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On the homogeneous ideal of collinear punctual subschemes of $\mathbf{P}^2$

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Abstract

Fix positive integers s, n_i, $1 \le i \le s$, a finite number of lines, $D_1, \dots, D_s$ of $\mathbf{CP}^2$, points $P_1, \dots, P_s$ with $P_i \in D_i$ for all i and let $Z(i)$ be the length n_i subscheme of D_i with support P_i. Set $Z := \cup_{1 \le i \le s} Z(i)$. Assume D_i and P_i general. Here we show (under mild assumptions on the integers n_i) that the homogeneous ideal of Z has the expected number of generators in each degree and hence we compute the minimal free resolution of Z.

Introduction

In the last few years several papers were written on the cohomological properties (e.g. the postulation, the degree of the generators of the homogeneous ideal or the minimal free resolution) of 0-dimensional subschemes of $\mathbf{P}^n$. In [3], using the so called “Horace method” (see [4]) Eastwood described completely the postulation of a general union $Z \subset \mathbf{P}^2$ of s schemes of finite length $Z(1), \dots, Z(s)$ with each $Z(i)$ connected and contained in a line. These schemes are called “multilinear” or “multijets”. His main result (see [3]. Th. 0) is that a general multilinear plane scheme has the cohomology as good as possible with the constraint given by the s integers $n(i) := \text{length}(Z(i))$. For an interpretation of this result in term of interpolation of polynomials, see the introduction of [3]. Here we study the minimal free resolution of such general multilinear subschemes of $\mathbf{P}^2$. For a 0-dimensional subscheme of $\mathbf{P}^2$ with good postulation (i.e. with maximal rank) and with critical value k this is equivalent to know the number of forms of degree $k+1$ needed to generate its homogeneous ideal (see e.g. [2]). If a suitable numerical condition is satisfied, we will show that the homogeneous ideal of the general multilinear plane scheme with given invariants $n(i){:}=\text{length}(Z(i))$ and critical value k is generated by forms of degree k. As in [3] we will work in characteristic 0. This numerical condition is also a necessary condition for multilinear schemes with maximal rank (see Remark 1.1). We will use in an essential way the results of [3]. To describe our result (see Theorem 0.1) we introduce the following notations. Let $\mathbf{n}$ be a finite sequence $n(1),\dots,n(s)$ of integers with $n(i)\ge n(j)$ if $i\ge j$. Set $n(j)=0$ for $j>s$ and $|\mathbf{n}|=\Sigma_i n(i)$. Let k be the unique integer such that $k(k+1)/2<|\mathbf{n}|\le(k+1)(k+2)/2$. Such an integer k is called the critical value of the datum $\mathbf{n}$. As in [3] we will work in characteristic 0. Let Z be the general multilinear scheme with s components of length $n(1),\dots,n(s)$. In [3], Th. 0, A. Eastwood gave a necessary and sufficient condition for the surjectivity of the restriction map $r(t,Z){:}\,H^0(\mathbf{P}^2,\ \boldsymbol{O}(t))\longrightarrow H^0(Z,\boldsymbol{O}_Z(t))\cong\boldsymbol{O}_Z$ for $t=k$ and the injectivity for $t=k-1$. By the Castelnuovo Mumford lemma if $r(k,Z)$ is surjective, then $r(t,Z)$ is surjective if $t>k$, and the homogeneous ideal of Z is generated in degree k and $k+1$. Set $D(k,\mathbf{n}){:}=\min_{1\le h\le k}\{2hk-h^2-2n(n(1)+\ldots+n(h))\}$ and $G(k,\mathbf{n}){:}=\min_{1\le h\le k}\{(2hk+3h-h^2)/2-(n(1)+\ldots+n(h))\}$. By [3], Th. 0, if $\mathbf{n}$ has critical value k and $G(k,\mathbf{n})\ge 0$, then Z is of maximal rank. In particular if $D(k,\mathbf{n})\ge 0$, then Z has maximal rank. The meaning of the function $D(k,\mathbf{n})$ and of the condition $D(k,\mathbf{n})\ge 0$ was explained in [3], necessity part of the proof of Prop. III. 1.1. For a feeling of the meaning of the condition $G(k,\mathbf{n})\ge 0$ the reader may look at Remark 1.1 below and the informal discussion at the beginning of the proof of part (b) of Theorem 0.1. The entire paper is devoted to the proof of the following result.

