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Ample vector bundles with sections vanishing along conic fibrations over curves

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Abstract

Let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on a complex projective manifold X of dimension $n = r+2$ having a section whose zero locus is a smooth surface Z. Pairs $(X, \mathcal{E})$ as above are classified under the assumption that (Z, L_Z) is a conic fibration over a smooth curve for some ample line bundle L on X.

0. Introduction

In the last 20 years the study of special varieties as ample divisors attracted the attention of many authors. This subject grows up from the classical studies concerning hyperplane sections.

Recently Lanteri and Maeda reconsidered this theme looking at smooth submanifolds defined through vanishing of sections of ample vector bundles. More precisely they discussed the following problem: let X be a smooth complex projective n-fold and let $\mathcal{E}$ be an ample vector bundle on X of rank $r \leq n-1$, satisfying the following assumption:

$(*)$ *There exists a section $s \in \Gamma(X, \mathcal{E})$ whose zero locus $Z = (s)_0$ is a smooth submanifold of X of dimension $n - r$.*

A generalization of the Theorem of Bertini [10, Thm. 3.8] shows that such an assumption is satisfied even for the general section of $\mathcal{E}$, if $\mathcal{E}$ is also spanned.

The aim of Lanteri and Maeda was to examine how the geometry of Z can condition the geometry of X. In particular they classified the pairs $(X, \mathcal{E})$ in some cases in which Z is a given "special" variety. For their results we refer to [5-8].

In this paper we focus on the situation above, assuming that Z is a surface which is a conic fibration through some polarization on X. More precisely our goal is to prove the following Theorem:

Theorem

Let X be a smooth complex projective manifold of dimension $n \geq 4$, and let $\mathcal{E}$ be an ample vector bundle of rank $n-2$ on X, such that the assumption $(*)$ is satisfied. Assume that there is an ample line bundle L on X whose restriction $l := L_Z$ gives (Z, l) the structure of a conic fibration over a smooth curve B. Then the pair $(X, \mathcal{E})$ is one of the following:

(a) *(X, L) is a scroll over B, and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}$ for every fiber F $(\cong \mathbb{P}^{n-1})$.*

(b) *(X, H) is a scroll over B for some ample line bundle $H \in \mathrm{Pic}(X)$, $L_F = 2H_F \cong \mathcal{O}_{\mathbb{P}}(2)$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber $F(\cong \mathbb{P}^{n-1})$, except when $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$, in which case (X, L) is a scroll over $\mathbb{P}^1$, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber $F(\cong \mathbb{P}^{n-1})$ and $l \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1)$.*

(c) *(X, L) is a hyperquadric fibration over B, and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)}$ for every general fiber F $(\cong \mathbb{Q}^{n-1})$.*

Moreover case (b) *can occur only if Z is a geometrically ruled surface over B.*

We can compare this Theorem with some results concerning ample divisors on 3-folds. By assumption, in this paper we are considering at least 4-dimensional manifolds X. Clearly, if X is a 3-fold, then Z is an ample divisor on X. This situation was already studied by Sommese [11, Thm. IV] and Ionescu [4, §2 Cor. 1].

On the other hand our Theorem supplements a result of Lanteri and Maeda. Indeed in [7, Thm. C] they studied the pair $(X, \mathcal{E})$ with the assumption that (Z, H_Z) is a hyperquadric fibration over a smooth curve B for some ample line bundle H on X and $\dim Z \geq 3$. So the case in which Z is 2-dimensional was not yet treated. This is exactly the case discussed here. Although the starting point is inspired by some ideas in [7], the method we use to prove the Theorem is somewhat different from that in [7, Thm. C], which relies on [9].

Here is a sketch of the proof. By $(*)$ and adjunction we see that $K_Z = (K_X + \det \mathcal{E})_Z$. But (Z, l) is a conic fibration by assumption, hence the adjoint bundle $K_Z + l$ is not ample. Combining these two facts, we obtain that the line bundle $K_X + \det \mathcal{E} + L$ is not ample as well. Thus $\mathcal{F} := \mathcal{E} \oplus L$ is an ample vector bundle of rank $n-1$ on X whose adjoint bundle is not ample. This allows us to apply results due to Ye, Zhang [12], and to Andreatta, Ballico and Wiśniewski [1], leading to a precise list of possibilities for the pair $(X, \mathcal{E})$, which we examine through a case-by-case analysis.

