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A remark on Fefferman-Stein's inequalities

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Abstract

It is proved that, for some reverse doubling weight functions, the related operator which appears in the Fefferman Stein's inequality can be taken smaller than those operators for which such an inequality is known to be true.

§ 1. The results

Let T be a classical operator (i.e. a fractional maximal operator or a Calderon-Zygmund operator or a fractional integral operator). The Fefferman-Stein's inequality related to T is

$$(1)\qquad \int_{\mathbb{R}^n} \big|(Tf)(x)\big|^p w(x)dx \leq C \int_{\mathbb{R}^n} \big|f(x)\big|^p (Sw)(x)dx \quad \text{for all functions} \quad f;$$

where w is a weight function (i.e. a nonnegative locally integrable functions) and S is a convenient operator which will be precised. In (1), it is assumed that $n \in \mathbb{N}^*$, $1 < p < \infty$ and $C > 0$ is a constant which does not depend on each f. The dual inequality associated to (1) is denoted as

$$(1^*)\quad \int_{\mathbb{R}^n} \big|(Tg)(x)\big|^{p'} (Sw)(x)^{1-p'} dx \leq C \int_{\mathbb{R}^n} \big|g(x)\big|^{p'} w(x)^{1-p'} dx \text{ for all functions } g;$$

where $p' = p/(p-1)$. As it is known [4] and as it will be alluded in this text, inequalities (1) and (1*) are useful tools in Harmonic Analysis and in Analysis of Partial Differential Equations.

Our purpose on this paper is to prove that, for some reverse doubling weight functions, the related operator S which appears in (1) and (1^*) can be taken smaller than those operators for which they are known to be true.

Recall that a weight function w satisfies *a reverse doubling condition* (or $w \in RD_\rho$) when there are $\rho > 0$, $c > 0$ for which

$$t^{n\rho} \int_{|y|<R} w(y)dy \leq c \int_{|y|<tR} w(y)dy$$

for all $R > 0$ and $t > 1$. In particular any doubling weight function satisfies the reverse doubling condition (see [10]). The growth condition

$$(\mathcal{H}) \qquad w(x) \leq c_0 |x|^{-n} \int_{\{c_1|x|<|z|<c_2|x|\}} w(z)dz \quad \text{for all} \quad x \neq 0$$

is also useful. Here $c_0, c_1, c_2 > 0$ are constants which do not depend on x. Condition ($\mathcal{H}$) is satisfied for instance for radial monotone weight functions.

The fractional maximal operator M_s of order s, $0 \leq s < n$, is defined as

$$(M_s f)(x) = \sup \left\{ |Q|^{s/n-1} \int_Q f(y)dy;\, Q \ \text{ cubes containing } \ x \right\}.$$

So $M = M_0$ is the Hardy-Littlewood maximal operator. For $1 < p < n/s$, then it is well-known that

$$(2) \qquad \int_{\mathbb{R}^n} (M_s f)^p(x) w(x)dx \leq C \int_{\mathbb{R}^n} f^p(x)(M_{sp}w)(x)dx \quad \text{for all} \quad f \geq 0\,.$$

Inequality (2) with $s = 0$ is the classical result due to C. Fefferman and E. Stein [4]. And for $s > 0$, then (2) can be derived by a Sawyer's result [9]. Inequality (2) is a crucial point to obtain vector-valued inequalities for maximal functions, which are for instance useful to study function spaces (see [4]). Although M_s is not a linear operator, a sort of dual version of (2) holds, since

$$(2^*) \qquad \int_{\mathbb{R}^n} (M_s g)^{p'}(x)(Sw)^{1-p'}(x)dx \leq C \int_{\mathbb{R}^n} g^{p'}(x) w^{1-p'}(x)dx \quad \text{for all} \quad g \geq 0\,.$$

with $(Sg)(x) = (S_t g)(x) = (M_{sp} g^t)^{1/t}(x)$ where $t > 1$. (2^*) can be used to get inequality (2) for singular integral operator (see for instance the proof of Theorem 2 below). For $s = 0$, Pérez [6] was able to get (2^*) with smaller operators S such as $c_1(Mw)(x) \leq (Sw)(x) \leq c_2(S_t w)(x)$. For $w \in RD_\rho \cap \mathcal{H}$ we can get a better result.