Theorem 0.1

Fix a datum $\mathbf{n}=\{n(1)\ge\ldots\ge n(s)\}$ *with critical value* k. *Set* $u'{:}=(k+2)(k+1)/2-|\mathbf{n}|$ *and* $u''{:}=(k+3)(k+2)/2-|\mathbf{n}|$.

(a) Assume $2u\le k(k+2)$, *i.e.* $3u'\ge u''$, *and that* $D(k,\mathbf{n})\ge 0$. *Then the homogeneous ideal of the general multijet* Z *associated to* $\mathbf{n}$ *is generated by* u' *forms of degree* k.

(b) Assume $2u>k(k+2)$ *(i.e.* $3u'\le u''$*),* $G(k,\mathbf{n})\ge 0$ *and the existence of a new datum* $\mathbf{n}'=\{n'(j)\}$ *with* $n'(j)\le n(j)$ *for every* $j,|\mathbf{n}'|=[(k^2+2k+1)/2]$ *and such that if* k *is even, then* $D(k,\mathbf{n}')\ge 0$, *while if* k *is odd* $D(k,\mathbf{n}')\ge -1$. *Then the homogeneous ideal of* Z *is generated by* u' *forms of degree* k *and* u'' *forms of degree* $k+1$.

The assumption $3u' \geq u''$ (resp. $3u' < u''$) in part (a) (resp. part (b)) of Theorem 0.1 are necessary conditions for multijets of maximal rank and critical value k, because if Z is such multijet we have $h^0(\mathbf{P}^2, \boldsymbol{I}_Z(k)) = u'$ (and hence $\dim(H^0(\mathbf{P}^2, \boldsymbol{O}(1)) \otimes H^0(\mathbf{P}^2, \boldsymbol{I}_Z(k))) = 3u'$ and $h^0(\mathbf{P}^2, \boldsymbol{I}_Z(k+1)) = u''$). The other numerical assumptions in the statement of 0.1 come from suitable restrictions to unions of lines. The assumption on the existence of the datum $\mathbf{n}'$ in part (b) of 0.1 is used heavily to apply the statement of part (a) in the proof of part (b).

As in [3] the proof of 0.1 uses an inductive method, the so-called "Méthode d'Horace". The main new trick of this paper is the use of reducible conics for the inductive steps.

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1. Proof of Theorem 0.1

We work always in characteristic 0. This is essential to use freely the degenerations proven in [3]. Set $\prod := \mathbf{P}^2, \boldsymbol{O} := \boldsymbol{O}_{\prod}, \Omega := \Omega^1_{\prod}$. For every subscheme Z of $\prod$ and every sheaf F on $\prod$, set $\boldsymbol{I}_Z := \boldsymbol{I}_{Z,\prod}, H^i(F) := H^i(\prod, F)$ and $h^i(F) := h^i(\prod, F)$. Look at the twists of the dual of the Euler sequence for $\mathrm{T}\mathbf{P}^2$:

$$0 \longrightarrow \Omega(t+1) \longrightarrow 3\boldsymbol{O}(t) \longrightarrow \boldsymbol{O}(t+1) \longrightarrow 0 \tag{1}$$

From (1) we obtain $h^0(\Omega(t)) = t^2 - 1$ for every $t \geq 0$ and $h^1(\Omega(j)) = 0$ for every integer $j \neq 0$. If $h \leq k$, set $S(k,h) := h^0(\Omega(k+1)) - h^0(\Omega(k+1-h)) = 2h(k+1) - h^2$.

