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# Normability of an S-ring

Marhrani El-Miloudi and Aamri Mohamed

*Université Hassan II Mohammedia, Faculté des Sciences Ben M'sik, B.P. 7955 Casablanca, Maroc*

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Abstract

We give some criteria of normability of an S-ring, and we study the properties of its norms.

## 1. Introduction

Let $A$ be a Hausdorff topological ring with identity and without zero divisors. We assume that $A$ is commutative.

$A$ is called an S-ring if it contains a sequence $(\lambda_n)_n$ of invertible elements which converges to 0 in $A$, and a symmetric closed neighborhood $e$ of 0 such that

$$1 \in e, \quad ee \subset e, \quad \text{and} \quad ea \subset a \qquad (S)$$

for every $a$ belonging to a fundamental system of neighborhoods of 0 in $A$. The S-ring notion was introduced in [1] and [2] in order to generalize some results on the closed graph theorem to topological modules. Some properties of these rings have been studied in [2]. In this paper, we give some criteria of normability of an S-ring, and we study the properties of its norms.

## 2. Notations

In the sequel, we assume that $A$ is an S-ring and we use the above notations. We will denote by $I(A)$ the set of invertible elements of $A$ and by $R$ the set of its topological nilpotents.

If $N$ is a norm on $A$, we denote by $C(N)$ the core of $N$:

$$C(N) = \left\{x \in A \;/\; N(x\,y) = N(x)\,N(y),\ \forall y \in A\right\},$$

and by $N_s$ the semi-norm on $A$ defined by

$$N_s(x) = \inf_{p \geq 0} N(x^p)^{1/p}.$$

## 3. Properties of normability of an S-ring

**Proposition 3.1**

*Let $B$ be a topological ring with identity and without zero divisors. We assume that $B$ is Hausdorff and commutative.*

*If $B$ contains a sequence of invertible elements which converges to 0, then $B$ is an S-ring if and only if 0 possesses a bounded neighborhood.*

*Proof.* If $B$ is an S-ring, it contains a bounded neighborhood of 0, by definition.

Conversely, let $V$ be a bounded neighborhood of 0 in $B$, $W = (V \cap (-V)) \cup \{-1, 1\}$ and $e = \{x \in B \;/\; x\,W \subset W\}$. $e$ is a neighborhood of 0 in $B$ which satisfies the condition $(S)$. □

**Proposition 3.2**

*The following assertions are equivalent*

1. *$A$ is normed.*
2. *$A$ contains an invertible topological nilpotent.*

*Proof.* 1. $\Rightarrow$ 2. If $N$ is a norm on $A$, there exists $n \in \mathbb{N}$ such that $N(\lambda_n) < 1$ and then $\lambda_n$ is an invertible topological nilpotent.

2. $\Rightarrow$ 1. Let $c$ be an invertible topological nilpotent in $A$; there exists $n_0 \in \mathbb{N}$ such that $c^{n_0}e + c^{n_0}e \subset e$.

Let $\alpha = c^{n_0}$ and $V_n = \alpha^{2\,n}e$; $(V_n)_{n\in\mathbb{Z}}$ is a fundamental system of neighborhoods of 0 in $A$, and we have $V_{n+1} + V_{n+1} \subset V_n$ for all $n \in \mathbb{Z}$. Moreover, we have $A = \cup_{n\in\mathbb{Z}} V_n$.

We define the mapping $g$ on $A$ by

$$\begin{cases} g(0) = 0\,, \\ g(x) = 2^{-n}\,, \quad \text{if } x \in V_n - V_{n+1}\,. \end{cases}$$

Let $N$ defined on $A$ by,

$$N(x) = \inf\left\{g(x_1) + \cdots + g(x_n) \;/\; x_1 + \cdots + x_n = x\right\},$$

$N$ is a norm on $A$, and since $V_n \subset N^{-1}([0, 2^{-n}]) \subset V_{n-1}$, for all $n \in \mathbb{N}$, $N$ defines the topology of $A$. $\square$

In the following, if $A$ is normed we denote by $N$ the norm defined as in the preceding proof.

**Proposition 3.3**

*Let $B$ be a commutative Hausdorff topological ring with identity and without zero divisors. The following assertions are equivalent*

1. *$B$ is an normed S-ring.*
2. *There exists a norm $M$ on $B$ such that $C(M)$ contains an invertible topological nilpotent.*

*Proof.* 1. $\Rightarrow$ 2. Let $e$ be a neighborhood of 0 in $B$ satisfying the condition $(S)$ and let $\alpha \in B$ defined as in the proof of the Proposition 3.2.

