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Effective constructions of separable quotients of Banach spaces

Marek Wójtowicz

Institute of Mathematics, Pedagogical University,

Pl. Słowiański 9, 65-069 Zielona Góra, Poland

Abstract

A simple way of obtaining separable quotients in the class of weakly countably determined (WCD) Banach spaces is presented (Theorem 1). A large class of Banach lattices, possessing as a quotient c_0, l_1, l_2, or a reflexive Banach space with an unconditional Schauder basis, is indicated (Theorem 2).

1. Introduction

In this paper we are concerned with effective constructions of separable quotients of Banach spaces, and we complete to the Banach space case general results obtained for locally convex spaces in [19]. We consider the so called Separable Quotient Problem, which P. Casazza states in the paper [3] as one of the more important and still unsolved problems in Functional Analysis:

> *Can every infinite dimensional Banach space be mapped continuously onto an infinite dimensional and separable Banach space?*

So far, most of the results obtained in that topic are purely existential in nature (see e.g., [7],[15], [19]; H.P. Rosenthal [14] has proved that all infinite dimensional $C(K)$-spaces have c_0 or l_2 as a quotient, cf. [8]), and they yield no information on the structure of separable quotients in general, if such exist. On the other hand, there are Banach spaces possessing plenty of complemented separable subspaces ([2], [4], [5], [18], [20], [22], [23]), but proofs of this fact are dependent on the toilsome and noneffective constructing of a continuous projection with separable range.

The purpose of the present paper is to give a simple method of obtaining separable quotients in the class of weakly countably determined Banach spaces without using the above mentioned projections with separable ranges (Theorem 1), and to indicate a wide class of Banach spaces having "nice" separable quotients (Theorem 2).

2. The WCD case

Let $E = (E, \tau)$ be a locally convex space, and let E' denote its topological dual. The terms "weak" and "weak*" refer to the topologies $\sigma(E, E')$ and $\sigma(E', E)$, respectively. If E is a Banach space, then $\chi(E)$ [$\chi^*(E")$, resp.] denotes the norm [weak*, resp.] density character of E [of E', resp.]; similar meanings have the respective density characters of subspaces of E and E', resp. If W is a nonempty subset of E', then $W_\perp$ denotes the annihilator (in E) of W. A Banach space E is said to be weakly compactly generated (WCG) provided that there is a linearly dense weakly compact subset C in E; and E is called weakly countably determined (WCD) whenever there is a sequence $\mathcal{A} = (A_n)$ of absolutely convex, closed and bounded subsets of E with $E = \bigcup_{I \in J(\mathcal{A})} \bigcap_{n \in I} A_n$, where $J(\mathcal{A}) = \{I \subset \mathbb{N} : \bigcap_{n \in I} A_n$ is weakly compact$\}$ (see [23], Definitions 1 and 2; or [21], p. 138).

In the proposition below we collect fundamental facts concerning WCG and WCD Banach spaces.

Proposition 1

(a) Every WCG space is a WCD space ([23], Lemma 2).
(b) The WCD property is inherited by quotient spaces [23], Theorem 3).
(c) If E is WCD then $\chi(E) = \chi^*(E')$ ([23], Corollary 2).
(d) Every reflexive space is WCG, and if E is WCG which can be continuously mapped onto a dense subspace of a Banach space G then G is WCG also ([9], Proposition 2.1).

The main theorem of this part presented below is inspired by a similar result for the so called countably WCG locally convex spaces ([19], Theorem 1).

Theorem 1

Let E be a WCD Banach space. If W is an infinite dimensional weak-separable and weak*-closed subset of E' then the quotient Banach space $E/W_\perp$ is separable and infinite dimensional.*

Proof. By Proposition 1, parts (b) and (c), we have $\chi(E/W_\perp) = \chi^*((E/W_\perp)') = \chi^*(W) = \aleph_0$. Since $(E/W_\perp)'$ is infinite dimensional, we have that $E/W_\perp$ is infinite dimensional as well. □

3. The Banach lattice case

In the second part of the paper we consider separable quotients of Banach lattices. The main result of this part (Theorem 2) is proved by means of weak*-Schauder basic sequences, and two propositions presented below contain all the needed facts for our purposes. For the convenience of the reader we recall successively fundamental results and notions both from theories of Schauder bases and linear lattices (=Riesz spaces). For more information in these topics we refer the reader to the monographs [1], [11], [12], [13], [16], [17].