We will call "datum" $\mathbf{n}$ a finite sequence $\{n(i)\}_{1 \leq i \leq s}$ (or just $\{n(i)\}$ for short) of integers $n(i) > 0$. Usually we will assume that $n(i) \geq n(j)$ for $i \geq j$; if this is true the datum $\mathbf{n}$ is said to be normalized. The length $|\mathbf{n}|$ of the datum $\{n(i)\}$ is $n(1) + \ldots + n(s)$. We will call "data" a pair $(k, \mathbf{n})$ with k integer, $k > 0$ and $\mathbf{n}$ a datum. For a normalized datum $\mathbf{n} = \{n(i)\}$ and all integers $k > h > 0$, set $C(k,h)(\mathbf{n}) = 2(\Sigma_{1 \leq i \leq h}\, n(i))$, with the convention that $n(i) = 0$ for $i > s$. For a normalized datum $\mathbf{n} = \{n(i)\}$ and integers $k > h > 0$ consider the following condition $Co(k,h)(\mathbf{n})$:

Condition $Co(k,h)(\mathbf{n}) : C(k,h)(\mathbf{n}) \leq S(k,h) := 2h(k+1) - h2$.

The assumption in part (a) of the statement of Theorem 0.1 says that for every h the condition $Co(k,h)(\mathbf{n})$ is satisfied.

Remark 1.1. We will check here that the conditions $Co(k,h)(\mathbf{n})$, $0 < h < k$, are necessary conditions in part (a) of Theorem 0.1 (assuming $G(k,\mathbf{n}) \geq 0$, i.e. by [3] that a general Z has maximal rank). Indeed by the interpretation via Koszul cohomology (or in $\mathbf{P}^2$ just by the Castelnuovo Mumford lemma and the twisted dual of the Euler sequence (1) as explained e.g. in [1] and [2]) the thesis of Theorem 0.1 for $Z = \cup Z(i), Z(i)$ contained in the line $L(i)$ with $L(i) \neq L(j)$ is equivalent to the surjectivity of the restriction map $r(k,Z) : H^0(\Omega(k+1)) \longrightarrow H^0(Z, \Omega(k+1)|Z)$. Set $Z[j] = \cup_{1 \leq i \leq j} Z(i)$ and $L[j] = \cup_{1 \leq i \leq j} L(i)$. If $r(k,Z)$ is surjective, then $r(k,Z[h])$ is surjective for all $h < k$ because $\dim(Z) = 0$. Hence if $r(k,Z)$ is surjective, then the restriction map: $H^0(L[h], \Omega(k+1)|L[h]) \longrightarrow H^0(Z[h], \Omega(k+1)|Z[h])$ is surjective. By the cohomology of $\Omega(t)$ we have $h^0(L[h], \Omega(k+1)|L[h]) = S(k,h)$, while $h^0(Z[h], \Omega(k+1)|Z[h]) = C(k,h)(\mathbf{n})$.

Lemma 1.2

Fix an integer $k \geq 2$. Fix a reducible conic $X = A \cup B$ of $\prod$ (A and B lines) and a 0-dimensional scheme $Z = U \cup V$ with $U \subset A, V \subset B, U \cap B = V \cap A = \emptyset$, length $(U) =$ length$(V) = k$. Then the restriction maps $r(k,Z) : H^0(\Omega(k+1)) \longrightarrow H^0(Z, \Omega(k+1)|Z)$, $r(k,X) : H^0(\Omega(k+1)) \longrightarrow H^0(X, \Omega(k+1)|X)$ and $r(k;X,Z) : H^0(X, \Omega(k+1)|X) \longrightarrow H^0(Z, \Omega(k+1)|Z)$ are surjective. Furthermore, $r(k;X,Z)$ is bijective.

