = \cup\{T_\sigma\colon \sigma < \rho\}$ whenever ρ is a limit ordinal (i.e. which has no immediate predecessor) and $\mu_\rho(\tau E) = \mu_\rho(E)$ for all $\tau \in T_\rho$. Then, if α has an immediate predecessor and $\sigma \in T\backslash T_\rho$, $\rho = \alpha - 1$, we put

$$\mu_\alpha(E) = \mathrm{Lim}_n\, \mu_\rho(\sigma^n E)$$

as in Proposition 8, for all $E \in \Sigma$ and $T_\alpha = T_\rho \cup \{\sigma\}$. If α is a limit ordinal we put

$$\mu_\alpha(E) = \lim_{\rho,\mathcal{U}} \mu_\rho(E) \qquad \text{and} \qquad T_\alpha = \bigcup_{\rho<\alpha} T_\rho \qquad (E \in \Sigma)$$

where $\mathcal{U}$ is an ultrafilter on $[1,\alpha)$ converging to α, i.e., containing the subsets $[\rho,\alpha)$ with $\rho < \alpha$.

It is easy to check that the measure $\mu_\alpha\colon \Sigma \to \mathbb{R}$ satisfies the identity $\mu_\alpha(\tau E) = \mu_\alpha(E)$ for all $\tau \in T_\alpha$ and $0 \le \mu_\alpha(E) \le \sup_{\tau\in T} \tau\mu(E)$ since T is a commutative group.

Finally, in the same way as in Proposition 8, it turns out that μ_α is a countably measure, and since there exists α such that $T_\alpha = T$, the theorem is obtained. $\square$

Now we are going to extend the proposition, with some restrictions, to non commutative group.

DEFINITION 2. A group G is called *perfect* [15] if for any finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_{i_0}\ \beta_1, \beta_2, \ldots, \beta_{j_0}\}$ of G we can find another subset of $G\,\{\xi_1, \xi_2, \ldots, \xi_{h_0}\,; \eta_1, \eta_2, \ldots, \eta_{k_0})$ such that a bijective mapping φ can be defined between the sets $\{(h,i)\colon 1 \le h \le h_0\,,\ 1 \le i \le i_0\}$ and $\{(k,j)\colon 1 \le k \le k_0\,,\ 1 \le j \le j_0\}$ in such a way that $\xi_h \alpha_i = \eta_k \beta_j$ whenever $(k,j) = \varphi(h,i)$.

It is easily seen that both commutative and finite groups are perfect.

DEFINITION 3. We will say that a family $\{G_\lambda\colon \lambda \in \Lambda\}$ of subsets of the group G is a *normal transfinite series* of G [15] if Λ is a well ordered set with last element verifying the following conditions:

1. $G_{\lambda_*} = \{e\}$ and $G_{\lambda^*} = G$, where e is the unit in G and λ_* and λ^* are first and last element of G respectively.

2. If λ is not a limit ordinal and $\lambda - 1$ is the immediate predecessor of λ, $G_{\lambda-1}$ is a normal divisor of G_λ.

3. If λ is a limit ordinal then $G_\lambda = \cup\{G_{\lambda'}\colon \lambda' < \lambda\}$.

The groups $G_{\lambda+1}/G_\lambda\ (\lambda < \lambda^*)$ are called the *quotient groups* of the transfinite normal series $\{G_\lambda\colon\ \lambda \in \Lambda\}$.

DEFINITION 4. A group G is said to be *accessible* (respectively *resoluble*) whenever there exists a normal transfinite series of G whose quotient groups are perfect (resp. commutative).