The proof is divided into two parts according as the genus $g(B)$ of the base curve B is positive (§2) or zero (§3).

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1. Notations and background material

In this paper we will only consider smooth projective varieties over the complex field $\mathbb{C}$, briefly called *manifolds*. Let X be a manifold. The tensor product between line bundles on X is denoted additively. A line bundle L on X is said to be *nef* (numerically effective) if $L \cdot C \geq 0$ for every curve C on X. A nef line bundle L is *big* if $L^{\dim X} > 0$. We say that a vector bundle $\mathcal{E}$ on X is *ample* if the tautological bundle $H(\mathcal{E})$ of the projective bundle $\mathbb{P}_X(\mathcal{E})$ is ample. A manifold X is called a *Fano manifold* if its anticanonical bundle $-K_X$ is ample.

A *polarized manifold* is a pair (X, L) consisting of a manifold X and an ample line bundle L on it. A polarized manifold (X, L) is called a *Del Pezzo manifold* if $K_X = -(\dim X - 1)L$. A polarized manifold (X, L) is said to be a *scroll* (respectively a *hyperquadric fibration*) over a manifold W if there exists a surjective morphism $\phi : X \to W$ with fiber $F \cong \mathbb{P}^m$ and $L_F \cong \mathcal{O}_{\mathbb{P}^m}(1)$ (respectively: with general fiber $F \cong \mathbb{Q}^m$ and $L_F \cong \mathcal{O}_{\mathbb{Q}^m}(1)$), where $m = \dim X - \dim W$.

(1.1) Note that, if (Z, l) is a 2-dimensional hyperquadric fibration over a smooth curve B via π, then any smooth fiber f of π is just a smooth plane conic, hence $f \cong \mathbb{P}^1$ and $l_f \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Moreover, the adjoint bundle $K_Z + l$ is restricted trivially to any fiber, hence

$$K_Z + l = \pi^* b \tag{1.1.1}$$

for some $b \in \mathrm{Pic}(B)$. Note that this would include the case when (Z, l) is a Del Pezzo surface with the anticanonical polarization. Since this case is already treated in [8] we will adopt the following stronger notion of 2-dimensional hyperquadric fibration, inspired by the adjunction theoretic definition [2, (3.3.1)].

DEFINITION (1.1.2). A polarized surface (Z, l) is said to be a *conic fibration* if a multiple of the adjoint bundle $K_Z + l$ defines a surjective morphism $\pi : Z \to B$ over a smooth curve B.

(1.1.3) This definition is equivalent to requesting for classical 2-dimensional hyperquadric fibrations (Z, l) that the line bundle b in (1.1.1) is ample, which exactly means that just the Del Pezzo surfaces with anticanonical polarization are excluded by (1.1.2).

Remark (1.1.4). Let (Z, l) be a conic fibration via $\pi : Z \to B$. Then Z is a ruled surface over B whose singular fibers consist just of two irreducible components. In other words, Z is the blow-up of a geometrically ruled surface $\pi_0 : Z_0 \to B$ at k points belonging to different fibers, where k denotes the number of singular fibers of $\pi : Z \to B$.

(1.2) Now we state the results which we will use to prove our Theorem. Let $\mathcal{L}$ be an ample line bundle on a manifold X and assume that $A := K_X + \mathcal{L}$ is nef. Then by the Kawamata-Shokurov base-point-free Theorem we know that there exists an integer $m \gg 0$ such that mA is spanned. If

(1.2.1) $$\phi := \phi_{|mA|} : X \to W_0 \subset \mathbb{P}^N$$

is the morphism associated to the linear system $|mA|$, we get through the Stein factorization a connected fiber morphism $\varphi : X \to W$ and a finite morphism $\eta : W \to W_0$.

