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On the LURWC property of Orlicz sequence space

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Abstract

Necessary and sufficient conditions for URWC points and LURWC property are given in Orlicz sequence space l_M.

In 1986 M.A. Smith [1] introduced the concept of local uniform rotundity in the direction of weakly compact sets (LURWC). Let X be a Banach space, $S(X), B(X)$ be the unit sphere and unit ball, respectively $x \in S(X)$ is called a URWC point provided $x_n \in B(X)$, $\|x_n + x\| \to 2$ and $x_n \xrightarrow{w} z$ imply $z = x$. X is said to be LURWC provided every point on $S(X)$ is a URWC point.

In this paper, we discuss the criteria for URWC points and LURWC property in Orlicz sequence space. Let $M(u)$ be a real, even, continuous and convex function on $(-\infty, +\infty)$, $M(0) = 0$, $\lim_{u\to\infty} M(u) = \infty$ and $M(u) > 0$ $(u > 0)$. Let $N(v)$ and $M(u)$ be a pair of complemented N-functions, $N(v) = \max_{u>0}\{u|v| - M(u)\}$. We say that $M(u)$ satisfies the Δ_2-condition provided there exist $u_0 > 0$ and $K \geq 0$ such that $M(2u) \leq kM(u)$ for every $u \in [0, u_0]$ ($M \in \Delta_2$ for short). An interval $[a, b]$ is called a structural affine interval of $M(u)$ if $M(u)$ is linear on $[a, b]$, but not on $[a - \varepsilon, b]$ and $[a, b + \varepsilon]$ for any $\varepsilon > 0$. Let the sequence consists of $\{[a_n, b_n]\}_{n=1}^{\infty}$ all structural affine intervals. Define

$$S_M := R \setminus \bigcup_{n=1}^{\infty} (a_n, b_n), \qquad S_M^0 := R \setminus \bigcup_{n=1}^{\infty} [a_n, b_n],$$

and let $\{a\}, \{b\}$ denote all left and right end points of the structural affine intervals of $M(u)$, respectively (a certain real number may belong to both $\{a\}$ and $\{b\}$). If $M(u)$ have no structural affine interval on $[c,d]$, we call $M(u)$ is strictly convex on $[c,d]$ ($M \in SC[c,d]$ for short).

On the space of all real sequences $x = (x(j))_{j=1}^{\infty}$ we denote the modular of x by $\rho_M(x) = \sum_{j=1}^{\infty} M(x(j))$. Linear set $l_M = \{x = (x(j))_{j=1}^{\infty} : \exists \lambda > 0,\ \rho_M(x/\lambda) < \infty\}$ becomes a Banach space if it is endowed with the Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho_M(x/\lambda) \le 1\right\}.$$

We call it the Orlicz sequence space and denote by l_M.

Theorem 1

If $x \in S(l_M)$, then x is URWC *point if and only if*

(i) there exists $0 < \tau < 1$, such that $\rho_M(x/(1-\tau)) < \infty$.

(ii) $\{j : |x(j)| \notin S_M\}$ is singleton at most.

(iii) If there exists j_0 such that $|x(j_0)| \in \{a\}$, then $\{j \ne j_0 : |x(j)| \in \{b\}\} = \emptyset$; if there exists j_0 such that $|x(j_0)| \in \{b\}$, then $\{j \ne j_0 : |x(j)| \in \{a\}\} = \emptyset$.

Proof. Necessity. We suppose without loss of generality that $x(j) \ge 0$ $(j = 1,2,\ldots)$. If (i) is false, then for any $\varepsilon > 0$, $\rho_M((1+\varepsilon)x) = \infty$. We may suppose as well that $x(j_0) > 0$. Put $y = (x(1),\ldots,x(j_0-1),-x(j_0),x(j_0+1),\ldots)$. Then $\|y\| = \|x\| = 1$, $\rho_M((x+y)/2) = \sum_{j\ne j_0} M(x(j)) < \rho_M(x) \le 1$, and for any $\varepsilon > 0$,

$$\rho_M\big((1+\varepsilon)(x+y)/2\big) = \rho_M\big((1+\varepsilon)x\big) - M\big((1+\varepsilon)x(j_0)\big) = \infty\,,$$

which shows that $\|(x+y)/2\| = 1$.

Obviously $y \ne x$, so this contradicts the fact that x is a URWC point.

Since any URWC point is surely an extreme point, by Theorem 7 [2] we obtain (ii) immediately.

