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Pairs of sets with convex union

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Abstract

In this paper the notion of convex pairs of convex bounded subsets of a Hausdorff topological vector space is introduced. Criteria of convexity pair are proved.

The space of pairs of convex compact sets has been investigated in many papers (see [3], [7], [8], [12]). Recently, this space has found an application in the quasidifferential calculus (see [1], [9]). A quasidifferential is represented as a pair of convex compact sets. Since this representation is not uniquely determined it is essential to find the minimal representation of this pair. The notion of minimal pairs was introduced in [4]. Some criteria of minimality are given in [2], [5] and [11].

In this paper we investigate pairs of convex compact sets with convex union. We show that for every pair of convex compact sets there exists an equivalent pair of sets with convex union. This observation allows us to introduce a new type minimality, called convex minimality.

In the sequel $X = (X, \tau)$ will be a Hausdorff topological vector space, $\mathcal{A}(X)$ will denote the family of all nonempty subsets of X, and let $\mathcal{B}(X)$ (resp. $\mathcal{K}(X)$) be the family of closed and bounded (resp. compact) convex sets in $\mathcal{A}(X)$. If $A, B \in \mathcal{A}(X)$, then let $A \overset{*}{+} B = \overline{A + B} = \overline{\bar{A} + \bar{B}}$, where $\bar{A}$ denotes the closure of A and $A+B$ is the usual algebraic Minkowski sum of A and B. It may be showed that $\mathcal{B}(X)$ satisfies the order cancellation law, i.e. for $A, B, C \in \mathcal{B}(X)$ the inclusion $A \overset{*}{+} B \subset B \overset{*}{+} C$ implies that $A \subset C$ (see [12]). Hence it follows that the commutative semigroup $(\mathcal{B}(X), \overset{*}{+})$ satisfies the law of cancellation.

Now let $\mathcal{B}^2(X) = \mathcal{B}(X) \times \mathcal{B}(X)$. The equivalence relation between pairs of bounded convex sets is given as follows $(A, B) \sim (C, D)$ iff $A \overset{*}{+} D = B \overset{*}{+} C$. The set

$\mathcal{B}^2(X)$ may be ordered by the relation: $(A, B) \leq (C, D)$ iff $A \subset C$ and $B \subset D$. For $A, B \in \mathcal{B}(X)$ we will use the notation $A \vee B := \overline{\text{conv}}\,(A \cup B)$. If $A, B, C \in \mathcal{B}(X)$, and $b \in X$, then $(A \vee B) \overset{*}{+} C = A \vee B \overset{*}{+} C$ and $A + \{b\} = A + b$. The interval $[a, b]$ is equal to $\{a\} \vee \{b\}$. A pair $(A, B) \in \mathcal{B}^2(X)$ is called *minimal* if for any pair (C, D) equivalent to (A, B) the relation $(C, D) \leq (A, B)$ implies that $(A, B) = (C, D)$. A pair $(A, B) \in \mathcal{K}^2(X)$ is called *convex* if $A \cup B$ is a convex set.

In this paper we will consider pairs of sets with convex union.

Let $A, B, S \in \mathcal{A}(X)$. We say that a set S *separates* sets A and B if $[a, b] \cap S \neq \emptyset$ for every $a \in A$ and $b \in B$.

Proposition 1

If $A, B \in \mathcal{B}(X)$ then $A \cap B$ separates sets A and B if and only if the union $A \cup B$ is a convex set.

Proof. Sufficiency. Let $a \in A$, $b \in B$. Then there exist $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ such that $x = \alpha \cdot a + \beta \cdot b \in A \cap B$. Hence $[a, b] = [a, x] \cup [x, b] \subset A \cup B$.
Necessity. Take arbitrary $a \in A$, $b \in B$. Let $[a, a_0] = [a, b] \cap A$. We have $[a, b] = [a, a_0] \cup [a_0, b]$. But $[a, b] \subset A \cup B$ so $[a_0, b] \subset B$. Hence $a_0 \in A \cap B$. □

Theorem 1

If $A, B \in \mathcal{B}(X)$, $A \cap B \neq \emptyset$ and $A + B \subset A \vee B \overset{*}{+} A \cap B$, then $A \cap B$ separates the sets A and B.