Theorem 1

Let $0 \le s < n$, $1 < p < n/s$. Assume w satisfying a reverse doubling condition and the growth condition $(\mathcal{H})$. Then there is $C > 0$ such that

$$\int_{\mathbb{R}^n} (M_s g)^{p'}(x)(M_{sp}w)^{1-p'}(x)dx \le C \int_{\mathbb{R}^n} g^{p'}(x)w^{1-p'}(x)dx.$$

The constant $C > 0$ in this result depends only on n, p and on the constants which appear in the reverse doubling condition and in condition $(\mathcal{H})$.

Next we consider the case of any Calderon-Zygmund operator T. Such an operator can be viewed as a linear operator taking $C_c^\infty(\mathbb{R}^n)$ into $L^1_{\text{loc}}(\mathbb{R}^n, dx)$, bounded on $L^2(\mathbb{R}^n, dx)$ and for which

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad a.e. \quad x \notin \operatorname{supp} f$$

for each $f \in L_c^\infty(\mathbb{R}^n, dx)$. Here $K(x,y)$ is a continuous function defined on $\{(x,y);\, x \ne y\}$ and satisfies the standard estimates: $|K(x,y)| \le C|x-y|^{-n}$ and $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\left(\frac{|x-x'|}{|x-y|}\right)^{\varepsilon}|x-y|^{-n}$ whenever $2|x-x'| \le |x-y|$, and where $C > 0$, $\varepsilon \in]0,1]$ do not depend on x, y, x'. By a classical result due to A. Córdoba and C. Fefferman [3] then

$$\int_{\mathbb{R}^n} \big|(Tf)(x)\big|^p w(x)dx \le C \int_{\mathbb{R}^n} \big|f(x)\big|^p (Sw)(x)dx \quad \text{for all} \quad f \tag{3}$$

with $(Sg)(x) = (S_t g)(x) = (Mg^t)^{1/t}(x)$ for any $t > 1$. Pérez [5] improved this result by proving that (3) is true with smaller operators S as in (2^*) with $s = 0$. Again a better result holds when $w \in RD_\rho \cap \mathcal{H}$.

Theorem 2

Let $1 < p < \infty$, and w a reverse doubling weight function satisfying the growth condition $(\mathcal{H})$. Then there is $C > 0$ such that

$$\int_{\mathbb{R}^n} \big|(Tg)(x)\big|^p w(x)dx \le C \int_{\mathbb{R}^n} \big|g(x)\big|^p (Mw)(x)dx\,.$$

Of course since T is a linear operator, then clearly this inequality is equivalent to

$$\int_{\mathbb{R}^n} |(T^*g)(x)|^{p'}(Mw)(x)^{1-p'}dx \le C \int_{\mathbb{R}^n} |g(x)|^{p'} w(x)^{1-p'}dx$$

where T^* is the dual of the operator T.

Finally we end with the case of the fractional integral operator I_s, $0 < s < n$,

$$(I_s f)(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)dy\,.$$

By a result due to D. Adams [1] then, for $1 < p < n/s$:

$$\int_{\mathbb{R}^n} (I_s f)^p(x)w(x)dx \leq C \int_{\mathbb{R}^n} f^p(x)(Sw)(x)dx \quad \text{for all} \quad f \geq 0 \tag{4}$$

with $(Sg)(x) = (S_{st}g)(x) = (M_{spt}g^t)^{1/t}(x)$ for any t with $1 < t < n/sp$. Recently Pérez [3] improved (4), by getting this inequality with $(Sf)(x) = (M_{sp}M^{[p]}w)(x)$ which is a smaller operator since $(M_\beta M^k w)(x) \leq (M_{\beta t}w^t)^{1/t}(x)$ (see [7]), here M^k is the k-iteration of the Hardy-Littlewood maximal operator. A better estimate can be obtained for weight functions $w \in RD_\rho \cap \mathcal{H}$.

Theorem 3

Let $0 < s < n$, $1 < p < n/s$. Assume w satisfying a reverse doubling condition and the growth condition $(\mathcal{H})$. Then there is $C > 0$ such that

$$\int_{\mathbb{R}^n} (I_s g)^p(x)w(x)dx \leq C \int_{\mathbb{R}^n} g^p(x)(M_{sp}w)(x)dx\,.$$

As indicated in [7], such inequality leads to

$$\int_{\mathbb{R}^n} \big|f(x)\big|^2 w(x)dx \leq C \int_{\mathbb{R}^n} \big|(\nabla f)(x)\big|^2 (M_2 w)(x)dx$$

which is useful in Partial Differential Equation (when solving a degenerate elliptic problem) and in quantum mechanics (when estimating eigenvalues of the operator $-\triangle - w$). Oubviously since I_s is a linear operator, then inequality in Theorem 4 is equivalent to

$$\int_{\mathbb{R}^n} (I_s g)^{p'}(x)(M_{sp}w)(x)^{1-p'}dx \leq C \int_{\mathbb{R}^n} g^{p'}(x)w(x)^{1-p'}dx\,.$$

In the next paragraph, we give the main estimates needed to prove the results in §3.

