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Tensor products and joint spectra for solvable Lie algebras of operators

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Abstract

Given two complex Hilbert spaces, H_1 and H_2, and two complex solvable finite dimensional Lie algebras of operators, L_1 and L_2, such that L_i acts on H_i (i= 1,2), the joint spectrum of the Lie algebra $L_1 \times L_2$, which acts on $H_1 \overline{\otimes} H_2$, is expressed by the cartesian product of $Sp(L_1, H_1)$ and $Sp(L_2, H_2)$.

1. Introduction

J.L. Taylor developed in [6] a notion of joint spectrum for an n-tuple a, $a = (a_1, ..., a_n)$, of mutually commuting operators acting on a Banach space E, i.e., $a_i \in \mathcal{L}(E)$, the algebra of all bounded linear operators on E, and $[a_i, a_j] = 0$, $1 \leq i, j \leq n$. This interesting notion, which extends in a natural way the spectrum of a single operator, has many important properties, among then, the projection property and the fact that $Sp(a, E)$ is a compact non empty subset of $\mathbb{C}^n$, where $Sp(a, E)$ denotes the joint spectrum of a in E.

One of the most remarkable results of Taylor joint spectrum is the one related with tensor products of tuples of operators. For example, in [2], Z. Ceausescu and F.H. Vasilescu proved the following result. Let H_i, $1 \leq i \leq n$, be complex Hilbert

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spaces, and a_i, $1 \le i \le n$, be bounded linear operators defined on H_i, respectively, $1 \le i \le n$. If we denote by H the completion of the tensor product $H_1 \otimes \ldots \otimes H_n$ with respect to the canonical scalar product, we may consider the n-tuple of operators $\tilde{a}$, $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$, where $\tilde{a}_i = 1 \otimes \ldots \otimes 1 \otimes a_i \otimes 1 \otimes \ldots \otimes 1$, $1 \le i \le n$, and 1 denotes the identity of the corresponding spaces. Then the following identity holds,

$$Sp(a, E) = Sp(a_1) \times \ldots \times Sp(a_n) .$$

Furthermore, in [3], Z. Ceausescu and F.H. Vasilescu showed that if H_1 (resp. H_2) is a complex Hilbert space and $a = (a_1, \ldots, a_n)$, (resp. $b = (b_1, \ldots, b_m)$), is a mutually commuting tuple of operators acting on H_1, (resp. H_2), then, the commuting tuple $(\tilde{a}, \tilde{b}) = (a_1 \otimes 1, \ldots, a_n \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_m)$ in $\mathcal{L}(H_1 \overline{\otimes} H_2)$, satisfies the relation,

$$Sp\big((\tilde{a}, \tilde{b}), H_1 \overline{\otimes} H_2\big) = Sp(a, H_1) \times Sp(b, H_2) ,$$

where 1 denotes the identity map of the corresponding Hilbert spaces, and $H_1 \overline{\otimes} H_2$ is the completion of the tensor product $H_1 \otimes H_2$ with respect to the canonical scalar product, see [3, Th 2.2].

In [1], we defined a joint spectrum for complex solvable finite dimensional Lie algebras of operators, L, acting on a Banach space E, and we denoted it by $Sp(L, E)$. We proved that $Sp(L, E)$ is a compact non empty subset of L^*, and that the projection property for ideals still holds. Besides, when L is a commutative algebra, our spectrum reduces to Taylor joint spectrum in the following sense. If $\dim L = n$, and if $\{a_i\}_{(1 \le i \le n)}$ is a basis of L, we consider the n-tuple $a = (a_1, \ldots, a_n)$, then $\{(f(a_1), \ldots, f(a_n)), f \in Sp(L, E)\} = Sp(a, E)$, i.e., $Sp(L, E)$ in terms of the basis of L^* dual of $\{a_i\}_{(1 \le i \le n)}$, coincides with the Taylor joint spectrum of the n-tuple a. Then, the following question arises naturally. If H_i, $i = 1, 2$, are two complex Hilbert spaces, and L_i, $i = 1, 2$, are two complex solvable finite dimensional Lie algebras of operators such that L_i acts on H_i, respectively, $i = 1, 2$, is there any relation between $Sp(L_1 \times L_2, H_1 \overline{\otimes} H_2)$ and $Sp(L_1, H_1) \times S(L_2, H_2)$.