A sequence (e_n) in a locally convex space (E,τ) is called a τ- Schauder [unconditional] basis of E if for each $x \in E$ there is a unique sequence (α_n) of scalars such that $x = \tau - \sum_{n=1}^{\infty} \alpha_n e_n$ [unconditional convergence] and the coefficient functionals (e_n^*) defined by $e_n^*(x) = \alpha_n$, $n = 1, 2, \ldots$, are τ-continuous. Two bases (e_n) and (f_n) of locally convex spaces E and F, resp., are said to be equivalent provided that the series $\sum_{n=1}^{\infty} \alpha_n e_n$ and $\sum_{n=1}^{\infty} \alpha_n f_n$ are convergent simultaneously; if E and F are Banach spaces then, by the Closed Graph Theorem, the linear operator $T : E \to F$ defined by $Te_n = f_n$, $n = 1, 2, \ldots$, is a homeomorphism. A sequence (e_n) in E is said to be a τ-Schauder basic sequence provided that (e_n) forms a Schauder basis for the space $[e_n]^\tau := \overline{\operatorname{lin}\{e_n : n \in \mathbb{N}\}}^\tau$. If E is a Banach space and τ is the norm topology on E, then we simply write $[e_n]$ instead of $[e_n]^\tau$. By using the Hahn-Banach theorem, it is easy to check that if (x_n^*) is a weak*- Schauder basic sequence in E' then there is a (not uniquely determined) sequence (x_n) in E such that the sequence (x_n, x_n^*), $n = 1, 2, \ldots$, is biorthogonal and for every $x^* \in [x_n^*]^{weak^*}$ we have $x^* = weak^* - \sum_{n=1}^{\infty} x^*(x_n) x_n^*$.

The next proposition summarizes elementary properties of weak*-Schauder basic sequences in Banach spaces.

Proposition 2

Let $E = (E, \|\ \|)$ be a Banach space.
(a) The space E has a separable quotient if and only if E' has a weak-Schauder basic sequence.*
(b) Let (x_n^*) be a weak-Schauder basic sequence in E', let (x_n) be a fixed, corresponding to (x_n^*), sequence in E' (as described above), and put $X = \bigcap_{n=1}^{\infty} \ker x_n^*$. Then*

(i) (x_n^*) *is a norm-Schauder basic sequence in* E'*;*
(ii) *if* Q *denotes the quotient map from* E *onto* E/X*, then the sequence* (Qx_n) *is a Schauder basis of* E/X *(endowed with the quotient norm); moreover,*
(iii) *if* (x_n^*) *is weak***-unconditional, then the basis* (Qx_n) *is unconditional.*

Proof. Part (a) is proved in ([19], Proposition 1), parts (i) and (ii) of (b) are proved in ([6], Proposition II.1); we shall show that part (iii) of (b) holds true as well.

Put $u_n = Qx_n$, $n = 1, 2, \ldots$, fix $u = \sum_{n=1}^{\infty} \alpha_n u_n$ in E/X, and let (t_n^o) be a sequence of reals with $|t_n^o| = 1$, $n = 1, 2, \ldots$. We claim that the series $\sum_{n=1}^{\infty} t_n^o \alpha_n u_n$ converges (in norm) in E/X. Since, by assumption, for every $x^* \in [x_n^*]^{weak^*}$ and $x \in E$ we have $\sum_{n=1}^{\infty} |x^*(x_n)||x_n^*(x)| < \infty$, the linear functional f, defined on the Banach space $l_\infty \oplus [x_n^*]^{weak^*} \oplus E$ by the rule $f((t_n), x^*, x) = \sum_{n=1}^{\infty} t_n x^*(x_n) x_n^*(x)$, is continuous. It follows that the number

(1) $C := \sup\{\| \sum_{n=1}^{m} t_n x^*(x_n) x_n^* \|: \ \| x^* \| \leq 1, \| (t_n) \|_{l_\infty} \leq 1, m \in \mathbb{N}\}$ is finite.