Theorem (1.3)

Let X be a manifold of dimension $n \geq 4$, and let $\mathcal{F}$ be an ample vector bundle of rank $n-1$ on X. Consider the adjoint bundle $A := K_X + \det \mathcal{F}$. Then:

(1.3.1) [12, Thm. 3] *A is nef unless $(X, \mathcal{F})$ is one of the following:*

(a) $(X, \mathcal{F}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)})$.
(b) $(X, \mathcal{F}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$.
(c) $(X, \mathcal{F}) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)})$.
(d) *X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve W, and $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fiber G ($\cong \mathbb{P}^{n-1}$) of $\psi : X \to W$.*

(1.3.2) [1, Thm. B)] *Assume that A is nef, and let $\varphi : X \to W$ be as in* (1.2). *Then A is big unless*

(a) *X is a Fano manifold and $\det \mathcal{F} = -K_X$.*
(b) *$\varphi : X \to W$ is a $\mathbb{P}^{n-1}$-bundle over a smooth curve W, and either $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ or $\mathcal{F}_G \cong T_G$, for every fiber G ($\cong \mathbb{P}^{n-1}$).*

(c) $\varphi : X \to W$ *is a hyperquadric fibration over a smooth curve* W*, and* $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)}$ *for the general fiber* G $(\cong \mathbb{Q}^{n-1})$.
(d) $\varphi : X \to W$ *is a* $\mathbb{P}^{n-2}$*-fibration over a smooth surface* W*, locally trivial in the complex topology, and* $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ *for every fiber* G $(\cong \mathbb{P}^{n-2})$.

(1.3.3) [1, Thm. C)] *Assume that* A *is nef and big but not ample. Then* $\sigma : X \to X'$ *is the blow-up of a manifold* X' *at* $m > 0$ *distinct points* $q_1, \dots, q_m$ *and there exists an ample vector bundle* $\mathcal{F}'$ *of rank* $n-1$ *on* X'*, whose adjoint bundle* $A' = K_{X'} + \det \mathcal{F}'$ *is ample, such that* $\mathcal{F} = \sigma^* \mathcal{F}' \otimes \mathcal{O}_X(-\sum_{i=1}^m E_i)$*, where* $E_i := \sigma^{-1}(q_i)$ *denotes the exceptional divisor of the blow-up.*

(1.4) In stating cases (b), (c) and (d) in (1.3.2), we inserted the description of $\mathcal{F}_G$, not explicitly mentioned in [1, Theorem]. It can be obtained as follows.

Let G be an arbitrary fiber of φ in cases (b) and (d), and a general fiber of φ in case (c). Then $K_G = (K_X)_G$. By the fibration Theorem there is an ample line bundle H on W such that $K_X + \det \mathcal{F} = \varphi^* H$. Hence we have

$$\mathcal{O}_G = (K_X + \det \mathcal{F})_G = K_G + \det \mathcal{F}_G,$$

i.e., $K_G + \det \mathcal{F}_G$ is trivial. Since $\operatorname{rank} \mathcal{F}_G \geq \dim G$, by [3, Main Theorem] the descriptions of $\mathcal{F}_G$ are as follows according to cases:

(b) $\mathcal{F}_G$ is either $\mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ or $T_{\mathbb{P}}$.
(c) $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)}$.
(d) $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$.

We will also use the following result:

Lemma (1.5) [6, Lemma (5.1)]

Let E *be an effective divisor on a manifold* X*, and assume that* $E \cong \mathbb{P}^{n-1}$ *and* $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}}(-1)$*. Let* $\phi : X \to X'$ *be its contraction. If* $\mathcal{E}$ *is an ample vector bundle of rank* $r \geq 2$ *on* X *and* $\mathcal{E}_E \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus r}$*, then there exists an ample vector bundle* $\mathcal{E}'$ *of rank* r *on* X' *such that*

$$\mathcal{E} \cong \phi^* \mathcal{E}' \otimes \mathcal{O}_X(-E).$$

2. Proof of the Theorem in the irrational case

(2.1) First, we will get more information on the structure of X through the Albanese variety. Let $\alpha_Z : Z \to \mathrm{Alb}(Z)$ and $\alpha_X : X \to \mathrm{Alb}(X)$ be the Albanese morphisms of Z and X, respectively. Due to functorial properties, the following commutative diagram holds:

$$\begin{array}{ccc} Z & \xrightarrow{\ i\ } & X \\ {\scriptstyle \alpha_Z}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_X} \\ \mathrm{Alb(Z)} & \xrightarrow{\mathrm{Alb}(i)} & \mathrm{Alb(X)}. \end{array}$$

