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Four-term recurrence relations for hypergeometric functions of the second order II

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Abstract

The recent work on four-term recurrence relations for the second order functions of hypergeometric type undertaken by the author (see [3]) is developed from a slightly different point of view also using generating functions. The direction of future prospects is indicated.

Introduction

A recent investigation carried out by the author, Exton (1996), is extended. As previously pointed out by Yáñez, Dehesa and Zarzo (1994), the recurrence relations of the important class of hypergeometric functions of the second order is incompletely known except for certain three-term recurrence relations. Here, certain generating functions which yield higher-order functions are considered. The required recurrences are then deduced by suitable specialisation.

In the previous study mentioned (Exton (1996)), the generating functions concerned involve elementary functions only. A slightly different avenue of attack is persued in this paper, where the elementary manipulation of series is now applied to combinations of exponential functions and second order hypergeometric functions.

Since hypergeometric functions figure to a considerable extent in the subsequent analysis, the definition of the single hypergeometric function of general order is given for convenience, namely

$$ {}_AF_B(a_1,\dots,a_A;b,\dots,b_B;x) = {}_AF_B\big((a);(b);x\big) = \sum_{n=0}^{\infty} \frac{\big((a),m\big)x^m}{\big[((b),m)m!\big]}\,, \tag{1.1}$$

where the Pochhammer symbol (a,n) is given by

$$(a,n) = a(a=1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}\,;\ (a,0) = 1\,. \tag{1.2}$$

The symbol $((a),n)$ denotes the sequence $(a_1,n)\dots(a_A,n)$ and for these preliminaries and further background, the reader should consult Exton (1996). Any parameters with any values leading to expressions which do not make sense are tacitly excluded.

2. Four-term recurrence relations for the function ${}_0F_1$

We begin by considering the generating function

$$V_1 = \exp\left(\frac{x}{t}\right) {}_1F_1(a;c;t) = \sum_{n=-\infty}^{\infty} t^n F_n\,, \tag{2.1}$$

which involves the confluent hypergeometric function ${}_1F_1$. The product of the exponential function and the confluent hypergeometric function is developed in powers of t:

$$\exp\left(\frac{x}{t}\right) {}_1F_1(a;c;t) = \sum_{r,s=0}^{\infty} \frac{(a,s)x^r t^{s-r}}{[(c,s)r!s!]}\,. \tag{2.2}$$

On replacing s by $n+r$, the series on the right becomes

$$\sum_{r=0}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(a,n+r)x^r t^n}{[(c,n+n,r)r!(n+r)!]}\,. \tag{2.3}$$

Within its domain of absolute convergence, this double series can be re-arranged as

$$\sum_{n=-\infty}^{\infty} t^n \left[\frac{(a,n)}{[(c,n)n!]}\sum_{r=0}^{\infty}\frac{(a+n,r)x^r}{[(c+n,r)(1+n,r)r!]}\right]. \tag{2.4}$$

Hence, by comparison with (2.1) and using (1.1), it follows that

$$F_n = F_n(a;c;x) = \frac{(a,n)}{[(c,n)n!]}\,{}_1F_2(a+n;c+n,1+n;x)\,. \tag{2.5}$$

The required recurrence relation can be obtained by taking partial derivatives of (2.1) with respect to t. We then have

$$\frac{\partial}{\partial t}V_1 = -xt^{-2}\exp\left(\frac{x}{t}\right){}_1F_1(a;c;t) + \exp\left(\frac{x}{t}\right)\frac{d}{dt}\,{}_1F_1(a;c;t) = \sum_{n=-\infty}^{\infty} nt^{n-1}F_n \tag{2.6}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2}V_1 &= 2xt^{-3}\exp\left(\frac{x}{t}\right){}_1F_1(a;c;t) + x^2t^{-4}\exp\left(\frac{x}{t}\right){}_1F_1(a;c;t)\\ &\quad - 2xt^{-2}\exp\left(\frac{x}{t}\right)\frac{d}{dt}\,{}_1F_1(a;c;t) + \exp\left(\frac{x}{t}\right)\frac{d^2}{dt^2}\,{}_1F_1(a;c;t)\\ &= \sum_{n=-\infty}^{\infty} n(n-1)t^{n-2}F_n\,.\end{aligned} \tag{2.7}$$

Hence, it may be seen that

$$\exp\left(\frac{x}{t}\right)\frac{d}{dt}\,{}_1F_1(a;c;t) = \frac{\partial}{\partial t}V_1 + xt^{-2}V_1 \tag{2.8}$$

and

$$\exp\left(\frac{x}{t}\right)\frac{d^2}{dt^2}\,{}_1F_1(a;c;t) = \frac{\partial^2}{\partial t^2}V_1 + 2xt^{-2}\frac{\partial}{\partial t}V_1 + \left(x^2t^{-4} - 2xt^{-3}\right)V_1\,. \tag{2.9}$$