If (iii) is false, we can assume that there are structural affine intervals $[a,b]$ and $[a',b']$ of $M(u)$, such that $x(1) = a, x(2) = b'$. Take t, t' small enough, satisfying $a + t \le b,\ b' - t' \ge a'$ and $M(a+t) - M(a) = M(b') - M(b'-t')$. Put

$$y = \big(a+t,\ b'-t',\ x(3),\ x(4),\ldots\big).$$

Then

$$\begin{aligned}\rho_M(y) &= M(a+t) + M(b'-t') + \sum_{j=3}^{\infty} M\big(x(j)\big)\\ &= M(a) + M(b') + \sum_{j=3}^{\infty} M\big(x(j)\big) = \rho_M(x)\,.\end{aligned}$$

Since we have proved that there is $0 < \tau < 1$ such that $\rho_M(x/(1-\tau)) < \infty$, we get $\rho_M(x) = 1$ hence $\rho_M(y) = 1$. This shows that $\|y\| = 1$. Similarly

$$\begin{aligned}\rho_M\big((x+y)/2\big) &= M\big((a+a+t)/2\big) + M\big((b'+b'-t')/2\big)\\ &\qquad + \sum_{j=3}^{\infty} M\big(x(j)\big)\\ &= \big(M(a)+M(a+t)\big)/2 + \big(M(b')+M(b'-t')\big)/2 + \sum_{j=3}^{\infty} M\big(x(j)\big)\\ &= \big(\rho_M(x)+\rho_M(y)\big)/2 = 1,\end{aligned}$$

which means $\|(x+y)/2\| = 1$. Obviously $y \neq x$, a contradiction.

Sufficiency. Let us still suppose that $x(j) \geq 0$ $(j = 1, 2, \ldots)$. Suppose that $x_n \in B(l_M)$ for any $n \in \mathbb{N}$ and $\|x_n + x\| \to 2$, $x_n \xrightarrow{w} z$.

In order to prove the equality $z = x$, it suffices to show that $x_n(j) \to x(j)$ $(j = 1, 2, \ldots)$. First we will prove that for any $\alpha \in [0,1]$,

$$\rho_M\big(\alpha x_n + (1-\alpha)x\big) \to 1 \qquad \big(n \to \infty\big). \tag{1}$$

From $\|x\| = 1$, $\|x_n\| \leq 1$ and $\|(x+x_n)/2\| \to 1$, we easily deduce that $\|\alpha x_n/2 + (1-\alpha/2)x\| \to 1$. Thus for any $\varepsilon > 0$, $\|(1+\varepsilon)(\alpha x_n/2 + (1-\alpha/2)x\| > 1$ for n large enough. Therefore

$$\begin{aligned}1 &\leq \rho_M\big((1+\varepsilon)(\alpha x_n/2 + (1-\alpha/2)x)\big)\\ &= \rho_M\big((1/2)(1+\varepsilon)[\alpha x_n + (1-\alpha)x] + [(1/2)(1-\varepsilon)(1+\varepsilon)/(1-\varepsilon)]x\big)\\ &\leq (1/2)(1+\varepsilon)\rho_M\big(\alpha x_n + (1-\alpha)x\big) + (1/2)(1-\varepsilon)\rho_M\big([(1+\varepsilon)/(1-\varepsilon)]x\big).\end{aligned}$$

Since $\rho_M(x\,(1-\tau)) < \infty$, setting $\varepsilon \to 0$, we have

$$1 \leq (1/2)\liminf_{n\to\infty} \rho_M\big(\alpha x_n + (1-\alpha)x\big) + (1/2)\rho_M(x),$$

which yields $\liminf_{n\to\infty} \rho_M(\alpha x_n + (1-\alpha)x) \geq 1$, and by the obvious inequality $\rho_M(\alpha x_n + (1-\alpha)x) \leq 1$, we get (1).

Now, we will prove that

$$\lim_{j_0\to\infty} \sup_n \sum_{j>j_0} M\big(x_n(j)\big) = 0.$$

If it is false, then there are $\varepsilon_0 > 0$ and $j_n \to \infty$ satisfying $\sum_{j>j_n} M(x_n(j)) \geq \varepsilon_0$ $(n = 1, 2, \ldots)$. (Here $\{x_n\}$ may be subsequence of sequence original $\{x_n\}$). Since $x_n \xrightarrow{w} z$, $\{x_n\}_{n=1}^{\infty}$ is weakly compact set in l_M, it is further l_{N^-} weakly compact set, from the well known result in [3], $\lim_{n\to\infty} \sup_n m\rho_M(x_n/m) = 0$. We may take m large enough satisfying

$$m\tau < m - 1 + \tau \quad \text{and} \quad m\rho_M\big(x_n/m\big) < \varepsilon_0/2 \quad \big(n = 1, 2, \ldots\big).$$