Because of [7, (1.1.8)], the morphism $\mathrm{Alb}(i)$ is an isomorphism. It is convenient to identify $\mathrm{Alb}(Z) = \mathrm{Alb}(X)$ and to write briefly $\alpha = \alpha_X$. Since $\alpha|_Z = \alpha_Z$, we have $\alpha(X) \supset \alpha(Z) = \alpha_Z(Z)$, which is isomorphic to B because of the irrationality of the base curve B. Hence we can assume that $\alpha_Z = \pi$. Now, by [7, (1.1.4)] we have that

$$h^0(\Omega^2_X) \le h^0(\Omega^2_Z) = p_g(Z) = 0,$$

which implies that $\omega_1 \wedge \omega_2 = 0$ for any two 1-forms ω_1, ω_2 on X, and so $\alpha(X)$ is 1-dimensional, i.e., $\alpha(X) \cong B$. In conclusion, we see that $\alpha : X \to B$ is a morphism with connected fibers such that $\alpha|_Z = \pi$.

(2.2) Now, consider the vector bundle

$$\mathcal{F} := \mathcal{E} \oplus L.$$

Note that $\mathcal{F}$ is an ample vector bundle of rank $n-1$ on X, and

$$\det \mathcal{F} = \det \mathcal{E} + L.$$

Let $A := K_X + \det \mathcal{F}$ denote the adjoint bundle of $\mathcal{F}$, and let us start assuming that A is not nef. In this case the pair $(X, \mathcal{F})$ satisfies the hypotheses of (1.3.1). First, note that cases (a), (b) and (c) in (1.3.1) cannot occur since $h^1(\mathcal{O}_X) = g(B) > 0$. The remaining possibility for $(X, \mathcal{F})$ led by (1.3.1) is that X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve W, and $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fiber G $(\cong \mathbb{P}^{n-1})$ of $\varphi : X \to W$. Let G be any fiber. By $\mathcal{E}_G \oplus L_G = \mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ we obtain

(2.2.1) $$\mathcal{E}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)},$$

$$L_G \cong \mathcal{O}_{\mathbb{P}}(1). \tag{2.2.2}$$

Let $s_G \in \Gamma(G, \mathcal{E}_G)$ denote the restriction to G of the section s defining Z, and set

$$f := Z \cap G = (s_G)_0.$$

Note that f is a fiber of π, since G is a fiber of φ and $\varphi|_Z = \pi$. Moreover by (2.2.1) we see that $f \in |\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}|$, which means that f is a linear subspace of G. Hence f is a smooth fiber of π, and due to (2.2.2) and the linearity of f in G we see that

$$\overbrace{f \cdot l}^{\text{in Z}} = \overbrace{f \cdot L_Z}^{\text{in Z}} = \overbrace{f \cdot L}^{\text{in X}} = \overbrace{f \cdot L_G}^{\text{in G}} = 1,$$

which contradicts the assumption that (Z, l) is a conic fibration. So this case doesn't occur as well.

Therefore A is nef. Thanks to [5, Lemma (1.2)] and by definition of conic fibration, its restriction to Z is given by

$$A_Z = (K_X + \det \mathcal{F})_Z = (K_X + \det \mathcal{E} + L)_Z = K_Z + L_Z = \pi^* b \tag{2.2.3}$$

for some $b \in \mathrm{Pic}(B)$. Through the injectivity of $\mathrm{Pic}(i) : \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ [7, (1.1.6)], we deduce that

$$A = \alpha^* b, \tag{2.2.4}$$

which in particular implies that

$$A^2 = 0. \tag{2.2.5}$$

This shows that A is not big: thus $(X, \mathcal{F})$ is one of the pairs listed in (1.3.2).