Proof. By the Euler sequence (1) we obtain $h^1(X, \Omega(t)) = 0$ for $t \neq 0$ and the surjectivity of $r(k,X)$. Note that $\Omega(k+1)|A \cong \boldsymbol{O}_A(k) \oplus \boldsymbol{O}_A(k-1)$ (and the same for B) and $h^0(X, \Omega(k+1)|X) = 4k$. Thus we see easily that $h^0(A, (\Omega(k+1)|A) \otimes \boldsymbol{I}_{U,A}) = h^0(B, (\Omega(k+1)|B) \otimes \boldsymbol{I}_{V,B}) = 1$. To check the lemma we need to check that any two non zero sections of $H^0(A, (\Omega(k+1)|A) \otimes \boldsymbol{I}_{U,A})$ and $H^0(B, (\Omega(k+1)|B) \otimes \boldsymbol{I}_{V,B})$ do not satisfy the gluing condition needed to obtain a section of $h^0(X, (\Omega(k+1)|X) \otimes \boldsymbol{I}_{Z,X})$. Since $(\Omega(k+1)|A) \otimes \boldsymbol{I}_{U,A} \cong \Omega(1)|A$ and $(\Omega(k+1)|B) \otimes \boldsymbol{I}_{V,B} \cong \Omega(1)|B$, this is equivalent to $h^0(X, (\Omega(1)|X) = 0$. By the dual Euler sequence we have $h^1(\Omega(-1)) = 0$. Hence we conclude. □

Remark 1.3. By the Euler sequence (1) we have $h^0(X, \Omega(2)) = 6$ and $h^1(X, \Omega) = 1$. Hence with the notations of 1.2 we see easily that $\dim(\mathrm{coker}(r(1,Z))) = 1$.

In [3] A. Eastwood introduced an operation ("collision de front") on the set of multijets and the corresponding operation for the set of all datum. This operation (operation (+)) sends a normalized datum $\mathbf{n} = (n(1), \ldots, n(s))$ with $s \geq 2$ to a normalized datum $\mathbf{n}' = (n(1)+n(2), n(3), \ldots, n(s))$ with $|\mathbf{n}'| = |\mathbf{n}|$. It was proved in [3] that all multijets with datum $\mathbf{n}'$ are the flat limit of a flat family of multijets with datum $\mathbf{n}$. Hence by semicontinuity if $\mathbf{n}'$ has $D(k, \mathbf{n}') = 0$ to prove part (a) of Theorem 0.1 for $\mathbf{n}$ it is sufficient to prove it for $\mathbf{n}'$. Now fix an integer k with $n(1) < k < n(1) + n(2)$. In [3], Th. II. 4.1, A. Eastwood proved the existence of the following degeneration ("collision de biais"). Let $D \subset \mathbf{P}^2$ be a line and Z'' a multijet of type $(n(3), \dots, n(s))$ with $Z'' \cap D = \emptyset$. Let Z' be a connected subscheme of the double line $2D \subset \mathbf{P}^2$ with length$(Z') = n(1) + n(2)$ and length$(Z' \cap D) = k$. Then ([3], Th. II. 4.1) $Z' \cup Z''$ is the flat limit of a flat family of multijets with datum $\mathbf{n}$. We will call operation (++) the use of this degeneration. The proof of 0.1 will be divided into several short parts, each of them numbered. The proof of part 1.4 below follows from the definitions. Part 1.4 implies parts 1.5, 1.6, 1.7, 1.8 and 1.9.

1.4. If $n(1) = n(2) = k$ and $Co(k, h+2)(\mathbf{n})$ is true, then the datum $\mathbf{m} = \{n(i)\}$, $3 \le i \le s$, satisfies $Co(k-2, h)(\mathbf{m})$.

1.5. Assume $n(1) = k$, $n(2) = k-1$, $n(3) > 0$ and let $\mathbf{m}$ be the normalized datum obtained from the $s-2$ integers $n(3)-1$ and $n(i)$, $4 \le i \le s$. If $Co(k, h+2)(\mathbf{n})$ is satisfied, then $Co(k-2, h)(\mathbf{m})$ is satisfied.

1.6. Assume $n(1) = k$, $n(2) = k-2$, $n(4) > 0$ and let $\mathbf{m}$ be the normalized datum obtained from the $s-2$ integers $n(3)-1$, $n(4)-1$ and $n(i), 5 \le i \le s$. If $Co(k, h+2)(\mathbf{n})$ is satisfied, then $Co(k-2, h)(\mathbf{m})$ is satisfied.