Hence

$$\sum_{j>j_n} M\big(x_n(j)/m\big) \leq \rho_M(x_n/m) < (1/2m)\varepsilon_0 \leq (1/2m) \sum_{j>j_n} M\big(x_n(j)\big) \quad (n = 1, 2, \ldots).$$

By condition (i) of sufficiency, when n is large enough

$$\sum_{j>j_n} M\big(x(j)/(1-\tau)\big) < \tau\,\varepsilon_0/\big(4(m-1+\tau)\big).$$

Combining this with (1), we obtain

$$\begin{aligned}
1 \leftarrow \rho_M&\Big(\frac{\tau}{m-1+\tau}x_n + \frac{(m-1)}{m-1+\tau}x\Big) \\
&= \sum_{j=1}^{j_n} M\Big(\frac{\tau}{m-1+\tau}x_n(j) + \frac{(m-1)}{m-1+\tau}x(j)\Big) \\
&\quad + \sum_{j>j_n} M\Big(\frac{m\tau}{m-1+\tau}\,\frac{x_n(j)}{m} + \frac{(m-1)(1-\tau)}{m-1+\tau}\,\frac{x(j)}{1-\tau}\Big) \\
&\leq \frac{\tau}{m-1+\tau}\sum_{j=1}^{j_n} M\big(x_n(j)\big) + \frac{m-1}{m-1+\tau}\sum_{j=1}^{j_n} M\big(x(j)\big) \\
&\quad + \frac{m\tau}{m-1+\tau}\sum_{j>j_n} M\big(x_n(j)/m\big) + \frac{(m-1)(1-\tau)}{m-1+\tau}\sum_{j>j_n} M\big(x(j)/(1-\tau)\big) \\
&\leq \frac{\tau}{m-1+\tau}\sum_{j=1}^{j_n} M\big(x_n(j)\big) + \frac{m-1}{m-1+\tau}\sum_{j=1}^{j_n} M\big(x(j)\big) \\
&\quad + \frac{m\tau}{m-1+\tau}\,\frac{1}{2m}\sum_{j>j_n} M\big(x_n(j)\big) + \frac{(m-1)(1-\tau)}{m-1+\tau}\sum_{j>j_n} M\big(x(j)/(1-\tau)\big)
\end{aligned}$$

$$\leq \frac{\tau}{m-1+\tau}\rho_M(x_n) - \frac{\tau}{2(m-1+\tau)}\sum_{j>j_n} M\big(x_n(j)\big)$$
$$+ \frac{m-1}{m-1+\tau}\rho_M(x) + \frac{\tau\varepsilon_0}{4(m-1+\tau)}$$
$$\leq 1 - \frac{\tau\varepsilon_0}{2(m-1+\varepsilon)} + \frac{\tau\varepsilon_0}{4(m-1+\tau)} = 1 - \frac{\tau\varepsilon_0}{4(m-1+\tau)}$$

a contradiction. Therefore (2) is true.

In the following we shall prove that if $x(j_0) \in S_M \setminus \{b\}$, then

$$\liminf_{n\to\infty}\ x_n(j_0) \geq x(j_0). \tag{3}$$

If $x(j_0) \in S_M \setminus \{a\}$, then

$$0 \leq \limsup_{n\to\infty} x_n(j_0) \leq x(j_0). \tag{4}$$

If $x(j_0) \in S_M^0$, then

$$\lim_{n\to\infty} x_n(j_0) = x(j_0). \tag{5}$$

In fact, if (3) is false, then there exists $\varepsilon_0 > 0$ such that (passing to a subsequence of necessary) we have $x_n(j_0) \leq x(j_0) - \varepsilon_0$.

Since $x(j_0)$ may only be the left end point of a structural affine interval of $M(u)$ but not the right one, hence $x_n(j_0)$ and $x(j_0)$ is not in the same structural affine interval of $M(u)$.