If $\alpha^{-1} \in e$, $(\alpha^{-n})_n$ would be bounded which is impossible; then $\alpha^{-1} \notin e$, which gives $1 \notin \alpha\, e$ and $\alpha^2 \in \alpha^2\, e - \alpha^4\, e = V_1 - V_2$. Consequently, $g(\alpha^2) = \frac{1}{2}$.

Let $d = \alpha^2$, $d$ is an invertible topological nilpotent and we have $g(d\,x) = g(d)\,g(x)$, for all $x \in B$. Indeed, if $x \in V_n - V_{n+1}$ we have $d\,x \in d\alpha^{2\,n}e - d\alpha^{2(n+1)}e = V_{n+1} - V_{n+2}$; and then $g(d\,x) = 2^{-(n+1)} = 2^{-1}\,g(x) = g(d)g(x)$.

If $x_1, \ldots, x_n \in B$ such that $x_1 + \cdots + x_n = d\,x$, we have $x = d^{-1}\,x_1 + \cdots + d^{-1}x_n$ which gives $N(x) \le g(d^{-1}x_1) + \cdots + g(d^{-1}x_n) = g(d^{-1})(g(x_1) + \cdots + g(x_n))$ and then, $N(x)\,N(d) \le g(x_1) + \cdots + g(x_n)$. Hence, $N(x)N(d) \le N(d\,x)$.

2. $\Rightarrow$ 1. Let $e = \{x \in B \;/\; M(x) \le 1\}$, $a_n = \{x \in B \;/\; M(x) \le \frac{1}{n}\}$ and $d$ be an invertible topological nilpotent which belongs to $C(M)$. We have $e\,a_n \subset a_n$, for all $n \in \mathbb{N}^\star$. Moreover, $(d^n)_n$ is a sequence of invertible elements which converges to 0. Then $B$ is an normed S-ring. $\square$

*Remark 3.4.* If $A$ is normed then $N(1) = 1$. Moreover, there exists an invertible topological nilpotent $d$ in $C(N)$ such that $e \subset \{x \in A \ / \ N(x) \leq 1\} \subset d^{-1}\, e$.

Indeed, for the last assertion one can see the proofs of the Proposition 3.2 and 3.3. We have $N(d) = N(d.1) = N(d)N(1)$ and then, $N(1) = 1$.

**Proposition 3.5**

*If $A$ is normed we have*

$$\big\{x \in A \,/\, N(x) < 1\big\} \subset e \subset \big\{x \in A \,/\, N(x) \leq 1\big\}\,.$$

*Proof.* We have already seen that $e \subset \{x \in A \ / \ N(x) \leq 1\}$. If $x \in A$ such that $N(x) < 1$ , there exists $x_1, \ldots, x_p \in A$ with $x = x_1 + \cdots + x_p$ and $g(x_1) + \cdots + g(x_p) < 1$.

Let $n_i \in \mathbb{Z}$ be such that $g(x_i) = 2^{-n_i}$. Since, $g(x_i) < 1$ we have $n_i > 0$ and thus $x = x_1 + \cdots + x_p \in \alpha^{2n_1}\, e + \cdots + \alpha^{2n_p}\, e \subset e$. □

**Corollary 3.6**

*Assume that $A$ is normed. Then for all $d \in C(N) \cap I(A)$ we have*

$$\big\{x \in A \ / \ N(x) < N(d)^n\big\} \subset d^n\, e \subset \big\{x \in A \ / \ N(x) \leq N(d)^n\big\} \quad \textit{for all} \quad n \in \mathbb{N}.$$

*Proof.* If $N(x) < N(d)^n$, we have $N(x\, d^{-n}) < 1$ which gives $xd^{-n} \in e$ and then, $x \in d^n\, e$.