Theorem 10

Let us suppose the following:

(i) T is a group of bijective mappings from Ω onto Ω.
(ii) $X = L_\infty(\Sigma)$ (or $\mathcal{B}(\Sigma)$).
(iii) P is the set whose elements are the functions are the functions $p\colon X \to \mathbb{R}$ such that

$$p(x+y) \le p(x) + p(y) \qquad (x, y \in X)$$

and

$$p(c\,\tau x) = c\,p(x) \qquad (x \in X)$$

where $c \in \mathbb{R}^+$ and $\tau \in T$.
(iv) F is the set of linear functionals $f\colon \mathcal{D}(f) \to \mathbb{R}$, where $\mathcal{D}(f) \subseteq X$ is the domain of f, such that

$$f(c\,\tau x) = c f(x) \qquad \big(x \in \mathcal{D}(f)\big).$$

for all $c \in \mathbb{R}$ *and* $\tau \in T$ *in such a way that* $x \in \mathcal{D}(f)$ *implies* $\tau\, x \in \mathcal{D}(f)$.

Then, if $p_0 \in P$ *and* $f_0 \in F$ *verify* $f_0(x) \leq p_0(x)$ *for all* $x \in \mathcal{D}(f_0)$ *we can find two functions* $p \in P$ *and* $f \in F$ *such that:*

1. $p \leq p_0$ *and* $f \supseteq f_0$ *(the latter means that* f *is an extension of* f_0*).*
2. *For each* $x \in X$ *and* $y \in \mathcal{D}(f)$ *it follows that*

$$f(y) = p(x+y) - p(x) = p(y)\,.$$

3. *If* $q \in P$ *satisfies the inequality* $f_0(x) \leq q(x)$ *for all* $x \in \mathcal{D}(f_0)$ *and* $q \leq p$, *then* $q = p$.
4. *If* $g \in F$ *is such that* $g(x) \leq p(x)$ *for all* $x \in \mathcal{D}(g)$, *then* $g \subseteq f$.
5. *Finally, if* T *is either an accessible or resoluble group, then* $\mathcal{D}(f) = X$.

Proof. The result follows from [15, Theorems 4 and 8].

Theorem 11

If (μ, T) *is uniformly countably additive, where* T *is either an accessible or resoluble group, then there exists a countably additive probability measure* $\nu\colon \Sigma \to \mathbb{R}$ *invariant under the group* T *verifying*

$$0 \leq \nu(E) \leq \sup_{\tau \in T} \tau\mu(E)$$

for each $E \in \Sigma$.

Proof. Let

$$p_0(x) = \sup_{\tau \in T} \int_\Omega \bigl|\tau\mu\bigr|\, d\mu$$

and

$$f_0\bigl(c\,\chi_\Omega\bigr) = c\,,$$

where $x \in X = L_\infty(\Sigma)$, $c \in \mathbb{R}$ and $\mathcal{D}(f_0) = \{c\,\chi_\Omega\colon c \in \mathbb{R}\}$.

Then, by Theorem 10, there exists a linear function $f \supseteq f_0$, with $f \in F$, such that $f(y) = p(y) \leq p_0(y)$ for nay $y \in \mathcal{D}(f) = X$. Therefore, if $\nu(E) = f(\chi_E)$ $(E \in \Sigma)$, it follows that $\nu(\Omega) = 1$.

If $E_1, E_2 \in \Sigma$ are disjoint, then

$$\nu\bigl(E_1 \cup E_2\bigr) = f\bigl(\chi_{E_1 \cup E_2}\bigr) = f(\chi_{E_1}) + f\bigl(\chi_{E_2}\bigr) = \nu(E_1) + \nu(E_2)\,.$$

From this, and taking into account that $\nu(E) \le p_0(\chi(E)) \le 1$, it follows that

$$\nu(E) = 1 - \nu(\Omega - E) \ge 0$$

and

$$\nu(\tau E) = f(\tau\chi_E) = f(\chi_E) = \nu(E)$$

for $E \in \Sigma$ and $\tau \in T$.