Now fix integers p, q with $p < q$, and $x \in X$ with

(2) $\| \sum_{n=p}^{q} \alpha_n u_n \| \geq (1/2) \| x + \sum_{n=p}^{q} \alpha_n x_n \|$,

and for a given $u' \in (E/X)'$ put $x^* = Q^*(u')$ and note that $x^* \in [x_n^*]^{weak^*}$ and $\| x^* \| \leq \| u' \|$. Since the sequence (x_n, x_n^*), $n = 1, 2, \ldots$, is biorthogonal and (x_n^*) is weak*-Schauder basic, by (1) and (2) we have

$$|u'(\sum_{n=p}^{q} t_n^o \alpha_n u_n)| = |x^*(x + \sum_{n=p}^{q} t_n^o \alpha_n x_n)| =$$
$$| \sum_{k=1}^{\infty} x^*(x_k) x_k^*(x + \sum_{n=p}^{q} t_n^o \alpha_n x_n)| =$$
$$| \sum_{k=1}^{q} t_n^o x^*(x_k) x_k^*)(x + \sum_{n=p}^{q} \alpha_n x_n)| \leq 2C \| \sum_{n=p}^{q} \alpha_n u_n \|$$

whenever $\| u' \| \leq 1$. It follows that $\| \sum_{n=p}^{q} t_n^o \alpha_n u_n \| \leq 2C \| \sum_{n=p}^{q} \alpha_n u_n \|$, where C does not depend on p, q, and (t_n^o), and therefore the series $\sum_{n=1}^{\infty} t_n^o \alpha_n u_n$ is norm-convergent in E/X, as claimed. $\square$

Now we come to Banach lattices. A topology τ on a linear lattice E is said to be locally solid provided that τ possesses a basis for 0 consisting of solid sets (a subset A of E is solid if $A = \bigcup_{x \in A} [-|x|, |x|]$). If, additionally, τ is locally convex, then (E, τ) is called locally convex-solid. If the norm $\| \ \|$ defined on a Riesz space E is monotonic (i.e., $|x| \leq |y|$ follows that $\| x \| \leq \| y \|$) and E is $\| \ \ \|$-complete, then $E = (E, \| \ \ \|)$ is said to be a Banach lattice; the topological dual E' of E is then a Banach lattice with respect to the dual norm and the ordering: $x^* \geq 0$ iff $x^*(x) \geq 0$ for all $x \in E^+$. The norm on E is said to be order continuous provided that $x_\alpha \downarrow 0$ in E follows that $\| x_\alpha \| \to 0$.

Proposition 3

If the norm of a Banach lattice $E = (E, \|\ \ \|)$ is order continuous, then every sequence (x_n^*) of positive and pairwise orthogonal elements of E' is an unconditional weak*-Schauder basic sequence.

Proof. The fact that (x_n^*) is a weak*-Schauder basic sequence is proved in ([19]; third proof of Proposition 3, pp. 181-182). We shall show that (x_n^*) is unconditional in $(E', weak^*)$. For sequences (α_n), (t_n) of reals with (α_n) arbitrary and $|t_n| = 1$, $n = 1, 2, \ldots$, we have

$$\Big|\sum_{n=1}^{m} t_n\alpha_n x_n^*\Big| = \sum_{n=1}^{m} |t_n\alpha_n| x_n^* = \Big|\sum_{n=1}^{m} \alpha_n x_n^*\Big|, \quad m = 1, 2, \ldots,$$

(since all x_n^*'s are pairwise orthogonal and positive). It follows that the series $\sum_{n=1}^{\infty} \alpha_n x_n^*$ and $\sum_{n=1}^{\infty} t_n\alpha_n x_n^*$ are convergent simultaneously in the locally-convex topology $|\sigma|(E', E)$ on E' defined by the collection of seminorms $p_x(x^*)=|x^*|(|x|)$, $x \in E$. Since the norm on E is order continuous, the topologies $\sigma(E', E)$ and $|\sigma|(E', E)$ are consistent ([1], Theorems 6.6 and 22.1) with $\sigma(E', E) \leq |\sigma|(E', E)$ ([1], p. 129); hence the above series are weak*-convergent simultaneously, and therefore (x_n^*) is weak*-unconditional. □

A Riesz space E is said to be σ-Dedekind complete provided that every countable subset of E, bounded (in order) from above, has a supremum. For our purposes we note that both duals of Banach lattices, weakly sequentially complete Banach lattices, Banach function spaces, and Banach lattices with order continuous norm are σ-Dedekind complete (see ([16], Theorems II.5.5, II.5.15, II.10.6), and ([13], Theorem 2.4.2)).