Conversely, if $x \in d^n\, e$, we have $x\, d^{-n} \in e$ and thus $N(x\, d^{-n}) = N(x)N(d)^{-n} \leq 1$. Consequently, $N(x) \leq N(d)^n$. □

**Corollary 3.7**

*We have the inclusion:*
$b(e) \subset \{x \in A \ / \ N(x) = 1\}$, *where $b(e)$ is the boundary of $e$.*

*Proof.* Let $x \in b(e)$. Since $e$ is closed, we have $x \in e$ and then $N(x) \leq 1$. If $N(x) < 1$, $x$ is an interior element of $e$, which is impossible. Consequently, $N(x) = 1$. □

**Proposition 3.8**

*$A$ is normed if and only if there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\lambda_n$ is a topological nilpotent.*

**Proposition 3.9**

*$A$ is normed if and only if $R$ is a neighborhood of 0.*

*Proof.* If $A$ is normed, the set $\{x \in A \,/\, N(x) < 1\}$ is a neighborhood of 0 contained in $R$. Hence, $R$ is a neighborhood of 0.

Conversely, if $R$ is a neighborhood of 0, there exists $n_0$ such that $\lambda_{n_0}$ belongs to $R$. Then, $A$ contains an invertible topological nilpotent and therefore it is normed. □

**Proposition 3.10**

*Assume that $A$ is normed and there exists a sequence $(x_n)_n$ of elements of $A$ which converges to 1 such that $N(x_n) < 1$, for all $n$. The following assertions hold*

1. $\{x \in A \,/\, N(x) < 1\}$ *is dense in* $\{x \in A \,/ N(x) \leq 1\}$.
2. $e = \{x \in A \,/\, N(x) \leq 1\}$.
3. $b(e) = \{x \in A \,/\, N(x) = 1\}$ *if and only if* $\mathrm{Int}(e) = \{x \in A \,/\, N(x) < 1\}$, *where* $\mathrm{Int}(e)$ *is the interior of* $e$.
4. *$R$ is dense in* $\left\{x \in A \,/\, N_s(x) \leq 1\right\}$.

*Proof.* 1. Let $x \in A$ be such that $N(x) \leq 1$; the sequence $(x_n x)_n$ converges to $x$ and, $N(x_n x) \leq N(x_n)N(x) < 1$ which gives the required result.

2. follows from 1.

3. follows from 2.

4. We have $\overline{R} \subset \left\{x \in A \,/\, N_s(x) \leq 1\right\}$. If $N_s(x) \leq 1$, we have $N_s(x_n x) < 1$, for all $n$ and the sequence $(x_n x)_n$ converges to $x$. Then, $x \in \overline{R}$. □

**Proposition 3.11**

*The following assertions are equivalent*

1. $1 \in \overline{R}$.
2. $e \subset \overline{R}$.
3. $\left\{x \in A \,/\, (x^n)_n \text{ is bounded}\right\} \subset \overline{R}$.

*Proof.* 2. ⇒ 1. Trivial.

1. ⇒ 3. Let $x \in A$ be such that $(x^n)_n$ is bounded. Since $1 \in \overline{R}$, there exists a sequence $(x_n)_n$ of elements of $R$ which converges to 1; $x_n\, x \in R$ for all $n$; and the sequence $(x_n x)_n$ converges to $x$. Hence, $x \in \overline{R}$.

3. ⇒ 2. If $x \in e$, the sequence $(x^n)_n$ is bounded and then $x \in \overline{R}$. □

**Proposition 3.12**

*Assume that the topology of $A$ is given by a spectral semi-norm. Then, $1 \in \overline{R}$ if and only if $\overline{R} = \{x \in A \ / \ (x^n)_n$ is bounded$\}$.*

*Proof.* We have $x \in \bar{R}$ if and only if $N_s(x) \leq 1$. Then

$$\bar{R} = \{x \in A \ / \ N_s(x^n) \leq 1 \ \forall n \in \mathbb{N}\} \ ;$$

which gives the required result. □

**Proposition 3.13**

*Assume that $A$ is complete and $1 \in \overline{R}$. $A$ is normed if and only if $I(A)$ is open.*

*Proof.* It is well known that if $A$ is complete and normed, $I(A)$ is open.

Conversely, if $I(A)$ is open, we have $I(A) \cap R \neq \emptyset$. And then, $A$ contains an invertible topological nilpotent. Consequently, it is normed. □

**Proposition 3.14**

*Let $x \in R$ be such that $x\,(e+e) \subset e$. Then, $(\sum_{k=0}^{n} x^k)_n$ is a Cauchy sequence in $A$.*

*Proof.* Let $S_n = 1 + x + \cdots + x^n$. Since, $x\,(e+e) \subset e$, we have $S_n \in e+e$, for all $n \in \mathbb{N}$; and then, $(S_n)_n$ is bounded.