1.7. Assume $n(1) = n(2) = k-1$, $n(4) > 0$ and let $\mathbf{m}$ be the normalized datum obtained from the $s-2$ integers $n(3)-1$, $n(4)-1$ and $n(i), 5 \le i \le s$. If $Co(k, h+2)(\mathbf{n})$ is satisfied, then $Co(k-2, h)(\mathbf{m})$ is satisfied.

1.8. Assume $n(1) = k-1$, $n(2) = k-2$, $n(5) > 0$ and let $\mathbf{m}$ be the normalized datum obtained from the $s-2$ integers $n(3)-1$, $n(4)-1, n(5)-1$ and $n(i), 6 \le i \le s$. If $Co(k, h+2)(\mathbf{n})$ is satisfied, then $Co(k-2, h)(\mathbf{m})$ is satisfied.

1.9. Assume $n(1) = k-2$, $n(2) = k-2$, $n(6) > 0$ and let $\mathbf{m}$ be the normalized datum obtained from the $s-2$ integers $n(3)-1$, $n(4)-1, n(5)-1, n(6)-1$ and $n(i), 7 \le i \le s$. If $Co(k, h+2)(\mathbf{n})$ is satisfied, then $Co(k-2, h)(\mathbf{m})$ is satisfied.

1.10. Assume $n(1) + n(2) \le k$ and let $\mathbf{m}$ be the normalized datum obtained from the $s-1$ integers $n(1)+n(2)$ and $n(i), 3 \le i \le s$. If $Co(k, h)(\mathbf{n})$ is satisfied, then $Co(k, h)(\mathbf{m})$ is satisfied. Indeed by assumption we have $2n(j) \le 2n(2) \le k$ for every $j > 2$. Thus $C(k, h)(\mathbf{m}) = C(k, h+1)(\mathbf{n}) \le 2k + (h-1)k = (h+1)k < 2h(k+1) - h^2$.

1.11. Consider the multijet $Z = \cup_{1 \le i \le s} Z(i)$ with $Z(i)$ contained in the line $L(i)$ and with support $P(i)$. Assume for instance $s \ge 8$. Since any two points of $\prod$ are collinear, Z may be considered general even if $\{P(5), P(6)\} \in (L(1) \cup L(2))_{\mathrm{reg}}$ and $\{P(7), P(8)\} \in (L(3) \cup L(4))_{\mathrm{reg}}$.

To prove part (a) of Theorem 0.1 we will consider de following situation. We start with a normalized dat $(k, \mathbf{n}) = \{n(i)\}_{1 \geq i \geq s}$ satisfying $Co(k,h)(\mathbf{n})$ for all h. We will make some operations of type (+) and at most 2 operations of type (++) to obtain a situation in which on each of the lines $L(1)$ and $L(3)$ there is a scheme of length k. We will take the residual scheme W with respect to $L(1) \cup L(3)$. W will be a multijet; call $\mathbf{m}$ its datum. W will be general. We will check that $Co(k-2,h)$ is true for all h with $0 < h < k-2$. Hence by induction on k we way assume Theorem 0.1 for W. Hence by the Horace method with respect to the reducible conic $L(1) \cup L(3)$ and Lemma 1.2 we will obtain that the minimal free resolution of Z is as wanted and by semicontinuity Theorem 0.1 holds for the data $(k, \mathbf{n})$. Hence by induction on k we will conclude.

1.12. First, using 1.10 we reduce to the case $n(1) \leq k$, $n(1)+n(2) > k$. In 1.17 we will consider all cases with $n(3) \geq k-2$. It is easy to check using 1.19 that we may even assume $n(3)+n(4) \geq k-2$. Hence in the following subsection 1.13 we will assume $n(1) \leq k-1$, $n(3) \leq k-3, n(1)+n(2) > k$, $n(3)+n(4) > k-2$. In 1.14 we will consider the easier case in which $n(1) = k$, $n(3) \leq k-3$, $n(1)+n(2) > k$, $n(3)+n(4) > k-2$.