§ 2. Basic tools

We will deduce the proofs of our theorems, by a recent result due to the author [8].

Proposition

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$, $0 \leq s < n$ *(when we deal with* I_s *we take* $0 < s < n$*). We assume*

$$(2.1)\qquad |x|^{sp'}\Big[\sup_{\{\frac{1}{4}|x|<|z|<4|x|\}} u(z)\Big] \leq cv(x) \quad \text{for almost all} \quad x\,;$$

$$(2.2)\qquad R^{s-n}\Big(\int_{|x|<R} v^{1-p}(x)dx\Big)^{1/p}\Big(\int_{|y|<R} u(y)dy\Big)^{1/p'} \leq c \quad \text{for all} \quad R>0;$$

$$(2.3)\qquad \Big(\int_{|x|<R} v^{1-p}(x)dx\Big)^{1/p}\Big(\int_{R<|y|} |y|^{(s-n)p'}u(y)dy\Big)^{1/p'} \leq c \quad \text{for all} \quad R>0.$$

Then for a constant $C > 0$

$$(2.4)\qquad \int_{\mathbb{R}^n} (M_s f)^{p'}(x)u(x)dx \leq C\int_{\mathbb{R}^n} f^{p'}(x)v(x)dx \quad \text{for all} \quad f \geq 0\,.$$

Similarly suppose (2.1), (2.2), (2.3) *and*

$$(2.3^*)\qquad \Big(\int_{|x|<R} u(x)dx\Big)^{1/p'}\Big(\int_{R<|y|} |y|^{(s-n)p}v^{1-p}(y)dy\Big)^{1/p} \leq c \quad \text{for all} \quad R>0\,.$$

Then

$$(2.5)\qquad \int_{\mathbb{R}^n} (I_s f)^{p'}(x)u(x)dx \leq C\int_{\mathbb{R}^n} f^{p'}(x)v(x)dx \quad \text{for all} \quad f \geq 0\,.$$

In fact the boundedness (2.4) implies (2.2) and (2.3), and (2.1) is also a necessary condition when both u and v satisfy condition $\mathcal{H}$ (see [8]).

Lemma 1

The Muckenhoupt condition (2.2) *implies the Hardy condition* (2.3) *(resp.* (2.3*)*) whenever* $v^{1-p} \in RD_\rho$ *(resp.* $u \in RD_\rho$*) for some* $\rho > 0$.

A proof can be found in [8]. Moreover it is also seen in this paper that the one side Fefferman-Phong condition

$$(2.6)\quad R^s\left(\frac{1}{R^n}\int_{|x|<R} v^{(1-p)t}(x)dx\right)^{1/tp}\left(\frac{1}{R^n}\int_{|y|<R} u(y)dy\right)^{1/p'} \leq c \text{ for all } R > 0,$$

(where $t > 1$) implies both (2.2) and (2.3). Consequently the two weight norm inequality (2.4) is true whenever the pointwise estimate (2.1) and the condition (2.6) are satisfied. Without (2.1), Pérez [6] obtained (2.4) from a similar one side Fefferman condition but this times relatively with arbitrary cubes. Although it seems restrictive, our result yields conditions easy to check. Indeed, for a given weight functions, we just have to make a pointwise inequality and integrations over arbitrary balls centered at the origin (which are more convenient for many usual weights, particularly for radial weights). Similarly the two side Fefferman-Phong condition

$$R^s\left(\frac{1}{R^n}\int_{|x|<R} v^{(1-p)t}(x)dx\right)^{1/tp}\left(\frac{1}{R^n}\int_{|y|<R} u^t(y)dy\right)^{1/tp'} \leq c$$

implies the boundedness (2.5) whenever (2.1) is assumed.