Proof. Let $\mathcal{U}$ be a base of neighborhoods of 0 in X. Given any $U \in \mathcal{U}$, define a balanced sequence $(V_n)_{n \geq 0}$ in $\mathcal{U}$ such that $V_0 + V_0 \subset U$ and $V_{n+1} + V_{n+1} \subset V_n$ for $n \geq 0$. From $A + B \subset A \vee B \overset{*}{+} A \cap B$ it follows that

$$A + B \subset \text{conv}\,(A \cup B) + A \cap B + V \text{ for any } V \in \mathcal{U},$$

in particular

$$A + B \subset \text{conv}\,(A \cup B) + A \cap B + V_n, \quad n \in \mathbb{N}.$$

Let $a \in A$ and $b \in B$. Then

$$a + b = \alpha_1 \cdot a_1 + \beta_1 \cdot b_1 + x_1 + v_1 \tag{1}$$

for some $a_1 \in A$, $b_1 \in B$, $x_1 \in A \cap B$, $v_1 \in V_1$, $\alpha_1, \beta_1 \geq 0$, $\alpha_1 + \beta_1 = 1$. Now, we consider $\beta_1 \cdot a + \alpha_1 \cdot a_1 \in A$ and $\alpha_1 \cdot b + \beta_1 \cdot b_1 \in B$. By the inclusion $A + B \subset \operatorname{conv}(A \cup B) + A \cap B + V_n$, we have:

$$\begin{gathered} \beta_1 \cdot a + \alpha_1 \cdot a_1 + \alpha_1 \cdot b + \beta_1 \cdot b_1 = \alpha_2 \cdot a_2 + \beta_2 \cdot b_2 + x_2 + v_2, \\ \beta_2 \cdot a + \alpha_2 \cdot a_2 + \alpha_2 \cdot b + \beta_2 \cdot b_2 = \alpha_3 \cdot a_3 + \beta_3 \cdot b_3 + x_3 + v_3, \\ \dots \\ \beta_k \cdot a + \alpha_k \cdot a_k + \alpha_k \cdot b + \beta_k \cdot b_k = \alpha_{k+1} \cdot a_{k+1} + \beta_{k+1} \cdot b_{k+1} + x_{k+1} + v_{k+1}, \\ \dots \\ \beta_n \cdot a + \alpha_n \cdot a_n + \alpha_n \cdot b + \beta_n \cdot b_n = \alpha_{n+1} \cdot a_{n+1} + \beta_{n+1} \cdot b_{n+1} + x_{n+1} + v_{n+1}, \end{gathered} \tag{2}$$

for some $\alpha_k + \beta_k = 1$, $\alpha_k, \beta_k \geq 0$, $a_k \in A$, $b_k \in B$, $x_k \in A \cap B$, $v_k \in V_k$, $1 \leq k \leq n+1$, $n \in \mathbb{N}$. From (1) and (2) we obtain

$$\begin{aligned} &(1 + \beta_1 + ... + \beta_n) \cdot a + (1 + \alpha_1 + ... + \alpha_n) \cdot b \\ &\quad = \alpha_{n+1} \cdot a_{n+1} + \beta_{n+1} \cdot b_{n+1} + x_1 + ... + x_{n+1} + v_1 + ... + v_{n+1}. \end{aligned} \tag{3}$$

Denote

$$\gamma_n = \frac{1}{n+1} \cdot (1 + \beta_1 + ... + \beta_n), \quad n \in \mathbb{N}.$$