First of all, we see that case (1.3.2)(a) cannot occur: in fact, if X is a Fano manifold, then $h^1(\mathcal{O}_X) = h^1(K_X + (-K_X)) = 0$ by the Kodaira Vanishing Theorem.

Case (1.3.2) (d) cannot occur as well. In fact we have that A gives X the structure of a $\mathbb{P}^{n-2}$-fibration over a smooth surface W, and $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fiber G ($\cong \mathbb{P}^{n-2}$). Then, recalling (1.2.1), we see that

$$mA = \phi^* \mathcal{O}_{W_0}(1).$$

Note however that $\mathcal{O}_{W_0}(1)^2 = (\mathcal{O}_{\mathbb{P}^N}(1)^2)_{W_0}$ is an effective non-trivial 0-cycle in W_0, which contradicts (2.2.5).

(2.3) Only cases (b) and (c) in (1.3.2) remain to be examined. First observe that, if

$$\varphi : X \to W$$

is a $\mathbb{P}^{n-1}$-bundle or a hyperquadric fibration over a smooth curve W, where φ is definite as in (1.2), then

$$mA = \phi^* \mathcal{O}_{W_0}(1) = \varphi^*(\eta^* \mathcal{O}_{W_0}(1)).$$

Comparing this with (2.2.4), we conclude that the fibers of α and those of φ must be the same. This implies that $\alpha = \varphi$ up to an isomorphism $W \cong B$. So cases (b) and (c) in (1.3.2) can be rephrased by using α and F instead of φ and G. From case (b):

> $\alpha : X \to B$ is a $\mathbb{P}^{n-1}$-bundle, and either $\mathcal{F}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ or $\mathcal{F}_F \cong T_F$ for every fiber F ($\cong \mathbb{P}^{n-1}$) of α.

Let F be any fiber of α. Clearly $\mathcal{F}_F \neq T_F$, since the tangent bundle of $\mathbb{P}^{n-1}$ is not decomposable, and there are two ways to write $\mathcal{F}_F$ as a direct sum of $\mathcal{E}_F$ and L_F: this gives two subcases of (b). Note that the same subcase happens for every fiber F of α.

(2.3.1) The first subcase is given by the following situation:

$$\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}, \quad L_F \cong \mathcal{O}_{\mathbb{P}}(1),$$

which implies that (X, L) is a scroll over B, leading to case (a) in the Theorem.

(2.3.2) The second one is given by:

$$\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}, \quad L_F \cong \mathcal{O}_{\mathbb{P}}(2).$$

Let f be any fiber of π, and consider the fiber F of α containing it. If $s_F \in \Gamma(F, \mathcal{E}_F) \cong \Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$ denotes the restriction of s to F, then $f = Z \cap F = (s_F)_0$. Thus f is a linear subspace of F, and so we conclude by arbitrariness of f that Z is a geometrically ruled surface. This gives case (b) in the Theorem.

(2.3.3) The remaining case (c) in (1.3.2) is immediate. We have that

> $\alpha : X \to B$ is a hyperquadric fibration, and $\mathcal{F}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)}$ for the general fiber F ($\cong \mathbb{Q}^{n-1}$) of α.

For any general fiber F of α, the only possibility for the summands of $\mathcal{F}_F$ is:

$$\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)}, \quad L_F \cong \mathcal{O}_{\mathbb{Q}}(1).$$

In particular (X, L) is a hyperquadric fibration as a polarized pair. This leads to case (c) in the Theorem, and concludes the proof when $g(B) > 0$. □

Remark (2.4). In case (b) in the Theorem, if $g(B) > 0$, then L and H can be chosen in such a way that $H_Z = h$, $L = 2H$, and so $l = 2h$, where h denotes the tautological bundle of Z. Indeed, if $h \in \mathrm{Pic}(Z)$ and $H \in \mathrm{Pic}(X)$ are ample line bundles making $\pi : Z \to B$ and $\alpha : X \to B$ scrolls over B, then $H_f = (H_Z)_f = \mathcal{O}_{\mathbb{P}}(1) = h_f$ for every fiber f of π, hence there exists a line bundle b over B such that $H_Z - h = \pi^* b$. Without loss of generality we can assume that $H_Z = h$. Suppose moreover to have chosen $L = 2H$: then $l = L_Z = 2H_Z = 2h$.