1.13. Here we assume $n(3) \leq k-3$, $n(1)+n(2) > k, n(3)+n(4) > k-2$. Set $c(1) := n(1)+n(2)-k \leq n(2)-1 \leq k-2$, $c(2) := n(3)+n(4)-k+2 \leq n(4)-1 \leq k-3$. As data $(k-2, \mathbf{m})$ we will take the normalized datum $\mathbf{m}$ corresponding to the $s-2$ integers $c(1)$, $c(2), n(5)-1, n(6)-1$, $n(j)$, $7 \leq j \leq s$. Indeed we will apply Horace with respect to $L(1) \cup L(3)$ taking (as explained in 1.11) $P(5) \in L(3)$ and $P(6) \in L(3)$. We will not use the other possibilities given by 1.11, although, taking $P(7) \in L(1)$ we could consider the integer $c(1)-1$ instead of $c(1)$. To check $Co(k-2,h)(\mathbf{m})$ we will distinguish several subcases according to which are the largest h integers between $c(1)$, $c(2)$, $n(5)-1$, $n(6)-1$, $n(j)$, $7 \leq j \leq h+6$. Note that $c(1) \geq c(2)$.

1.13.1. First assume that both $c(1)$ and $c(2)$ are among the first h integers of $c(1), c(2), n(5)-1$, $n(6)-1$, $n(j)$, $7 \leq j \leq h+6$. Then $Co(k-2,h)(\mathbf{m})$ follows from $Co(k,h+2)(\mathbf{n})$ because $S(k,h+2) - S(k-2,h) = 4k = 2(n(1)+n(2)+n(3)+n(4)-c(1)-c(2))$.

1.13.2. Now we will assume that neither $c(1)$ nor $c(2)$ are among the first h integers of $c(1)\, c(2)$, $n(5)-1$, $n(6)-1$, $n(j)$, $7 \leq j \leq h+6$. We distinguish 3 subcases.

1.14.1. Assume that neither $n(5)-1$ nor $n(6)-1$ are in the first h integers of $\mathbf{m}$. This implies $n(5) = n(j)$ for all j with $6 \leq j \leq h+6$. Hence $C(k-2,h)(\mathbf{m}) =$

$2hn(5)$ and $C(k,h+6)(\mathbf{n}) = 2(n(1)+n(2)+n(3)+n(3)+n(4))+2(h+2)n(5)$. We have $S(k,h+6) = 2(h+6)(k+1)-(h+6)^2 = 2hk-h^2+12k+12+2h-12h-36 = 2hk-h^2-10h-24 = S(k-2,h)+12k-10h-24$. Hence the result holds if $2(n(1)+n(2)+n(3)+n(4))+4n(5) \geq 12k-10h-24$. If this inequality is not satisfied, then $12n(5) \leq 12k-10h-26$. Hence $6C(k-2,h)(\mathbf{m}) = 12hn(5) \leq 12hk-10h^2-26h \leq 6S(k-2,h)$, as wanted.

1.14.2. Assume that $n(5)-1$ is one of the first h integers of $\mathbf{m}$ but $n(6)-1$ is not. Hence $n(5) > n(6) = n(j)$ for all j with $7 \leq j \leq h+5$. We have $C(k-2,h)(\mathbf{m}) = 2n(5)-2+2(h-1)n(6)$ and $C(k,h+5)(\mathbf{n}) = 2(n(1)+n(2)+n(3)+n(4))+2n(5)+2hn(6) = C(k-2,h)(\mathbf{m})+2+2(n(1)+n(2)+n(3)+n(4))+2n(6)$. We have $S(k,h+5) = 2(h+5)(k+1)-(h+5)^2 = 2hk-h^2+10k+2h+10-10h-25 = 2hk-h^2+10k-8h-15 = S(k-2,h)+10k-10h-15$. Hence we conclude if $2+2(n(1)+n(2)+n(3)+n(4))+2n(6) \geq 10k-10h-15$. If this inequality is not satisfied, then $4n(5)+n(6) \leq 5k-5h-10$ and $5n(6) \leq 5k-5h-14$, i.e. $n(6) \leq k-h-3$. Hence $10C(k-2,h) = 20n(5)-20+10(h-1)n(6) \leq 25k-25h-52+(20h-25)n(6) \leq 25k-25h-52+(20h-25)(k-h-3) = 25k-25h-52+20hk-20hk-20h^2-60h-25k+25h+75 = 20hk-20h-60h+23 < 10S(k-2,h)$, as wanted.