Bearing in mind that the confluent hypergeometric function ${}_1F_1(a;c;t)$ is a solution of the differential equation

$$ty'' + (c-t)y' - ay = 0\,, \tag{2.10}$$

we have

$$\begin{aligned}\exp\left(\frac{x}{t}\right)&\left[\frac{td^2}{dt^2}\,{}_1F_1(a;c;t) + (c-t)\frac{d}{dt}\,{}_1F_1(a;c;t) - a\,{}_1F_1(a;c;t)\right] = 0\\ &= \frac{t\partial^2}{\partial t^2}V_1 + \frac{2xt^{-1}}{t}V_1 + \left(x^2t^{-3} - 2xt^{-2}\right)V_1\\ &\quad + \frac{(c-t)}{t}V_1 + x\left(ct^{-2} - t^{-1}\right)V_1 - a\,V_1\\ &= \frac{t\partial^2}{\partial t^2}V_1 + \left(2xt^{-1} + c - t\right)\frac{\partial}{\partial t}V_1\\ &\quad + \left[x^2t^{-3} + (c-2)xt^{-2} - xt^{-1} - a\right]V_1 = 0.\end{aligned} \tag{2.11}$$

From the generating function and its partial derivatives, equating the coefficients of t^n and after a little algebra, it follows that

$$x^2 F_{n+3} + x(2n+c+2)F_{n+2} + \big[n^2 + (c+1)n + c - x\big]F_{n+1} - (n+a)F_n = 0\,. \qquad (2.12)$$

This case can be specialised, so that F_n becomes a hypergeometric function of the second order. Let $c = a$, when we have

$$F_n = \frac{{}_0F_1(-;1+n;x)}{n!}, \qquad (2.13)$$

$$x^2 F_{n+3} + x(2n+a+2)F_{n+2} + \big[n^2 + (a+1)n + a - x\big]F_{n+1} - (n+a)F_n = 0 \qquad (2.14)$$

and

$$x^2\,{}_0F_1(-;4+n;x) + 2x(n+1)(n+3)\,{}_0F_1(-;3+n;x) + \big[n^2 + n - x\big](n+3)(n+2)\,{}_0F_1(-;2+n;x) - (n+3)(n+2)(n+1)n\,{}_0F_1(-;1+n;x) = 0\,, \quad (215)$$

where any terms involving the parameter a have been eliminated since this parameter does not appear in the associated hypergeometric functions.

Proceeding along the same lines, when we put $a = 1$, it now follows that

$$F_n = \frac{{}_0F_1(-;c+n;x)}{(c,n)} \qquad (2.16)$$

and

$$x^2\,{}_0F_1(-;c+n+3;x) + x(2n+c+2)\,{}_0F_1(-;c+n+2;x) + \big[n^2 + (c+1)n + c - x\big](c+n+2)(c+n+1)\,{}_0F_1(-;c+n+1;x) - (n+1)(c+n+2)(c+n+1)(c+n)\,{}_0F_1(-;c+n;x) = 0\,. \qquad (2.17)$$

The function ${}_0F_1$ is of particular interest on account of its close relationship with the Bessel function. See Erdélyi (1953) Vol. II.

3. Recurrence relations for the confluent hypergeometric function ${}_1F_1$

Similar methods may be employed to obtain new four-term recurrence relations applicable to certain confluent hypergeometric functions. Beginning with the generating function

$$V_2 = \exp\left(\frac{x}{t}\right) {}_2F_1(a,b;c;t) = \sum_{n=-\infty}^{\infty} t^n G_n\,, \tag{3.1}$$

it may easily be shown that

$$G_n = G_n(a,b;c;x) = (a,n)(b,n)\,\frac{{}_2F_2(a+n,b+n;c+n,1+n;x)}{[(c,n)n!]} \tag{3.2}$$

$$\exp\left(\frac{x}{t}\right)\frac{d}{dt}\,{}_2F_1(a,b;c;t) = \frac{\partial}{\partial t}V_2 + xt^{-2}V_2 \tag{3.3}$$

and

$$\exp\left(\frac{x}{t}\right)\frac{d^2}{dt^2}\,{}_2F_1(a,b;c;t) = \frac{\partial^2}{\partial t^2}V_2 + 2xt^{-2}\frac{\partial}{\partial t}V_2 + \left(x^2t^{-4} - 2xt^{-3}\right)V_2\,. \tag{3.4}$$