In this paper we answer this question in the affirmative. Moreover, by a refinement of the argument of Z. Ceausescu and F.H. Vasilescu in [3], we extend the main results of [2] and [3] for complex solvable finite dimensional Lie algebras and its joint spectrum. In order to describe in more detail our main theorem we need to introduce a definition. If H_i and L_i are as above, $i = 1, 2$, we consider the direct product of L_1 and L_2, i.e., the complex solvable finite dimensional Lie algebra $L_1 \times L_2$ defined by,

$$L_1 \times L_2 = \big\{x_1 \otimes 1 + 1 \otimes x_2, x_i \in L_i, i = 1, 2\big\} ,$$

where 1 is as above. Then, its is clear that $L_1 \times L_2$ is a Lie algebra of operators which acts on $H_1 \overline{\otimes} H_2$, and our main theorem may be stated as follows,

$$Sp\big(L_1 \times L_2, H_1 \overline{\otimes} H_2\big) = Sp\big(L_1, H_1\big) \times Sp\big(L_2, H_2\big)\,,$$

where the above sets are considered as subsets of $(L_1 \times L_2)^*$ under the natural identification $(L_1 \times L_2)^* \cong L_1^* \times L_2^*$.

The paper is organized as follows. In Section 2 we review several definitions and results of [1], and we also prove a proposition which is an important step to our main result. Finally, in Section 3, we prove our main theorem.

2. Preliminaries

We briefly recall several definitions and results, related to the spectrum of a complex solvable Lie algebra of operators, see [1]. From now on L denotes a complex solvable finite dimensional Lie algebra, and H a complex Hilbert space on which L acts as right continuous operators, i.e., L is a Lie subalgebra of $\mathcal{L}(H)$ with the opposite product. If $\dim(L) = n$, and if f is a character of L, i.e., $f \in L^*$ and $f(L^2) = 0$, where $L^2 = \{[x, y]; x, y \in L\}$, let us consider the following chain complex, $(H \otimes \wedge L, d(f))$, where $\wedge L$ denotes the exterior algebra of L, and $d_p(f)$ is as follows,

$$d_p(f)\colon H \otimes \wedge^p L \to H \otimes \wedge^{p-1} L\,,$$

$$\begin{aligned} d_p(f) e\big\langle x_1 \wedge \ldots \wedge x_p\big\rangle &= \sum_{k=1}^{k=p} (-1)^{k+1} e\big(x_k - f(x_k)\big)\big\langle x_1 \wedge \ldots \wedge \hat{x_k} \wedge \ldots \wedge x_p\big\rangle \\ &\quad + \sum_{1 \leq k < l \leq p} (-1)^{k+l} e\big\langle [x_k, x_l] \wedge x_1 \ldots \hat{x_k} \ldots \hat{x_l} \ldots x_p\big\rangle\,, \end{aligned}$$

where $\hat{}$ means deletion. If $p \leq 0$ or $p \geq n+1$, we define $d_p(f) = 0$.

If we denote by $H_*((H \otimes \wedge L, d(f)))$ the homology of the complex $(H \otimes \wedge L, d(f))$, we may state our first definition.

Definition 1. With L and f as above, the set $\{f \in L^*, f(L^2) = 0, H_*((H \otimes \wedge L, d(f))) \neq 0\}$ is the joint spectrum of L acting on H, and it is denoted by $Sp(L, H)$.

As we have said, in [1] we proved that $Sp(L, E)$ is a compact non empty subset of L^* which reduces to Taylor joint spectrum when L is a commutative algebra, in the sense explained in the introduction. Besides, if I is an ideal of L, and π denotes the projection map from L^* to I^*, then,

$$Sp(I, H) = \pi\big(Sp(L, H)\big)\,,$$

i.e., the projection property for ideals still holds. With regard to this property, we ought to mention the paper of C. Ott, see [5], who pointed out a gap in the proof of this result, and give another proof of it. In any case, the projection property remains true.

We observe that the set $H \otimes \wedge L$ has a natural structure of Hilbert space, so that the sets $H \otimes \langle x_{i_1} \wedge \ldots \wedge x_{i_p} \rangle$, $1 \leq i_1 < \ldots < i_p \leq n$, $0 \leq p \leq n$, are orthogonal subspaces of $H \otimes \wedge L$, and if $<,>$ denotes the inner product of H, $< a\langle x_{i_1} \wedge \ldots \wedge x_{i_p} \rangle, b\langle x_{i_1} \wedge \ldots \wedge x_{i_p} \rangle >=< a, b >$.