Now we can prove the second result mentioned in the Abstract. It sharpens Corollary 2 of [19] by indicating more precisely the Banach lattices which can be obtained as separable quotients of σ-Dedekind complete Banach lattices (it is interesting itself that these quotients have lattice structure as well).

Theorem 2

Every σ-Dedekind complete Banach lattice E has a quotient with an unconditional Schauder basis. More precisely, E can be mapped continuously onto one of the spaces: c_o, l_1, l_2, or a reflexive Banach space with an unconditional Schauder basis.

In particular, this statement holds true in the following classes of Banach lattices: duals of Banach lattices, weakly sequentially complete Banach lattices, Banach lattices with order continuous norm, Banach function spaces (and so, in Orlicz spaces also).

Proof. Consider two cases: (a) l_∞ embeds (as a closed sublattice) into E, and (b) l_∞ does not embed into E.

In case (a) there is a contractive projection in E onto the lattice copy of l_∞ ([16]; Corollary 1, p. 110), and since l_∞ has l_2 as a quotient ([10], p. 111), E can be mapped continuously onto l_2.

In case (b), by ([16], Theorem II.5.14), the norm of E is order continuous, so, by Proposition 3 and Proposition 2(b)(iii), E has a quotient E/X with an unconditional Schauder basis (u_n), say. We have to consider the nonreflexive case only. By the classical James' result (see e.g. [10]; Theorems 1.c.9, 1.c.10, 1.c.12), there exist a strictly increasing sequence of integers $(m(n))_{n=1}^{\infty}$ and a sequence of reals (θ_k) such that the vectors $y_n = \sum_{k=m(n)+1}^{m(n+1)} \theta_k u_k$, $n = 1, 2, \ldots$, are of norm 1 (in the quotient norm $\| \quad \|_X$ on E/X), and the basic sequence (y_n) is equivalent to the unit vectors of c_o or l_1. We shall show that in both cases the Banach space $Y = [y_n]$ is complemented in E/X, and this fact will finish the proof of our theorem. In the c_o-case, Y is complemented by Sobczyk's theorem ([10], Theorem 2.f.5). For the l_1-case, we renorm $E/X = [u_n]$ such that $|||\sum_{n=1}^{\infty} t_n \alpha_n u_n||| \leq |||\sum_{n=1}^{\infty} \alpha_n u_n|||$ whenever $|t_n| \leq 1$, $n = 1, 2, \ldots$, i.e. $|||\sum_{n=1}^{\infty} \alpha_n u_n|||$ we define as $\sup\{\| \sum_{n=1}^{\infty} \theta_n \alpha_n u_n \|_X \colon |\theta_n| \leq 1,\ n = 1, 2, \ldots\}$. The norms $\| \quad \|_X$ and $||| \quad |||$ are equivalent: we have $\| u \|_X \leq |||u||| \leq c \| u \|_X$, where c is the unconditional constant of the basis (u_n) ([17], Theorem 16.1; [10], Proposition 1.c.7). By the definition of the norm $||| \quad |||$, we can assume that in expansions of y_n's the coefficients θ_k, $k = 1, 2, \ldots$, are ≥ 0. All this follows that $(E/X, ||| \quad |||)$, endowed with the coordinatewise ordering, is a Banach lattice; moreover, y_n's are all positive and pairwise orthogonal, and therefore Y is a closed sublattice of E/X. The equivalence of (y_n) and the unit vectors of l_1 follows that Y is order and topologically isomorphic to the AL-space l_1. Corollary 1, p. 120, in [16] asserts that in this case Y is complemented in the Banach lattice $(E/X, ||| \quad |||)$, and therefore in the Banach space $(E/X, \| \quad \|)$. $\square$

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