The function ${}_2F_1(a,b;c;t)$ satisfies the differential equation

$$t(1-t)y'' + \left[c-(1+a+b)t\right]y' - aby = 0\,, \tag{3.5}$$

and, as in the previous section,

$$\begin{aligned}\frac{(t-t^2)^2}{t^2}V_2 &+ \frac{[2xt^{-1}-2x+c-(1+a+b)t]}{tV_2} \\ &+ \left[x^2t^{-3} - (2+c+x)t^{-2} + (3+a+b)xt^{-1} - ab)\right]V_2 = 0\,.\end{aligned} \tag{3.6}$$

From the generating relation (3.1) and its derivatives we obtain the result

$$\begin{aligned}x^2G_{n+3} &+ x(2n+2-c-x)G_{n+2} \\ &- \left[n^2 + (c+1-2x)n + c + (1+a+b)x\right]G_{n+1} \\ &+ \left[(1+a+b)n - ab\right]G_n = 0\,.\end{aligned} \tag{3.7}$$

If $c = a$,

$$G_n = (b,n)\,\frac{{}_1F_1(b+n;1+n;x)}{n!}\,, \tag{3.8}$$

and we see that (3.7) takes the from

$$
\begin{aligned}
&x^2(b+n+2)(b+n+1)(b+n)\,{}_1F_1(b+n+3;4+n;x)\\
&\quad+x(2n+2-x)(b+n+1)(b+n)(n+1)\,{}_1F_1(b+n+2;3+n;x)\\
&\quad+\big[n^2+(1-2x)n+(1+b)x\big](b+n)(n+2)(n+1)\,{}_1F_1(b+n+1;2+n;x)\\
&\quad+(n+a)(n+3)(n+2)(n+1)\,{}_1F_1(b+n;1+n;x)=0\,. \qquad (3.9)
\end{aligned}
$$

Similarly, if $a=1$,

$$
G_n=(b,n)\,{}_1F_1(b+n;c+n;x) \qquad (3.10)
$$

and we have

$$
\begin{aligned}
&x^2(b+n+2)(b+n+1)(b+n)\,{}_1F_1(b+n+3;c+n+3;x)\\
&\quad+x(2n+2-c-x)(b+n+1)(b+n)(1+n)\,{}_1F_1(b+n+2;c+n+2;x)\\
&\quad+\big[n^2+(c+1-2x)n+c+(2+b)x\big](b+n)(2+n)(1+n)\\
&\quad\times\,{}_1F_1(b+n+1;c+n+1;x)\\
&\quad+(n+1)(3+n)(2+n)(1+n)\,{}_1F_1(b+n;c+n;x)=0\,. \qquad (3.11)
\end{aligned}
$$

If Kummer's first theorem (Erdélyi (1953), Vol. I),

$$
{}_1F_1(a;c;x)=\exp(x)\,{}_1F_1(c-a;c;-x) \qquad (3.12)
$$

is applied to (3.9) and (3.11) respectively, two further four-term recurrence relations for confluent hypergeometric functions arise, namely

$$
\begin{aligned}
&x^2(b+n+2)(b+n+1)(b+n)\,{}_1F_1(1-b;4+n;-x)\\
&\quad+x(2n+2-x)(b+n+1)(b+n)(n+3)\,{}_1F_1(1-b;3+n;x)\\
&\quad+x\big[n^2+(1-2x)n\big](b+n+1)(b+n)(n+3)(n+2)\,{}_1F_1(1-b;2+n-x)\\
&\quad+\big[(1+b)n-ab\big](n+3)(n+2)(n+1)\,{}_1F_1(1-b;1+n;-x)=0 \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&x^2(b+n+2)\,{}_1F_1(c-b;c+n+3;-x)+x(2n+2-c-x)(c+n+2)\\
&\quad\times\,{}_1F_1(c-b;c+n+2;-x)\\
&\quad+\big[n^2+(c+1-2x)n+c+(2+b)x\big](c+n+2)(c+n+1)\\
&\quad\times\,{}_1F_1(c-b;c+n+1;-x)\\
&\quad+\big[(2+b)n-b\big](c+n+2)(c+n+1)(c+n)\\
&\quad\times\,{}_1F_1(c-b;c+n;-x)=0\,. \qquad (3.14)
\end{aligned}
$$

4. A recurrence relation for a polynomial of the form ${}_2F_0$

A rather different type of generating function is now discussed in which a hypergeometric polynomial of the third order arises. This function may be specialised so as to give a polynomial of the second order for which a four-term recurrence relation is presented. Consider the expression

$$V_3 = \exp(xt)\,{}_2F_1(a,b,c;t) = t^n P_n\,, \tag{4.1}$$

and by series manipulation of a similar type to that used in the previous sections, it is found that

$$P_n = P_n(a,b;c;x) = x^n\,\frac{{}_3F_1(a,b,-n;c;-\frac{1}{x})}{n!}\,. \tag{4.2}$$