3. Proof of the Theorem in the rational case

(3.1) As in (2.2), we consider the ample vector bundle of rank $n-1$ on X defined by $\mathcal{F} := \mathcal{E} \oplus L$. The adjoint bundle $A = K_X + \det \mathcal{F}$ is not ample, since its restriction to Z is

(3.1.1) $$A_Z = \pi^* b$$

for some $b \in \mathrm{Pic}(B)$, as we saw in (2.2.3). So we need to examine all the cases in (1.3).

(3.2) First assume that A is not nef. By (1.3.1) we obtain the following four possibilities for the pair $(X, \mathcal{F})$.

From (1.3.1) (a): $(X, \mathcal{F}) \cong \big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}\big)$. Then, since $\mathcal{F} = \mathcal{E} \oplus L$, we get

$$(X, \mathcal{E}, L) \cong \big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{P}}(1)\big),$$

which gives $Z \cong \mathbb{P}^2$, hence a contradiction.

From (1.3.1) (b): $(X, \mathcal{F}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$. So there are two possibilities for the triple $(X, \mathcal{E}, L)$: either

$$(X, \mathcal{E}, L) \cong \big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{P}}(2)\big), \quad \text{or}$$
$$(X, \mathcal{E}, L) \cong \big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}}(1)\big).$$

The former case leads to the same contradiction as before. In the latter case we obtain

(3.2.1) $$(Z, l) \cong \big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)\big)$$

which is a scroll (with respect to any rulings of Z) and not a conic fibration. In conclusion, also case (b) in (1.3.1) doesn't occur.

From (1.3.1) (c): $(X, \mathcal{F}) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)})$. Arguing as in the previous cases, we get

$$(X, \mathcal{E}, L) \cong \big(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{Q}}(1)\big),$$

which gives again (Z, l) as in (3.2.1), and so leads to a contradiction.

Finally, in case (1.3.1) (d) we have that X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve W, and $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fiber G ($\cong \mathbb{P}^{n-1}$) of $\psi : X \to W$.

Then (X, L) is a scroll over W and $\mathcal{E}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber G. Let $s_G \in \Gamma(G, \mathcal{E}_G) \cong \Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$ be the restriction of s to G. Then $g := Z \cap G = (s_G)_0$ is a positive dimensional linear subspace of G. Therefore $g \cong \mathbb{P}^1$, since $g \subset Z$ and $Z \ncong \mathbb{P}^2$, and $l_g = (L_G)_g \cong \mathcal{O}_{\mathbb{P}}(1)$. Consequently (Z, l) is a scroll via the projection $\psi|_Z : Z \to W$. But (Z, l) is also a conic fibration via the projection $\pi : Z \to B$. So Z admits two different fibrations:

$$\begin{array}{ccc} Z & \xrightarrow[\psi|_Z]{\text{with fiber } \mathbb{P}^1} & W \\ {\scriptstyle \text{with general fiber } \mathbb{P}^1}\Big\downarrow{\scriptstyle \pi} & & \\ B. & & \end{array}$$

In particular $W \cong \mathbb{P}^1$, and so $K_Z^2 = 8$. On the other hand, considering Z as a geometrically ruled surface Z_0 with base curve B blown-up at k points (1.1.4), we obtain $K_Z^2 = K_{Z_0}^2 - k = 8 - k$. Therefore $k = 0$ and $(Z, l) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1))$. This gives the exception in case (b) of the Theorem. See also [5, Thm. B, case (3)].

(3.3) Now, assume that A is nef but not big: thus we have to examine the four possibilities for $(X, \mathcal{F})$ listed in (1.3.2).

In case (1.3.2) (a) X is a Fano manifold and $\det \mathcal{F} = -K_X$. Then

$$K_Z = (K_X + \det \mathcal{E})_Z = -L_Z = -l,$$

i.e. (Z, l) is a Del Pezzo surface with the anticanonical polarization. But these surfaces are excluded from our discussion, as we noted in (1.1.3).