Let $p > q$ it is easy to see that

$$\sum_{k=q}^{p} x^k = x^q S_{p-q} \,. \tag{3.1}$$

Since $e+e$ is bounded there exists $n_0 \in \mathbb{N}$ such that

$$\lambda_{n_0}\, e\,(e+e) \subset \lambda_n\, e \,. \tag{3.2}$$

On the other hand, there exists $N \in \mathbb{N}$ such that $x^q \in \lambda_{n_0}\, e$, for all $q > N$. Combining 3.1 and 3.2 we obtain $\sum_{k=q}^{p} x^k \in \lambda_n\, e$, for all $p > q > N$. Which gives the required result. □

**Corollary 3.15**

*Assume that $A$ is complete. Then for all $x \in R$ such that $x\,(e+e) \subset e$, $1-x$ is invertible and we have $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$.*

*Proof.* By Proposition 3.14, $(\sum_{k=0}^{n} x^k)_n$ is a Cauchy sequence; then it is convergent in $A$.

On the other hand we have $(1-x)\sum_{k=0}^{n} x^k = 1 - x^{n+1}$, which converges to 1. Then, $1-x$ is invertible and we have $(1-x)^{-1} = \displaystyle\sum_{k=0}^{\infty} x^k$. □

**Corollary 3.16**

*Assume that $A$ is complete. Then for all $x \in R$, $1-x$ is invertible in $A$.*

*Proof.* Since $x \in R$ there exists $n \in \mathbb{N}^\star$ such that $x^n\,(e+e) \subset e$; and by Corollary 3.15 $1-x^n$ is invertible in $A$. Consequently there exists $y \in A$ such that $y(1-x^n) = 1$. Which completes the proof. □

Definition 3.17. A topological ring $B$ is called sequentially retrobounded if for every sequence $(x_n)_n$ of invertible elements of $B$, $(x_n)_n$ converges to 0 or $(x_n^{-1})_n$ possesses a bounded subsequence.

Example 3.18: Every valued ring is sequentially retrobounded.

**Lemma 3.19**

*$A$ is locally compact if and only if $e$ is compact.*

*Proof.* Assume that $A$ is locally compact and, let $V$ be a compact neighborhood of 0 in $A$. There exists $n \in \mathbb{N}$ such that $\lambda_n\, e \subset V$; then $e \subset \lambda_n^{-1}\, V$, and since $e$ is closed, it is compact.

Conversely, if $e$ is compact, $A$ is trivially locally compact. □

**Proposition 3.20**

*Assume that $A$ is locally compact and sequentially retrobounded. Then it is normed.*

*Proof.* Let $c \in e \cap I(A)$ be such that $c^{-1} \notin e$. For all natural integer $n$ we have $c^n \in e$, and since $e$ is compact, there exists a subsequence $(c^{n_k})_k$ of $(c^n)_n$ which converges to an element $\alpha \in e$.

We claim that $\alpha = 0$. Indeed, if $\alpha \neq 0$, $(c^{-n_k})_k$ contains a bounded subsequence. And since $A$ is locally compact, it contains a subsequence which converges to some $\gamma \in A$. We have $\alpha\,\gamma = 1$, and then $\alpha$ is invertible.

Let $m_k = n_{k+1} - n_k$ (we may assume that $m_k > k$ for all $k$), we have $c^{m_k} \in e$, for all $k \in \mathbb{N}^*$ and $(c^{m_k})_k$ converges to 1. Let $y \in e$, we have $y\,c^{m_k} \in c\,e$ and $(yc^{m_k})_k$ converges to $y$. Hence, $y \in c\,e$. Consequently, $e = c\,e$, which is impossible since $c^{-1} \notin e$. It follows that $\alpha = 0$.

On the other hand, we have $c^n = c^{n-n_k}.c^{n_k}$ which converges to 0 since $(c^{n-n_k})_{n-n_k}$ is bounded. Consequently, $c$ is an invertible topological nilpotent; and by Proposition 3.2 $A$ is normed. □

**Corollary 3.21**

*Assume that every sequence $(x_n)_n$ of invertible elements of $A$ converges to 0 or $(x_n^{-1})_n$ contains a convergent subsequence. Then $A$ is normed.*

*Proof.* $A$ is sequentially retrobounded. Let us prove that $A$ is locally compact.