1.14.3. Assume that both $n(5)-1$ and $n(6)-1$ are among the first h integers in $\mathbf{m}$. We have $C(k,h+4)(\mathbf{n}) - C(k-2,h) = 2(n(1)+n(2)+n(3)+n(4))+4$ and $S(k,h+4) = 2(h+4)(k+1)-(h+4)^2 = 2hk+2h+8k+8-h^2-8h-16 = 2hk-h^2+8k-6h-8 = S(k-2,h)+8k-8h-8$. Hence we conclude if $2(n(1)+n(2)+n(3)+n(4)) \geq 8k-8h-12$. If this inequality fails, then $n(5) \leq k-h-3$. Hence $C(k-2,h)(\mathbf{m}) \leq -4+2hn(5) \leq 2hk-2h^2-6h-4 < S(k-2,h)$, as wanted.

1.15. Here we will consider the case in which $n(1) = k$. Now to apply Horace we will exploit $L(1) \cup L(2)$. First, note that as in 1.10 if $n(2)+n(3) \leq k$ we may take as new datum the $s-1$ integers $k, n(2)+n(3)$, $n(j), 4 \leq j \leq s$. Hence we will assume $n(2)+n(3) > k$. If $n(2) = k$, we conclude by Lemma 1.2. If $n(2) = k-1$, we take $P(3) = L(2) \cap L(3)$ and conclude again by 1.11 and Lemma 1.2. If $n(2) = k-2$, we take $P(3) = L(2) \cap L(3)$ and $P(4) = L(2) \cap L(4)$. Hence we may assume $n(2) \leq k-3$ and $n(2)+n(3) \geq k+1$. Set $c(1) = n(2)+n(3)-k \leq n(3)-3 \leq k-6$. Call $\mathbf{m}$ the new datum corresponding to the normalization of the $s-3$ integers $c(1)$, and $n(j)$, $4 \leq j \leq s$. Fix an integer h with $0 < h \leq k-3$. We distinguish 2 subcases.

1.15.1. Assume that $c(1)$ is among the first h integers of $\mathbf{m}$. Then $C(k-2,h)(\mathbf{m})$ follows from $C(k,h+2)$ as in 1.13.1.

1.15.2. Assume that $c(1)$ is not among the first h integers of $\mathbf{m}$. Hence $C(k,h+3)(\mathbf{n}) = 2k+2n(2)+2n(3)+C(k-2,h)(\mathbf{m})$. We have $S(k,h+3) =$

$2(h+3)(k+1)-(h+3)^2 = 2hk-h^2+6k+2h+6-6h-9 = S(k-2,h)+6k-2h-3$. Hence we win if $2n(2)+2n(3) \geq 4k-2h-3$. If $2n(2)+2n(3) \leq 4k-2h-4$, we have $2n(4) \leq 2k-h-2$. Hence $C(k-2,h) \leq 2kh-h^2-2h = S(k-2,h)$, as wanted.