The measure ν is countably additive. Indeed, it is finitely additive and if (E_n) is a sequence in Σ such that $E_n \searrow \emptyset$, we have $\lambda(E_n) \to 0$ and therefore

$$0 \le \nu(E_n) \le p_0(\chi_{E_n}) = \sup_{\tau\in T} \int_\Omega \left|\tau\chi_{E_n}\right| d\mu = \sup_{\tau\in T} \tau\mu(E_n) \to 0 . \ \square$$

Theorem 12

Let Ω be a topological group $G = T$, Σ the σ-algebra of the Borel subsets of G, $\mathcal{H}$ a class consisting of closed subsets of G, μ a Borel measure such that (μ, T) is uniformly countably additive and $\mu(G) = 1$. Then, if λ is the measure of Proposition 7 and ν is a T-invariant finite Borel measure which is $\mathcal{H}$-inner regular on G, and verifies that $\nu(E\backslash Z)$is a λ-continuous measure in Σ for a set $Z \in \Sigma$ such that $\nu(Z) > 0$ and $\lambda(Z) = 0$, and if in addition $\mathcal{H}_0$ is the class of the sets $H \in \mathcal{H}$ such that $H \subseteq Z$ and $\nu(H) > 0$, then we have:

(i) For any $H \in \mathcal{H}_0$ we have $ZH^{-1} = G$.

(ii) If $H \in \mathcal{H}_0$ and $(xH)_{x\in A}$ is a family of pairwise disjoint sets, then A is finite.

(iii) For each $H \in \mathcal{H}_0$ there exists a finite set $A \subseteq G$ such that $AHH^{-1} = G$. Therefore, if ν is a Radon measure, G is compact.

(iv) If Z is a countable union of compact sets, then int $HH^{-1} \ne \emptyset$ for all $H \in \mathcal{H}_0$. We have also that G is compact.

(v) If Z is a countable union of compact sets, the family $(HH^{-1}HH^{-1}\colon H \in \mathcal{H}_0)$ is a fundamental system of neighborhoods of the unit element e of G.

(vi) Z is a dense subset of G.

Proof. By Radon-Lebesgue theorem, there exists an integrable function f such that

$$\nu(E\backslash Z) = \int_E f d\lambda \quad \forall E \in \Sigma .$$

(i) Since $\nu(xH\backslash Z) = \nu(xZ\backslash Z) = 0$ for all $x \in G$ and $H \in \mathcal{H}_0$, then

$$\nu(xH \cap Z) = \nu(xH) - \nu(xH\backslash Z) = \nu(H) > 0 .$$

Therefore, $xH \cap Z \neq \emptyset$ for any $x \in G$ and $ZH^{-1} = G$.

(ii) Since $(xH \cap Z)_{x \in A}$ is a family of pairwise disjoint subsets of Z such that $\nu(xH \cap Z) = \nu(H) > 0$, then card $A \leq \nu(Z)/\nu(H)$.

(iii) It follows from (ii) that there exists a finite subset $A \subseteq G$ such that $AH \cap xH \neq \emptyset$ for any $x \in G$, therefore we have $AHH^{-1} = G$.

(iv) Since Z is σ-compact, every $H \in \mathcal{H}_0$ contains a compact set $H' \in \mathcal{H}_0$ and so we can assume that H is compact. Then it follows from (iii) that $G = \cup_{x \in A} xHH^{-1}$ with A finite and therefore there exists $x \in A$ such that int $xHH^{-1} \neq \emptyset$. As a result of this int $HH^{-1} \neq \emptyset$ and $G = AHH^{-1}$ is compact.

(v) If $H \in \mathcal{H}_0$ and $x \in$ int HH^{-1}, there exists a neighborhood V of e such that $xV \subseteq HH^{-1}$ and $V^{-1}x^{-1} \subseteq HH^{-1}$. Then $V^{-1}V \subseteq HH^{-1}HH^{-1}$ and $HH^{-1}HH^{-1}$ is a neighborhood of e. Then for each neighborhood V of e there exists $x \in Z$ and $H \in \mathcal{H}_0$ such that $H \subseteq xV$. Therefore $HH^{-1}HH^{-1} \subseteq xVV^{-1}VV^{-1}x^{-1}$ from which (v) follows.