Let $(x_n)_n$ be a sequence of elements of $e$ which does not converge in $A$; then $(x_n^{-1})_n$ contains a subsequence $(x_{n_k}^{-1})_k$ which converges to some $\alpha$ in $A$. We have $\alpha \neq 0$; and consequently, $(x_n)_n$ contains a subsequence which converge in $e$. It follows that $e$ is compact. □

**Lemma 3.22**

*Assume that for all $x$ in $I(A)$, $x \in e$ or $x^{-1} \in e$. Then $A$ is sequentially retrobounded.*

*Proof.* Assume that $(x_n^{-1})_n$ contains a subsequence $(x_{n_k}^{-1})_k$ which is not bounded. Then for every $p$, there exists a subsequence $(x_{p_j}^{-1})_j$ of $(x_{n_k}^{-1})_k$ such that $x_{p_j}^{-1} \notin \lambda_p^{-1}\,e$ for all $j$. Hence, $\lambda_p^{-1}x_{p_j} \in e$ i.e. $x_{p_j} \in \lambda_p\,e$. Consequently, every neighborhood of 0 in $A$ contains a subsequence of $(x_n)_n$; we conclude then that 0 is an accumulation point of $(x_n)_n$. □

**Proposition 3.23**

*Assume that $A$ is locally compact and that for every $x \in I(A)$, $x \in e$ or $x^{-1} \in e$. Then $A$ is normed.*

*Proof.* It suffices to use Lemma 3.22 and Proposition 3.20. □

**Proposition 3.24**

*Assume that $A$ is locally compact and, for all $x, y \in A$ such that $xy \in e$ we have $x \in e$ or $y \in e$. Then the topology of $A$ is given by a spectral norm.*

*Moreover, $e = \overline{R}$ if and only if $1 \in \overline{R}$.*

*Proof.* Let $x \in I(A)$. Since $xx^{-1} = 1 \in e$, we have $x \in e$ or $x^{-1} \in e; \forall$ and then by Proposition 3.23 $A$ is normed.

Let $x \in R$, there exists $n \in \mathbb{N}^\star$ such that $N(x^n) < 1$, which gives, $x^n \in e$. And from the hypotheses we obtain $x \in e$. It follows that $R \subset e$. Consequently $R$ is bounded and $N$ is equivalent to $N_s$.

Assume that $1 \in \overline{R}$. By Proposition 3.11 we have $e \subset \overline{R}$ and then $e = \bar{R}$. □

**Proposition 3.25**

*Assume that $A$ is normed and let $M$ be a spectral norm which is equivalent to $N$ on $A$. The following assertions are equivalent*

1. *$C(M) \cap C(N)$ contains an invertible topological nilpotent.*
2. *There exists $\alpha > 0$ such that $M = N_s^\alpha$.*
3. *$C(M) = C(N_s)$.*

*Proof.* 2. $\Rightarrow$ 3. Trivial.

3. $\Rightarrow$ 1. $C(N) \cap C(M) = C(N) \cap C(N_s) = C(N)$ which contains an invertible topological nilpotent.

1. $\Rightarrow$ 2. Let $c$ be an invertible topological nilpotent which belongs to $C(N) \cap C(M)$, and let $d = c^{-1}$. We have $N_s(d) > 1$.

Let $\alpha = (\ln M(d))(\ln N_s(d))^{-1}$. For all $x \in A - \{0\}$, there exists $s \in \mathbb{R}$ such that $M(x) = M(d)^s$. Let $r \in \mathbb{Q}$, we have $r = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}^\star$. If $r > s$, we have $M(x) < M(d)^{p/q}$ and, $M(x)M(d)^{-p/q} < 1$ which gives $M(x^p d^{-q}) < 1$. Hence, $N_s(x^p d^{-q}) < 1$ and then $N_s(x) < N_s(d)^r$.