Lemma 2

Let $1 < p < n/s$ *with* $0 \leq s < n$ *(we take* $1/0 = \infty$*). Define* u *as* $u = (M_{sp}w)^{1-p'}$. *There is a constant* $C > 0$ *such that*

$$(2.7)\quad u(x) = (M_{sp}w)^{1-p'}(x) \leq C\left(R^{sp-n}\int_{|y|<R} w(y)dy\right)^{1-p'} \quad \textit{for all} \quad |x| < R\,,$$

and

$$(2.8)\qquad |x|^{sp'} \sup_{\frac{1}{4}|x|<|y|<4|x|} u(y) \leq Cv(x) = Cw^{1-p'}(x),$$

for all x, y *with* $1/4|x| < |y| < 4|x|$.

Inequality (2.7) is true since

$$R^{sp-n}\int_{|y|<R} w(y)dy \le R^{sp-n}\int_{|x-y|<2R} w(y)dy \le c(M_{sp}w)(x)\,.$$

To see (2.8) first note that for $1/4|x| < |y| < 4|x|$:

$$u(y) = (M_{sp}w)^{1-p'}(y) \le c\Big(|x|^{sp-n}\int_{c_1|x|<|z|<c_2|x|} w(z)dz\Big)^{1-p'}\,,$$

since

$$|x|^{sp-n}\int_{c_1|x|<|z|<c_2|x|} w(z)dz \le c'|y|^{sp-n}\int_{|y-z|<(1+4c_2)|y|} w(z)dz \le c(M_{sp}w)(y)\,.$$

Then using the growth condition $(\mathcal{H})$, the conclusion is obtained as follows:

$$\begin{aligned}|x|^{sp'}u(y) &\le c|x|^{sp'}\Bigg(|x|^{sp-n}\int_{c_1|x|<|z|<c_2|x|} w(z)dz\Bigg)^{1-p'}\\ &= c\Bigg(|x|^{-n}\int_{c_1|x|<|z|<c_2|x|} w(z)dz\Bigg)^{1-p'} \le c\,c_0^{p'-1}w^{1-p'}(x)\,.\end{aligned}$$

§ 3. Proofs of results

Proof of Theorem 1. Let $u = (M_{sp}w)^{1-p'}$, $v = w^{1-p'}$ so that $v^{1-p} = w$. By (2.8), the pointwise estimate (2.1) is satisfied. On the other hand the Muckenhoupt condition (2.2) appears by using (2.7). By the hypothesis then $v^{1-p} = w \in RD_\rho$, and consequently by Lemma 1, the Hardy condition (2.3) is satisfied. Thus by our Proposition, the two weight norm inequality (2.4) is true, which is nothing else than the inequality in Theorem 1. □

Proof of Theorem 2. Since T is a linear operator, then it is equivalent to get

$$\int_{\mathbb{R}^n}\big|(T^*g)(x)\big|^{p'}(Mw)(x)^{1-p'}dx \le C\int_{\mathbb{R}^n}\big|g(x)\big|^{p'}w(x)^{1-p'}dx$$

where T^* is the dual of the operator T. First observe that $u = (Mw)^{1-p'}$ satisfies the usual Muckenhoupt condition A_∞ (see [4] for the definition). Indeed choosing

$t > p'$ or $\frac{p'-1}{t-1} < 1$, then by the standard technique [4]: $(Mw)^{(p'-1)/(t-1)} \in A_1$ and $u = [(Mw)^{(p'-1)/(t-1)}]^{1-t} \in A_t$. Next by a result due to Coifman [3] ($u \in A_\infty$) then

$$\int_{\mathbb{R}^n} \big|(T^*g)(x)\big|^{p'} (Mw)(x)^{1-p'} dx \leq C \int_{\mathbb{R}^n} \big|Mg(x)\big|^{p'} (Mw)(x)^{1-p'} dx$$

and consequently we are reduced to the inequality in Theorem 1 (with $s = 0$). □

Proof of Theorem 3. Since I_s is a linear operator, then it is equivalent to obtain

$$\int_{\mathbb{R}^n} \big|(I_s g)(x)\big|^{p'} (M_{sp}w)(x)^{1-p'} dx \leq C \int_{\mathbb{R}^n} \big|g(x)\big|^{p'} w(x)^{1-p'} dx\,.$$

Again taking $u = (M_{sp}w)^{1-p'}$ and $v = w^{1-p'}$ then, as in the proof of Theorem 1, all conditions (2.1), (2.2) and (2.3) are satisfied. Since $u = (M_{sp}w)^{1-p'} \in A_{p'}$ then u is a doubling weight function and consequently $u \in RD_\rho$ (see [10]) for some $\rho > 0$. So again by Lemma 1, the dual condition (2.3*) is satisfied and consequently, using the Proposition, Theorem 3 is now proved. □

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