In case (1.3.2) (b) $\varphi : X \to W$ is a $\mathbb{P}^{n-1}$-bundle over a smooth curve W, and $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber G ($\cong \mathbb{P}^{n-1}$). By (1.2) we have

$$mA = \phi^* \mathcal{O}_{W_0}(1) = \varphi^*(\eta^* \mathcal{O}_{W_0}(1)).$$

Taking the restriction of mA to Z and recalling (3.1.1), we get

$$mA_Z = (\varphi|_Z)^*(\eta^* \mathcal{O}_{W_0}(1))_{\varphi(Z)} = \pi^* mb,$$

which means that the two morphisms, $\pi : Z \to B$ ($\cong \mathbb{P}^1$) and $\varphi|_Z : Z \to W$, have the same fibers. We can assume that $\varphi|_Z = \pi$ and so consider φ as an extension of

π to X. Let F be a fiber of φ: there are two ways to write $\mathcal{F}_F = \mathcal{E}_F \oplus L_F$ according to the following two cases:

$$(\mathcal{E}_F, L_F) \cong \big(\mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}}(1)\big)$$
$$\big(\mathcal{E}_F, L_F\big) \cong (\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{P}}(2)\big).$$

They are quite analogous to the situations discussed in (2.3.1) and (2.3.2) when $g(B) > 0$, and give rise to cases (a) and (b) in the Theorem.

In (1.3.2) (c) $\varphi : X \to W$ is a hyperquadric fibration over a smooth curve W, and $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)}$ for the general fiber G ($\cong \mathbb{Q}^{n-1}$). The same argument used in the previous situation shows that $\varphi|_Z = \pi$ up to an isomorphism $W \cong B$, hence we can consider φ as an extension of π to X. Then

$$\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)} \quad \text{and} \quad L_F \cong \mathcal{O}_{\mathbb{Q}}(1)$$

for the general fiber F of φ, and so we obtain case (c) in the Theorem.

From the last case (d) in (1.3.2) $\varphi : X \to W$ is a $\mathbb{P}^{n-2}$-fibration over a smooth surface W, and $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fiber G ($\cong \mathbb{P}^{n-2}$). Exactly as before, we see that $\varphi|_Z$ and π have the same fibers, and so

(3.3.1) $$\varphi(Z) = B.$$

On the other hand, $\mathcal{F}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ implies that $\mathcal{E}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$. Thus, if $s_G \in \Gamma(G, \mathcal{E}_G) \cong \Gamma(\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$ denotes the restriction of s to G, then $G \cap Z = (s_G)_0 \neq \emptyset$ for any fiber G of φ. This means that $\varphi|_Z : Z \to W$ is surjective, i.e.

(3.3.2) $$\varphi(Z) = W.$$

Clearly (3.3.1) and (3.3.2) are in contradiction, since B is a curve and W is a surface, an so this case cannot occur.

(3.4) The last possibility to consider for $(X, \mathcal{F})$ is that A is nef and big but not ample. Then, by (1.3.3), $\sigma : X \to X'$ is the blow-up of a manifold X' at $m > 0$ distinct points $q_1, \dots, q_m$ and there exists an ample vector bundle $\mathcal{F}'$ of rank $n-1$ on X', whose adjoint bundle

(3.4.1) $$A' = K_{X'} + \det \mathcal{F}' \quad \text{is ample},$$

such that

(3.4.2) $$\mathcal{F} = \sigma^* \mathcal{F}' \otimes \mathcal{O}_X\Big(-\sum_{i=1}^{m} E_i\Big),$$

where $\sum_{i=1}^m E_i := \sum_{i=1}^m \sigma^{-1}(q_i)$ denotes the exceptional divisor of the blow-up. In particular, by definition of blow-up,

$$E_i \cong \mathbb{P}^{n-1} \quad \text{and} \quad \mathcal{O}_{E_i}(E_i) \cong \mathcal{O}_{\mathbb{P}}(-1) \quad \text{for} \quad i = 1, \dots, m. \tag{3.4.3}$$