If $r < s$, we have $M(x) > M(d)^r = M(d)^{p/q}$ and then $M(x^p d^{-q}) > 1$. Hence, $N_s(x^p d^{-q}) > 1$. It follows that $N_s(x) > N_s(d)^r$. We conclude by density of $\mathbb{Q}$ in $\mathbb{R}$ that $N_s(x) = N_s(d)^s$. Consequently, $M(x) = N_s(x)^\alpha$, which gives the required result. □

**Corollary 3.26**

*Assume that $A$ is normed and let $v$ be an absolute value on $A$ which is equivalent to $N$ on $A$. Then, $N_s$ is an absolute value on $A$.*

*Moreover, there exists $r > 0$ such that $N_s = v^r$ and $N$ is equivalent to $N_s$.*

*Proof.* $v$ is a spectral norm on $A$; and $C(N) \cap C(v) = C(N)$ which contains an invertible topological nilpotent. By Proposition 3.25 there exists $r > 0$ such that $N_s = v^r$. Consequently, $N_s$ is an absolute value.

The last assertion is trivial. □

**Corollary 3.27**

*If $v_1$ and $v_2$ are two equivalent absolute values on $A$, there exists $r > 0$ such that $v_2 = v_1^r$.*

**Proposition 3.28**

*Assume that $I(A)$ is open. If every ideal of $A$, which is different from $\{0\}$, is dense in $A$ then $A$ is a division ring.*

*Proof.* Let $x \in A - \{0\}$ and $a = \{\alpha\, x \;/\; \alpha \in A\}$. $a$ is an ideal of $A$ which is different from 0. Then, $a$ is dense in $A$. Since $I(A)$ is a non empty open subset of $A$, we have $I(A) \cap a \neq \emptyset$. Hence, $a$ contains an invertible element of $A$. Consequently, $a = A$ and then $x$ is invertible. Which finishes the proof. □

**Corollary 3.29**

*Assume that $A$ is normed, complete and that every ideal which is different from $\{0\}$ in $A$ is dense in $A$. Then $A$ is a division ring.*

*Proof.* Since $A$ is normed and complete, $I(A)$ is a non empty open set. □

**Proposition 3.30**

*Assume that $A$ is locally compact and that its topology is given by a spectral norm $M$. Then $A$ is a division ring.*

*Proof.* Let $c$ be an invertible topological nilpotent in $C(N)$; $c$ is also in $C(N_s)$. On the other hand, $M$ and $N_s$ have the same topological nilpotents. Then, $M$ and $N_s$ are equivalent. So, we may assume that $M = N_s$.

Let $a \in A$ be such that $N_s(a) = 1$. The sequence $(a^n)_n$ is bounded, and then it contains a subsequence $(a^{n_k})_k$ which converges to some $\alpha \in A$.

We may assume that $m_k = n_{k+1} - n_k > k$, for all $k$.

We have $N_s(a^n) = (N_s(a))^n = 1$, for all $n \in \mathbb{N}$. Then, $N_s(\alpha) = 1$; and consequently, $\alpha \neq 0$. $(a^{m_k})_k$ contains a subsequence $(a^{m_{k_j}})_j$ which converges to some $\gamma \in A$.

Since $a^{n_{k+1}} = a^{n_{k+1}-n_k}\, a^{n_k} = a^{m_k} a^{n_k}$, we have, $a^{n_{k_j+1}} = a^{m_{k_j}} a^{n_{k_j}}$; consequently, $\alpha = \alpha\, \gamma$. And then $\gamma = 1$.

$A$ is locally compact then it is complete and, since it is normed, $I(A)$ is open. It follows that there exists $j \in \mathbb{N}$ such that $a^{m_{k_j}} \in I(A)$ and then, $a$ is invertible.

Let $b \in A$ be such that $N_s(b) > 1$ and $x \in A - \{0\}$.

If $N_s(x) < 1$, there exists two subsequences $(n_k)_k$ and $(m_k)_k$ of natural integers such that $\lim_{k\to+\infty} N_s(x)^{n_k} N_s(b)^{m_k} = 1 = \lim_{k\to+\infty} N_s(x^{n_k} b^{m_k})$.

Consequently, $(x^{n_k} b^{m_k})_k$ contains a subsequence $(x^{n_{k_j}} b^{m_{k_j}})_j$ which converges to some $y \in A$. We have $N_s(y) = 1$; then, $y$ is invertible and, since $I(A)$ is open, there exists $j \in \mathbb{N}$ such that $x^{n_{k_j}} b^{m_{k_j}} \in I(A)$. It follows that $x$ is invertible.

If $N_s(x) \geq 1$, there exists $m \in \mathbb{N}$ such that $N_s(\lambda_n\, x) < 1$. Then, $\lambda_n\, x$ is invertible. Which finishes the proof. □

**Corollary 3.31**

*Assume that $A$ is locally compact. Then $A$ is valued if and only if it is a division ring.*

*Proof.* It is well known that every locally compact division ring is valued.