1.16. Now we will consider the case $n(1) \leq k-1$, $c(1)+k := n(1)+n(2) \geq 0$, $n(3) \leq k-3, c(2)+k-2 := n(3)+n(4) \geq k-2$ and in which among the first h integers of $\mathbf{m}$ there is $c(1)$ but not $c(2)$. We have $c(1) \leq n(2)-1 \leq k-2$. Hence $C(k,h+3)(\mathbf{n}) = C(k-2,h)(\mathbf{m})+2k+2(n(3)+n(4))$. Since $S(k,h+3) = S(k-2,h)+6k-2h-3$, we win if $2(n(3)+n(4)) \geq 4k-4h-3$. Assume $n(3)+n(4) \leq 2k-h-2$. Hence $2n(5) \leq 2k-h-2$. Thus we have $C(k-2,h) \leq (h-1)(2k-h-2)+c(1) \leq S(k-2,h)$, as wanted.

1.17. To conclude the proof of part (a) of 0.1 we discuss all the cases with $n(3) \geq k-2$. Here we will exploit the divisor $L(1) \cup L(2)$ to apply Horace taking (see 1.11 and Lemma 1.2) $P(j) \in L(1)$ for $3 \leq j \leq k-n(1)+2$, $P(j) \in L(2)$ for $k-n(1)+3 \leq j \leq 2k-n(1)-n(2)+2$, without making any operation of type (++).

Proof of part (b) of 0.1. Fix general multijets Z' and Z respectively of type $\mathbf{n}'$ and $\mathbf{n}$ and with $Z' \subseteq Z$. By the generality of Z' and [3] we may assume that Z' has maximal rank. Assume k even. We have $3(h^0(\prod, \boldsymbol{I}_Z(k))) = h^0(\prod, \boldsymbol{I}_Z(k+1))$. Thus by part (a) the multiplication map $m(Z',k) : H^0(\prod, \boldsymbol{O}(1)) \otimes H^0(\prod, \boldsymbol{I}_{Z'}(k)) \longrightarrow H^0(\prod \boldsymbol{I}_{Z'}(k+1))$ is bijective. Hence the multiplication map $m(Z,k) : H^0(\prod, \boldsymbol{O}(1)) \otimes H^0(\prod, \boldsymbol{I}_Z(k)) \longrightarrow H^0(\prod, \boldsymbol{I}_Z(k+1))$ is injective. Since Z has maximal rank, part (b) holds. Now assume k odd. Now we have $3(h^0(\prod, \boldsymbol{I}_Z(k))) = h^0(\prod, \boldsymbol{I}_Z(k+1))+1$. Hence, roughly speaking, to check the injectivity of the multiplication map $m(Z',k)$ (and hence the injectivity of the corresponding multiplication map $m(Z,k)$ for Z) we may "loose one condition". Hence at one step of the proof we may use the Horace method taking as supporting divisor a line instead of a reducible conic and then apply Lemma 1.2. Since $G(k,\mathbf{n}) \geq 0$, we have $n'(1) \leq n(1) \leq k+1$. If $n'(1) = k+1$, we apply Horace using as divisor $L(1)$. Call $\mathbf{m}$ the datum formed by the integers $n'(j)$ with $2 \leq j \leq s$. Since $D(k,\mathbf{n}') \geq -1$, we obtain $D(k-1,\mathbf{m}) \geq 0$ and we apply part (a) to $\mathbf{m}$. Since $2|\mathbf{n}'| = h^0(\Omega(k+1))+1$, we have $2|\mathbf{m}| = h0(\Omega(k))$. Hence by when we apply apart (a) to $\mathbf{m}$ we obtain $H^0(\Omega \otimes \boldsymbol{I}_{Z''}) = 0$ for a general Z'' of type $\mathbf{m}$. Hence by Horace $H^0(\Omega \otimes \boldsymbol{I}_{Z'})$ for a general Z' of type $\mathbf{n}'$. Thus the same is true for a general Z of type $\mathbf{n}$ with $Z' \subseteq Z$. If $n'(1) \leq k$, we reduce to the previous cases making operations of type (+) and at most one operation of type (++) involving each time only the first 2 integers, $n''(1)$ and $n''(2)$. To check that in this way we find $D(k-1,\mathbf{m}) \geq 0$, see the part of the proof o part (a) in which only one operation of type (++) is made. □

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