(vi) It is straightforward that if $x \in G \backslash \bar{Z}$, then there exists a neighborhood V of e such that $xV \cap Z = \emptyset$. On the other hand, there exists $H \in \mathcal{H}_0$ and $x_0 \in Z$ such that $H \subseteq x_0 V$. We conclude that $xx_0^{-1}H \cap Z = \emptyset$ and $ZH^{-1} \neq G$, which contradict (i) (see, [11], [14] and [18]). □

As a result of this theorem, one can see that the fact that ν is not a λ-continuous measure imposes certain conditions. ν is λ-continuous or $Z = G$? On the other hand, Theorem 11, which is true for accessible groups, is probably false for arbitrary groups. On constructing Haar measures, the algebraic structure of the group does not exert any influence, which implies its existence for arbitrary groups. Note that this construction was extended in [17] to non locally compact groups in the following fashion:

Definition 5. A *topological measure* (resp. a *topological outer measure*) is a τ-additive Borel measure (resp. a τ-additive Borel outer measure) that in addition is locally finite, semifinite and outer regular.

A *Haar measure* (resp. a *Haar outer measure*) on a topological group, is a non-zero topological measure (resp. a non-zero topological outer measure) invariant under left translations [17].

In the particular case of locally compact group, this concept coincides with the usual one.

Theorem 13 [17, Theorem 1]

A topological group G has a Haar measure μ if and only if it is a dense subgroup of a locally compact group G_0 such that if μ_0^* is a Haar outer measure on G_0, then there exists an open set $U \subseteq G$ with $0 < \mu_0^*(U) < \infty$.

Theorem 14 [17, Theorem 2]

On a topological group there is at most one Haar measure (up to a constant).

Theorem 15 [17, Theorem 5]

If G is a topological group endowed with a Haar measure μ, then the two following classes of sets (V_1) and (V_2) form a fundamental system of neighborhoods of the unit element $e \in G$:

(V_1) $W_p(f, \varepsilon) = \{x \in G\colon \|xf - f\|_p < \varepsilon\}$ *for every* $\varepsilon > 0$ *and every function* $f \in L^p(\mu)$ $(1 \le p < \infty)$.

(V_2) AA^{-1} *for every μ-measurable subset A with* $0 < \mu(A) < \infty$.

Corollary 16 [17, Theorem 6]

Every Haar measure on a topological group G determines the topology of G in a unique way.

Remark. The additive group G of real numbers has two topologies that turn it into two locally compact groups with two different Haar measures. Indeed, one of them is the usual topology and the other is the discrete one. The corresponding Haar measures are then the Lebesgue measure and the one that associates each set with its numbers of elements (either finite or infinite).

The classic proof of the existence of non Lebesgue measurable sets on the real line using the axiom of choice works to prove that there are no ultracomplete countably additive probability measures on the unit circle, that is, defined on the set of all subsets of the unit circle T and invariant under rotations around the origin 0. However, according to Solovay's axiom, which is consistent with Zermelo-Fraenkel axiomatic, there exists an ultracomplete countably additive probability measure invariant under such rotations. In the same way in which Theorem 11 was proved using [15, Theorems 4 and 8], it follows that there exists an ultracomplete finitely additive probability measure on a set Ω invariant under a accessible group T.

On the other hand, if μ is an ultracomplete finitely additive (f.a.) probability measure on the unit sphere $S \subset \mathbb{R}^3$, we proved in [16] that for every $\varepsilon > 0$ there exists a subset $E \subset S$ and a rotation τ whose axis meets the origin, such that

$$\mu(E) < \varepsilon \quad \text{and} \quad \mu(\tau E) > 1 - \varepsilon\,.$$

Therefore, the group consisting of such rotations is neither accessible nor measurable according to von Neumann. Banach-Tarski's paradox [3] is nothing but another consequence of this fact.