From $|x_n(j_0)| \leq M^{-1}(1)$ and $x_n(j_0) \leq x(j_0) - \varepsilon_0$, there is $\delta > 0$ satisfying

$$M\big((x_n(j_0) + x(j_0))/2\big) \leq (1-\delta)\big[M(x_n(j_0)) + M(x(j_0))\big]/2, \quad (n = 1, 2, \ldots),$$

and by (1), we have

$$0 \leftarrow \frac{\rho_M(x_n) + \rho_M(x)}{2} - \rho_M\Big(\frac{x_n + x}{2}\Big)$$
$$= \sum_{j=1}^{\infty}\Big[\frac{M(x_n(j)) + M(x(j))}{2} - M\Big(\frac{x_n(j) + x(j)}{2}\Big)\Big]$$
$$\geq \frac{M(x_n(j_0)) + M(x(j_0))}{2} - M\Big(\frac{x_n(j_0) + x(j_0)}{2}\Big)$$
$$\geq (\delta/2)\big[M(x_n(j_0)) + M(x(j_0))\big] \geq (\delta/2)M\big(x(j_0)\big)$$

a contradiction, which shows that (3) is true.

The proof of (4) is analogous to (3). From (3) and (4) we obtain (5) simply. Now we will prove the theorem itself. There may be the following four cases according to (ii) and (iii)

$$x(j) \in S_M^0 \quad (j = 1, 2, \ldots). \tag{I}$$

By (5), we easily get $\lim\limits_{n\to\infty} x_n(j) = x(j) \ \ (j = 1, 2, \ldots)$.

$$x(j) \in S_M^0, \quad (j \neq j_0). \tag{II}$$

By (5), $\lim\limits_{n\to\infty} x_n(j) = x(j) \ \ (j \neq j_0)$. Moreover by (2), $\lim\limits_{n\to\infty} \sum_{j\neq j_0} M(x_n(j)) = \sum_{j\neq j_0} M(x(j))$, thus

$$\begin{aligned}\lim_{n\to\infty} M\big(x_n(j_0)\big) &= \lim_{n\to\infty} \rho_M(x_n) - \lim_{n\to\infty} \sum_{j\neq j_0} M\big(x_n(j)\big) \\ &= 1 - \sum_{j\neq j_0} M\big(x(j)\big) = M\big(x(j_0)\big),\end{aligned}$$

hence $\lim\limits_{n\to\infty} |x_n(j_0)| = x(j_0)$.

Since we can directly deduce that $\lim\limits_{n\to\infty} x_n(j_0) = -x(j_0)$, which contradicts the condition $\rho_M((x_n + x)/2) \to 1$. Therefore $\lim_{n\to\infty} x_n(j_0) = x(j_0)$.

$$x(j) \in S_M^0 \bigcup \{b\} \setminus \{a\}. \tag{III}$$

From (5) and (4), $0 \leq \limsup\limits_{n\to\infty} x_n(j) \leq x(j)$, thus $\limsup\limits_{n\to\infty} M(x_n(j)) \leq M(x(j))$, $(j = 1, 2, \ldots)$. If there is j_0 such that $\limsup\limits_{n\to\infty} M(x_n(j_0)) < M(x(j_0))$, combining (2), we get

$$1 = \limsup_{n\to\infty} \rho_M(x_n) < \rho_M(x) = 1,$$

a contradiction. Hence $\limsup\limits_{n\to\infty} x_n(j) = x(j) \ \ (j = 1, 2, \ldots)$. Since the above equation holds for any subsequence of $\{x_n\}$, therefore

$$\lim_{n\to\infty} x_n(j) = x(j) \quad (j = 1, 2, \ldots).$$

$$x(j) \in S_M^0 \bigcup \{a\} \setminus \{b\}. \tag{IV}$$

The proof is analogous as in case III. □

Theorem 2

The space l_M *is* LURWC *if and only if*

(i) $M \in \Delta_2$,

(ii) $M \in SC[0, M^{-1}(1/2)]$.

Proof. Necessity. Since LURWC implies strict rotundity, by Theorem 3 [2], we get the result immediately.

Sufficiency. For any $x \in S(l_M)$, by $M \in \Delta_2$, we get $\rho_M(x/(1-\tau)) < \infty$ for any $\tau > 0$. Since $M \in SC[0, M^{-1}(1/2)]$ and $\rho_M(x) = \sum_{j=1}^{\infty} M(x(j)) = 1$, we know that if $|x(j)| \notin S_M$, so $M(x(j)) \geq 1/2$. Thus, either there is only one j_0 such that $|x(j_0)| \notin S_M$ or there are $j_0 \neq j_1$, such that $|x(j_0)| = |x(j_1)| = M^{-1}(1/2)$. However $M^{-1}(1/2)$ may only belong to $\{a\}$ but not to $\{b\}$, so (ii) and (iii) of Theorem 1 hold. This means that x is a URWC point. □

References

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