We shall have occasion to use the direct product of two complex solvable finite dimensional Lie algebras, and its action on the tensor product of two complex Hilbert spaces. We recall here the main facts which we need for our work. If H_i, i =1,2, are two complex Hilbert spaces, $H_1 \overline{\otimes} H_2$ denotes the completion of the tensor product $H_1 \otimes H_2$ with respect to the canonical scalar product. Now, if L_i, i=1,2, are two complex solvable finite dimensional Lie algebras of operators, such that L_i acts on H_i, respectively, i=1,2, we consider the algebra $L_1 \times L_2$, the direct product of L_1 and L_2, which acts in a natural way on $H_1 \overline{\otimes} H_2$, and it is defined by,

$$L_1 \times L_2 = \left\{ x \otimes 1 + 1 \otimes y; x \in L_1 + y \in L_2 \right\},$$

where 1 denotes the identity of the corresponding space.

It is clear that $L_1 \times L_2$, defined as above, is a complex solvable finite dimensional Lie subalgebra of $\mathcal{L}(H_1 \overline{\otimes} H_2)$. Moreover, by the structure of the Lie bracket in $L_1 \times L_2$, we have two distinguished ideals, L_1' and L_2', which we define as follows,

$$L_1' = \left\{ x \otimes 1; x \in L_1 \right\}, \qquad L_2' = \left\{ 1 \otimes y; y \in L_2 \right\}.$$

In addition, if we consider the natural identification $\tilde{K} : (L_1 \times L_2)^* \cong L_1^* \times L_2^*$, $\tilde{K}(f) = (f \circ i_1, f \circ i_2)$, where $f \in (L_1 \times L_2)^*$, and $i_j \colon L_j \to L_1 \times L_2$, $j = 1, 2$, are the canonical inclusions, as $(L_1 \times L_2)^2 = L_1^2 \times L_2^2$, we have that the set of characters of $L_1 \times L_2$ is the cartesian product of the sets of characters of L_1 and L_2.

The following proposition is an important step to our main theorem.

Proposition 1

Let H_i, $i = 1, 2$, be two complex Hilbert spaces, and L a complex solvable finite dimensional Lie algebra of operators acting on H_1. Let L'_1, (resp. L'_2), be the Lie algebra of operators $\{x \otimes 1, x \in L\}$, (resp. $\{1 \otimes x, x \in L\}$), which acts on $H_1 \overline{\otimes} H_2$, (resp. $H_2 \overline{\otimes} H_1$), where 1 denotes the identity map of H_2, (resp. H_1). Then,

i) $Sp\big(L_1', H_1 \overline{\otimes} H_2\big) = Sp\big(L_2', H_2 \overline{\otimes} H_1\big)$,

ii) $Sp\big(L_1', H_1 \overline{\otimes} H_2\big) \subseteq Sp\big(L, H_1\big)$,

iii) $Sp\big(L_2', H_2 \overline{\otimes} H_1\big) \subseteq Sp\big(L, H_1\big)$.

Proof. It is clear that $iii)$ is a consequence of $i)$ and $ii)$. Let us prove $i)$.

If f is a character of L, we consider C_1, (resp. C_2), the Koszul complex associated to the Lie algebra $L_1^{'}$, (resp. $L_2^{'}$), and f, then,

$$C_1 = \left(H_1 \overline{\otimes} H_2 \otimes \wedge L_1^{'}, d^1(f)\right), \qquad\qquad C_2 = \left(H_2 \overline{\otimes} H_1 \otimes \wedge L_2^{'}, d^2(f)\right),$$

where the maps $d^i(f)$, i= 1,2, are as above.

In addition, a elementary calculation shows that the map μ,

$$\mu_p\colon H_1 \overline{\otimes} H_2 \otimes \wedge^p L_1^{'} \to H_2 \overline{\otimes} H_1 \otimes \wedge^p L_2^{'}\,,$$
$$\mu_p\big(e_1 \otimes e_2 \langle x_1, \ldots, x_p\rangle\big) = e_2 \otimes e_1 \big\langle x_1, \ldots, x_p\big\rangle\,,$$

defines an isomorphism which commutes with $d^i(f)$, i=1,2, i.e., μ is an isomorphism of chain complexes. Then,

$$Sp\big(L_1^{'}, H_1 \overline{\otimes} H_2\big) = Sp\big(L_2^{'}, H_2 \overline{\otimes} H_1\big)\,.$$

In order to verify $ii)$, let us consider the Koszul complex associated to L and f, $C = (H_1 \otimes \wedge^p L, d(f))$, and $\tilde{C}$ the following chain complex,

$$\tilde{C} = \big((H_1 \otimes \wedge L) \overline{\otimes} H_2, d(f) \otimes 1\big)\,,$$

where 1 denotes the identity map of H_2.