Conversely, if $A$ is valued, then it is a division ring by Proposition 3.30. □

**Corollary 3.32**

*Assume that $A$ is normed. If $(A, N_s)$ is locally compact then, $A$ is a division ring and $N_s$ is an absolute value.*

*Proof.* $(A, N_s)$ is an S-ring and its topology is given by a spectral norm. Then, $A$ is a division ring by Proposition 3.30, and since $(A, N_s)$ is locally compact, it is valued. We conclude that $N_s$ is an absolute value. □

**Proposition 3.33**

*Assume that $A$ is normed. The following assertions are equivalent*

1. $e = \overline{R}$.
2. $1 \in \overline{R}$; *and for all $x \in A$, such that $x^n \in e$ for some $n \in \mathbb{N}^*$, we have $x \in e$.*

*Proof.* 1. $\Rightarrow$ 2. We have $1 \in \overline{R}$ and since $R$ is bounded, $N$ and $N_s$ are equivalent. Moreover, we have $\overline{R} = \{x \in A \ / \ N_s(x) \le 1\}$.

If $x^n \in e$, then $N_s(x^n) \le 1$; and then, $N_s(x) \le 1$. It follows that $x \in \overline{R} = e$.

2. $\Rightarrow$ 1. Let $x \in R$. There exists $n \in \mathbb{N}$ such that $N(x^n) < 1$. On the other hand, we have $e \subset \{x \in A \ / \ N(x) \le 1\} \subset \{x \in A \ / \ N_s(x) \le 1\} = \overline{R}$, which gives the required result. □

**Proposition 3.34**

*Let $B$ be a locally convex unitary real algebra. $B$ is an S-ring if and only if $B$ is normed.*

*Proof.* Let $N$ be a norm on $B$, and set

$$a_n = \left\{x \in A \ / \ N(x) \le \frac{1}{n}\right\} \quad \text{and} \quad e = \left\{x \in A \ / \ N(x\,y) \le 1 \ \forall y \in a_1\right\}.$$

$(a_n)_n$ is a fundamental system of neighborhoods of 0 in $B$; moreover, $e$ satisfies the condition $(S)$ with $\lambda_n = \frac{1}{n}$.

Conversely, assume that $B$ is an S-ring and let $e$ be a neighborhood of 0 satisfying the condition $(S)$.

Let $F$ be the absolutely convex hull of $e$ and $p$ the gauge of $F$.

As $F$ is absolutely convex, bounded and absorbent, $p$ is a norm on the vector space $B$. Moreover, we have $F = \{x \in B \ / \ p(x) \le 1\}$. Then $p$ defines the topology of $B$.

On the other hand, if $p(x) = \alpha$ and $p(y) = \beta$, we have $p(\alpha^{-1}\,x) = p(\beta^{-1}\,y) = 1$. Then, $x \in \alpha\,F$ and $y \in \beta\,F$. Consequently, $x\,y \in \alpha\beta F$. Which gives $p(x\,y) \le \alpha\,\beta = p(x)\,p(y)$. □

## 4. Case where $A$ is a division ring

In this section we assume that $A$ is a division ring.

**Proposition 4.1**

*The following assertions are equivalent*

1. *$A$ is normed.*
2. *$R$ is open.*
3. *$R$ is different from $\{0\}$.*

DEFINITION 4.2. $A$ is said to be locally retrobounded if there exists a fundamental system $(V_n)_n$ of neighborhoods of 0 in $A$, such that $(A - V_n)^{-1}$ is bounded for all $n$.

**Proposition 4.3**

*$A$ is locally retrobounded if and only if there exists a neighborhood $V$ of 0, which is bounded and such that $(A - V)^{-1}$ is bounded.*

**Proposition 4.4**

*The following assertions are equivalent*

1. *$(A - e)^{-1} \subset e$.*
2. *For all $x, y \in A$ such that $xy \in e$ we have $x \in e$ or $y \in e$.*
3. *For all $x \in A - \{0\}$ we have $x \in e$ or $x^{-1} \in e$.*

*Proof.* 2. $\Rightarrow$ 3. If $x \notin e$, then $x^{-1} \in e$, since $x\,x^{-1} = 1 \in e$.