So the following problems arise:

Problem 1. Let G be a group of permutations of a set Ω and μ_0 be a c.a. probability measure (resp. f.a. probability measure) defined on a σ-algebra (resp. an algebra) and invariant under G. Is the existence of an ultracomplete extension μ of μ_0 invariant under G consistent with Zermelo-Fraenkel axiomatics?

In case of countable additively, it is of course necessary that the cardinal of Ω to be real-measurable.

Problem 2. Give a necessary and sufficient condition on a group G^* in order the following holds: If μ_0 is an arbitrary f.a. measure (resp. c.a.) defined on an algebra (resp. σ-algebra) of subsets of a set (resp. real-measurable set) Ω that in addition is invariant under a permutations group G of Ω isomorphic to G^*, then there exists an ultracomplete f.a. (resp. c.a.) extension μ of μ_0 invariant under G. For the case of finite measurability we gave the solution in [16, Theorem 17], valid even for relatively invariant measures like the Lebesgue measure with respect to the group of similarities: The solution is that G^* must be measurable, i.e., there exists an ultracomplete f.a. probability measure on G^* invariant under the group of left translations in G^*.

In the same fashion as in Theorem 11 we have:

Theorem 17

If (μ, T) is uniformly countably additive and T is measurable group, then there exists a c.a. probability measure $\nu\colon \Sigma \to \mathbb{R}$ T-invariant with $0 \le \nu(E) \le \sup_{\tau\in T} \tau\mu(E)$ for any $E \in \Sigma$.

Proof. As T is measurable group, there exists a finitely additive measure $\mu_0 \ge 0$ with $\mu_0(T) = 1$ invariant under left translations on $\mathcal{P}(T)$. Then

$$\nu\colon\ \nu(E) = \frac{1}{\mu(\Omega)} \int_T \tau\mu(E)\, d\mu_0(\tau) \qquad (E \in \Sigma)$$

is a finitely additive measure with $\nu(\Omega) = 1$ that is also T-invariant since

$$\begin{aligned} \nu(\sigma E) &= \frac{1}{\mu(\Omega)} \int_T \tau\mu(\sigma E)\, d\mu_0(\tau) \\ &= \frac{1}{\mu(\Omega)} \int_T \sigma^{-1}\tau(E)\, d\mu_0(\tau) \\ &= \frac{1}{\mu(\Omega)} \int_T \tau\mu(E)\, d\mu_0(\tau) \\ &= \nu(E) \end{aligned}$$

for all $E \in \Sigma$ and $\sigma \in T$. If in addition $E_n \searrow \emptyset$, according to Proposition 7 we have $\sup_{\tau\in T} \tau\mu(E_n) \to 0$ and therefore $\nu(E_n) \to 0$. Then ν is a countably additive measure. $\Box$

Theorem 11 is a particular case of this theorem since by using [16, Corollary 16] any accessible group is measurable.

Remark. Conversely, if ν and μ_0 are countably additive and

$$F(E) = \big(\tau\mu(E)\big)_{\tau\in T} \in L^\infty(\mu_0) \qquad (E \in \Sigma)\,,$$

then by Pettis theorem [5, I.2, Theorem 1], it turns out that F is a ν-continuous measure if and only if F is countably additive. Indeed, if F is countably additive, then since $\nu(E) = 0$ implies $\tau\mu(E) = 0$ for almost every $\tau \in T$ and $F(E) = 0$, it follows that

$$\lim_{\nu(E)\to 0} F(E) = 0\,.$$

Conversely, since ν is countably additive, this identity implies that F is countably additive.

On the other hand, since any infinite group G contains a countable infinite subgroup, in contrast with Solovay's axiom, from the axiom of choice if can be deduced that any ultracomplete c.a. finite measure on an infinite group T, invariant by left translation, is zero.

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