We observe that if η is the map defined by,

$$\eta\colon (H_1 \overline{\otimes} H_2) \otimes \wedge^p L_1^{'} \to (H_1 \otimes \wedge^p L) \overline{\otimes} H_2\,,$$
$$\eta_p\big(e_1 \otimes e_2 \langle x_1, \ldots, x_p\rangle\big) = e_1 \big\langle x_1, \ldots, x_p\big\rangle \otimes e_2\,,$$

an easy calculation shows that η defines an isomorphism of chain complexes between C_1 and $\tilde{C}$.

Now, if f does not belong to $Sp(L, H_1)$, by [8; L.2.4], the complex $\tilde{C}$ is exact. As η is an isomorphism of chain complexes, C_1 is exact, then f does not belong to $Sp(L_1^{'}, H_1 \overline{\otimes} H_2)$. $\square$

3. The main result

We now state our main result.

Theorem

Let H_1 and H_2 be two complex Hilbert spaces, and L_1 and L_2 be two complex solvable finite dimensional Lie algebras of operators, such that L_i acts on H_i, respectively, i= 1,2. Let us consider the complex solvable finite dimensional Lie algebra $L_1 \times L_2$, which acts on $H_1 \overline{\otimes} H_2$ and it is defined by,

$$L_1 \times L_2 = \big\{ x_1 \otimes 1 + 1 \otimes x_2, x_i \in L_i,\ i = 1,2 \big\} ,$$

where 1 denotes the identity of the corresponding spaces.

Then,

$$Sp\big(L_1 \times L_2, H_1 \overline{\otimes} H_2\big) = Sp\big(L_1, H_1\big) \times Sp\big(L_2, H_2\big) ,$$

where, in the above equality, the set $Sp(L_1 \times L_2, H_1 \overline{\otimes} H_2)$ is considered as a subset of $L_1^* \times L_2^*$ under the natural identification $\tilde{K} \colon (L_1 \times L_2)^* \cong L_1^* \times L_2^*$ of Section 2.

Proof. In order to prove that $Sp(L_1 \times L_2, H_1 \overline{\otimes} H_2)$ is contained in the cartesian product of $Sp(L_1, H_1)$ and $Sp(L_2, H_2)$, let $L_1^{'}$, (resp.$L_2^{'}$), be the ideal of $L_1 \times L_2$ defined by $\{x \otimes 1, x \in L_1\}$, (resp. $\{1 \otimes y, y \in L_2\}$), where 1 is as above, see Section 2. Then, by the projection property of the joint spectrum, Proposition 1, and by the identification $\tilde{K}$,

$$\begin{aligned} Sp\big(L_1 \times L_2, H_1 \overline{\otimes} H_2\big) &\subseteq Sp\big(L_1^{'}, H_1 \overline{\otimes} H_2\big) \times Sp\big(L_2^{'}, H_1 \overline{\otimes} H_2\big) \\ &\subseteq Sp\big(L_1, H_1\big) \times Sp\big(L_2, H_2\big) . \end{aligned}$$

Let us prove the converse inclusion. We consider, first, the following preliminary facts.

Let f_i be a character of L_i, and let K_i be the Koszul complex of L_i acting on H_i, associated to f_i, for $i = 1, 2$,

$$K_1 = \big(H_1 \otimes \wedge L_1, d^1(f_1)\big) , \qquad K_2 = \big(H_2 \otimes \wedge L_2, d^2(f_2)\big) .$$

We observe that there is a natural identification between the spaces $(H_1 \otimes H_2) \otimes \wedge(L_1 \times L_2)$ and $(H_1 \otimes \wedge L_1) \otimes (H_2 \otimes \wedge L_2)$. If we denote by ψ this identification, ψ is the following map,

$$\psi \colon (H_1 \otimes H_2) \otimes \wedge(L_1 \times L_2) \to (H_1 \otimes \wedge L_1) \otimes (H_2 \otimes \wedge L_2) ,$$

$$\psi\big(e_1\otimes e_2\langle x_1\wedge\ldots\wedge x_p\wedge y_1\wedge\ldots\wedge y_q\rangle\big)=e_1\langle x_1\wedge\ldots\wedge x_p\rangle\otimes e_2\langle y_1\wedge\ldots\wedge y_q\rangle\,,$$

where $e_1\in H_1$, $e_2\in H_2$, $p\in[\![1,n]\!]$, $q\in[\![1,m]\!]$, and $n=\dim(L_1)$, $m=\dim(L_2)$.