3. $\Rightarrow$ 2. If $x\,y \in e$ and $x \notin e$, we have $x^{-1} \in e$; and then, $y = x^{-1}\,(x\,y) \in e$.

1. $\Rightarrow$ 3. If $x \notin e$, we have $x \in A - e$. Hence, $x^{-1} \in e$.

3. $\Rightarrow$ 1. If $x \in (A - e)^{-1}$, we have $x^{-1} \notin e$ which gives $x \in e$. □

**Proposition 4.5**

*$A$ is valued if and only if $A$ is normed and locally retrobounded.*

*Proof.* If $A$ is valued, it is normed. Let now $v$ be an absolute value on $A$ and set $W = \{x \in A \ / \ v(x) \leq 1\}$. $W$ is a bounded neighborhood of 0.

If $x \notin W$, we have $v(x) > 1$ and $v(x^{-1}) < 1$. It follows that $x^{-1} \notin W$. Consequently, $(A - W)^{-1} \subset W$ and then $A$ is locally retrobounded.

Conversely, if $A$ is normed it contains an invertible topological nilpotent and since it is locally retrobounded, $A$ is valued (see for instance [4] Th.19.14). □

Let $P = \{x \in A - \{0\} \ / \ x \notin R \text{ and } \ x^{-1} \notin R\}$.

**Proposition 4.6**

*If $R \cup P = e$ then $A$ is valued.*

*Proof.* If $R = \{0\}$, we have $P \cup \{0\} = e = \{x \in A - \{0\} \ / \ x \notin R \textit{ and } x^{-1} \notin R\} \cup \{0\} = A$; which is impossible. Then, $R \neq \{0\}$ and $A$ is normed.

If $x \notin e$ we have $x \notin R \cup P$ and then, $x^{-1} \in R \subset e$. We conclude that $A$ is locally retrobounded. And then it is valued. □

**Proposition 4.7**

*Assume that $1 \in \overline{R}$. The following assertions are equivalent*

1. $(A - e)^{-1} \subset e$.
2. *There exists an absolute value $v$ on $A$ such that $e = \{x \in A \ / \ v(x) \leq 1\}$.*
3. $e = R \cup P$.
4. $e = \overline{R}$ *and $P$ is bounded.*

*Proof.* Since $1 \in \overline{R}$, we have $R \neq \{0\}$ and then $A$ is normed.

1. $\Rightarrow$ 2. $A$ is locally retrobounded and normed then it is valued. If $v$ is an absolute value on $A$, we have

$$R = \{x \in A \ / \ v(x) < 1\} \quad \text{and} \quad P \cup R = \{x \in A \ / \ v(x) \leq 1\}.$$

Let us prove that $e = \{x \in A \ / \ v(x) \leq 1\}$. We have $e \subset \{x \in A \ / \ v(x) \leq 1\}$. If $v(x) < 1$ and $x \notin e$ we obtain $x^{-1} \in e$, and then $v(x^{-1}) \leq 1$. Hence, $v(x) \geq 1$ which is impossible. It follows that $\{x \in A \ / \ v(x) < 1\} \subset e$; and since $1 \in \overline{R}$, we have $\{x \in A \ / \ v(x) \leq 1\} \subset \overline{e} = e$, which gives the required result.

2. $\Rightarrow$ 3. We have $e = R \cup P = \{x \in A \ / \ v(x) \leq 1\}$.

3. $\Rightarrow$ 1. If $x \notin e$ we have $x \notin R \cup P$ and then $x^{-1} \in R$. Hence, $x^{-1} \in e$.

3. $\Rightarrow$ 4. $1 \in \overline{R}$ then $e \subset \overline{R}$ and, since $e = R \cup P$ we have $e = \{x \in A \ / \ v(x) \leq 1\} \subset \overline{R}$. Or, $\overline{R} \subset \{x \in A \ / \ v(x) \leq 1\} = e$. Then $e = \overline{R}$.

4. $\Rightarrow$ 1. $R$ is a neighborhood of 0 and $R \cup P$ is bounded then $A$ is valued. Let $v$ be an absolute value on $A$, we have $e = \overline{R} = \{x \in A \ / \ v(x) \leq 1\}$. If $x \notin e$, we have $v(x) > 1$ and then $v(x^{-1}) < 1$. It follows that $x^{-1} \in R \subset e$ and hence, $(A - e)^{-1} \subset e$. □

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