Taking first and second partial derivatives of (4.1) with respect to t, we have

$$\frac{2}{t}V_3 = x\;\exp(xt)\,{}_2F_1(a,b;c;t) + \exp(xt)\frac{d}{dt}\,{}_2F_1(a,b;c;t) \tag{4.3}$$

and

$$\frac{2}{t^2}V_3 = x^2\;\exp(xt)\,{}_2F_1(a,b;c;t) + 2x\;\exp(xt)\frac{d}{dt}\,{}_2F_1(a,b;c;t) + \exp(xt)\frac{d^2}{dt^2}\,{}_2F_1(a,b;c;t) \tag{4.4}$$

Hence,

$$\exp(xt)\frac{d}{dt}\,{}_2F_1(a,b;c;t) = \frac{2}{t}V_3 - xV_3 \tag{4.5}$$

and

$$\exp(xt)\frac{d^2}{dt^2}\,{}_2F_1(a,b;c;t) = \frac{2}{t^2}V_3 - \frac{2x}{t}V_3 + x^2V_3\,. \tag{4.6}$$

From the associated hypergeometric equation (3.5), after a little reduction, we have

$$\frac{(t-t^2)^2}{2}V_3 + \frac{\bigl[c-(1+a+b+2x)t+2xt^2\bigr]}{t}V_3 - \bigl\{x^2t^2 - [x^2+x(1+a+b)]t + cx + ab\bigr\}V_3 = 0\,. \tag{4.7}$$

It then follows that

$$\bigl[n^2+(c+1)n+c\bigr]P_{n+1} - \bigl[n^2+(a+b+2x)n+xc+ab\bigr]P_n + \bigl[2xn+a+b+1-2x+x^2\bigr]P_{n-1} - x^2P_{n-2} = 0\,. \tag{4.8}$$

If $c = a$,

$$P_n = x^n \, \frac{{}_2F_0(b, -n; -; -\frac{1}{x})}{n!} \,, \tag{4.9}$$

so that

$$\begin{aligned} & n(n+1)x^2 \, {}_2F_0\Big(b, -n-1; -; -\frac{1}{x}\Big) \\ & \quad - \big[n^2 + (b+2x)n\big](n+1)x \, {}_2F_0\Big(b, 2-n; -; -\frac{1}{x}\Big) \\ & + \big[2xn + b + 1 - 2x + x^2\big](n+1)n \, {}_2F_0\Big(b, 1-n; -; -\frac{1}{x}\Big) \\ & \quad - (n+1)\Big(n(n-1)x \, {}_2F_0\big(b, 2-n; -; -\frac{1}{x}\big)\Big) = 0 \,. \end{aligned} \tag{4.10}$$

The polynomial ${}_2F_0(b, -n; -; -\frac{1}{x})$ is close related to the Charlier polynomial; compare Exton (1996).

5. Prospects and conclusion

Numerous other recurrence relations of hypergeometric functions of the second order with various numbers of terms can be deduced from appropriate generating functions. An example is

$$\Big(\frac{1-x}{t}\Big)^{-d} {}_1F_1(a; c; t) = t^n K_n(x) \,, \tag{5.1}$$

where

$$K_n(x) = (a, n) \, {}_2F_2 \frac{(d, a+n; c+n, 1+n; x)}{[(c, n)n!]} \,. \tag{5.2}$$

Another avenue of approach consists of developing the product of an elementary function and a hypergeometric function of higher order, such as

$$\exp\Big(\frac{x}{t}\Big) \, {}_3F_2(a, b, c; f, g; t) = t^n M_n(x) \,. \tag{5.3}$$

It is found that

$$M_n(x) = (a, n)(b, n)(c, n) \, \frac{{}_3F_3(a+n, b+n, c+n; f+n, g+n, 1+n; x)}{[(f, n)(g, n)n!]} \,. \tag{5.4}$$

The algebra involved becomes rapidly more complicated with increasing order of the hypergeometric functions concerned. In the case of (5.3), an associated differential equation of the third order must be taken into account. As above, second order functions and their recurrences may be deduced by appropriate adjustment of the parameters.

The results obtained in this paper cannot be deduced from known three-term recurrencerelations of the second-order hypergeometric functions. Four-term recurrences of the type obtained above are of general interest in a number of applications and the reader should consult the references in de Lange and Raab (1991), for example. See also the discussion by Dehesa and Yáñez (1994), which might possibly be enlarged using similar methods to those employed above.

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