As

$$(H_1\overline{\otimes}H_2)\otimes\wedge^k(L_1\times L_2)=\oplus_{p+q=k}(H_1\otimes\wedge^p L_1)\overline{\otimes}(H_2\otimes\wedge^q L_2)\,,$$

if we consider $H_i\otimes\wedge L_i$, i=1,2, $(H_1\overline{\otimes}H_2)\otimes\wedge(L_1\times L_2)$ and $(H_1\otimes\wedge L_1)\overline{\otimes}(H_2\otimes\wedge L_2)$ with their natural structure of Hilbert spaces, a straightforward calculation shows that the map ψ may be extended to an isometric isomorphism from $(H_1\overline{\otimes}H_2)\otimes\wedge(L_1\times L_2)$ onto $(H_1\otimes\wedge L_1)\overline{\otimes}(H_2\otimes\wedge L_2)$.

Besides, if f is a character of $L_1\times L_2$, and if we define $f_j=f\circ i_j\in L_j^*$, where $i_j\colon L_j\to L_1\times L_2$ are the canonical inclusions and $j=1,2$, i.e., if we consider f decomposed under the natural identification $\tilde{K}\colon(L_1\times L_2)^*\cong L_1^*\times L_2^*$, if K is the Koszul complex of $L_1\times L_2$ acting on $H_1\overline{\otimes}H_2$ associated to f,

$$K=\big((H_1\overline{\otimes}H_2)\otimes\wedge(L_1\times L_2),d^k(f)\big)\,,$$

an easy calculation shows that,

$$\psi d^k(f)=\big(d^1(f_1)\otimes 1+\xi\otimes d^2(f_2)\big)\psi\,,$$

where ξ is the map,

$$\xi\colon\oplus_{p=0}^n(H_1\otimes\wedge^p L_1)\to\ \oplus_{p=0}^n(H_1\otimes\wedge^p L_1)\,,$$
$$\xi=\ \oplus_{p=0}^n(-1)^p\,,$$

and where 1 is the identity map of H_1, and $d^k(f)$ is the boundary map of the Koszul complex K, equivalently, if we consider the algebraic tensor product of the complexes K_1 and K_2, $K_1\otimes K_2$, and its natural completion $K_1\overline{\otimes}K_2$, the map ψ provides an isometric isomorphism of chain complexes, from K onto $K_1\overline{\otimes}K_2$.

Moreover, if T_i, $i=1,2$, and T_k are the maps,

$$T_i=d^i(f_i)+d^i(f_i)^*,\qquad\qquad T_k=d^k(f)+d^k(f)^*,$$

as ξ is a selfadjoint map, an easy calculation shows that,

$$\psi T_k=\big(T_1\otimes 1+\xi\otimes T_2\big)\psi\,.$$

Let us return to the proof. If f does not belong to $Sp(L_1\times L_2,H_1\overline{\otimes}H_2)$, by [7; L.3.1], the operator T_k is an invertible map. On the other hand, if f_i, $i=1,2$, belongs to $Sp(L_i,H_i)$, $i=1,2$, as T_i is a selfadjoint operator, $i=1,2$, there exist by [7; L.3.1] and by [4; II,31] two sequences of unit vectors, $(a_n^i)_{n\in\mathbb{N}}$, $i=1,2$, such that $\parallel a_n^i\parallel=1$, $a_n^i\in H_i\otimes\wedge L_i$, and $T_i(a_n^i)\xrightarrow[n\to\infty]{}0$, for $i=1,2$. However, as ψ and ψ^{-1} are isometric isomorphisms, by elementary properties of the tensor product in Hilbert spaces, we have that: $\parallel a_n^1\otimes a_n^2\parallel=1$, $\parallel\psi^{-1}(a_n^1\otimes a_n^2)\parallel=1$, and $T_k(\psi^{-1}(a_n^1\otimes a_n^2))\xrightarrow[n\to\infty]{}0$, equivalently, by [4; II,31], $0\in Sp(T_k)$, which is impossible by our assumption. □

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