Let $j \in \{1, \dots, m\}$ be any fixed index. By (3.4.2)

$$\mathcal{F}_{E_j} \cong \mathcal{O}_{\mathbb{P}}^{\oplus(n-1)} \otimes \mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)},$$

hence

$$(\mathcal{E}_{E_j}, L_{E_j}) \cong \big(\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{P}}(1)\big). \tag{3.4.4}$$

This allows us to apply Lemma (1.5), which tells us that there exists an ample vector bundle $\mathcal{E}'$ of rank $n-2$ on X' such that

$$\mathcal{E} \cong \sigma^*\mathcal{E}' \otimes \mathcal{O}_X\Big(-\sum_{i=1}^m E_i\Big). \tag{3.4.5}$$

For every $j = 1, \dots, m$, let $s_{E_j} \in \Gamma(E_j, \mathcal{E}_{E_j}) \cong \Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$ denote the restriction of s to E_j. Then $e_j := E_j \cap Z = (s_{E_j})_0$ is a positive dimensional linear subspace of E_j which is contained in Z, therefore $e_j \cong \mathbb{P}^1$. Note that the e_j are disjoint (-1)-curves in Z: actually $e_i \cap e_j \subset E_i \cap E_j = \emptyset$ for $i \neq j$, and $e_j^2 = \mathcal{O}_{E_j}(E_j) \cdot e_j = \mathcal{O}_{\mathbb{P}}(-1) \cdot e_j = -1$ by (3.4.3). In particular this gives $K_Z \cdot e_j = -1$. On the other hand, since $l \cdot e_j = L_{E_j} \cdot e_j = 1$ by (3.4.4), we get

$$e_j \cdot \pi^* b = (K_Z + l) \cdot e_j = 0,$$

which shows that e_j is contained in a fiber of π. In conclusion any curve e_j is an irreducible component of a singular fiber f_j of $\pi : Z \to B$. So we can look at the restriction $\sigma|_Z$ of σ to Z as the contraction of the (-1)-curves e_j for $j = 1, \dots, m$, and then $Z' := \sigma(Z)$ is again a smooth ruled surface over B.

Now we note by (3.4.3) and (3.4.4) that $K_X + (n-1)L$ is restricted trivially to every E_j. Hence the reduction morphism of (X, L) factors through σ and then there exists an ample line bundle $L' \in \operatorname{Pic}(X')$ such that $L = \sigma^*L' - \mathcal{O}_X(\sum_{i=1}^m E_i)$. Thus, by (3.4.5):

$$\sigma^*(\mathcal{E}' \oplus L') = \sigma^*\mathcal{E}' \oplus \sigma^*L' \cong \Bigg(\mathcal{E} \otimes \mathcal{O}_X\Big(\sum_{i=1}^m E_i\Big)\Bigg) \oplus \Bigg(L \otimes \mathcal{O}_X\Big(\sum_{i=1}^m E_i\Big)\Bigg)$$

$$= (\mathcal{E} \oplus L) \otimes \mathcal{O}_X\Big(\sum_{i=1}^m E_i\Big) = \mathcal{F} \otimes \mathcal{O}_X\Big(\sum_{i=1}^m E_i\Big) = \sigma^*\mathcal{F}'.$$

Therefore

$$\mathcal{E}' \oplus L' = \mathcal{F}'.$$

Note that $(Z', L'_{Z'})$ is a conic fibration over B as well. Let $s' \in \Gamma(X', \mathcal{E}')$ be the section corresponding to $s \in \Gamma(X, \mathcal{E})$ under σ^* and the isomorphism given by (3.4.5). Then $Z' = (s')_0$.

In conclusion we get a new situation, with X', $\mathcal{E}'$, L' and Z', satisfying the assumption of the Theorem. Then by (3.1) applied to $(X', \mathcal{E}', L')$ we see that the adjoint bundle $K_{X'} + \det \mathcal{F}'$ is not ample, contradicting (3.4.1). This makes the case when A is nef and big impossible, and so the proof of the Theorem is concluded. □

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