We have $0 \leq \gamma_n \leq 1$ and

$$1 - \gamma_n = \frac{1}{n+1} \cdot (\alpha_1 + ... + \alpha_n).$$

By (3) we have

$$\gamma_n \cdot a + (1 - \gamma_n) \cdot b = \frac{c_n}{n+1} - \frac{b}{n+1} + x'_n + v'_n, \tag{4}$$

where

$$c_n = \alpha_{n+1} \cdot a_{n+1} + \beta_{n+1} \cdot b_n, \;\; x'_n = \frac{1}{n+1} \cdot (x_1 + ... + x_{n+1}),$$

$$x'_n \in A \cap B, \;\; v'_n = \frac{1}{n+1} \cdot (v_1 + ... + v_{n+1}), \quad n \in \mathbb{N}.$$

By the convexity of $A \cap B$, we have $x'_n \in A \cap B$. In virtue of the compactness of the interval $[0, 1]$ it follows that there exists a subsequence $(\gamma_{n_k})_{k=1}^{\infty}$ such that $\lim_{k \to \infty} \gamma_{n_k} = \alpha$. Denote $z_n = \gamma \cdot a + (1 - \gamma_n) \cdot b$. Then

$$\lim_{k \to \infty} z_{n_k} = \alpha \cdot a + (1 - \alpha) \cdot b = z.$$

Now from (4) we obtain

$$z = z - z_{n_k} + \frac{c_{n_k}}{n_k + 1} - \frac{b}{n_k + 1} + x'_{n_k} + v'_{n_k}. \tag{5}$$

By the boundedness of $A \vee B$, we have

$$\lim_{k\to\infty} \frac{c_{n_k}}{n_k + 1} = 0.$$

Now we observe that

$$v'_n = \frac{1}{n+1} \cdot (v_1 + ... + v_{n+1}) \in V_1 + ... + V_n + V_n \subset V_1 + ... + V_{n-1} \subset V_0.$$

for any $n \in \mathbb{N}$. By (5), for sufficiently large k, it must be

$$\alpha \cdot a + (1 - \alpha) \cdot b \in A \cap B + V_1 + V_2 + V_2 + V_0 \subset A \cap B + V_1 + V_1 + V_0 \subset A \cap B + U.$$

Hence

$$[a, b] \cap (A \cap B + U) \neq \emptyset \tag{6}$$

for every $U \in \mathcal{U}$. Now suppose that $[a, b] \cap A \cap B = \emptyset$. Since the interval $[a, b]$ is compact and $A \cap B$ is closed, there exists a neighborhood $U \in \mathcal{U}$ (see [10]) such that

$$([a, b] + U) \cap (A \cap B + U) = \emptyset.$$

This contradicts condition (6). Hence $[a, b] \cap A \cap B \neq \emptyset$. □

Analogously as in the case of locally convex space (see [7]), we can show

Lemma 1

If $A, B, C \in \mathcal{A}(X)$ and C is convex, then

$$\operatorname{conv}(A \cup B) + C = \operatorname{conv}[(A + C) \cup (B + C)].$$

Proof. Let $x \in \operatorname{conv}(A \cup B) + C$. Then

$$x = \sum_{i=1}^{n} \alpha_i \cdot a_i + \sum_{j=1}^{m} \beta_j \cdot b_j + c,$$

for some $a_i \in A$, $b_j \in B$, $c \in C$, $\alpha_i, \beta_j \geq 0$ with

$$\sum_i^n \alpha_i + \sum_j^m \beta_j = 1,\ 1 \leq j \leq n,\ 1 \leq j \leq m.$$

We have

$$x = \sum_i^n \alpha_i \cdot (a_i + c) + \sum_j^m \beta_j \cdot (b_j + c),$$

i.e.

$$x \in \text{conv}\,[(A + C) \cup (B + C)].$$

Hence

$$\text{conv}\,(A \cup B) + C \subset \text{conv}\,[(A + C) \cup (B + C)].$$

The inverse inclusion is obvious. □

Lemma 2

If $A, B, C \in \mathcal{B}(X)$, then $A \vee B \overset{*}{+} C = (A \overset{*}{+} C) \vee (B \overset{*}{+} C)$.

Proof. We have

$$A \vee B \overset{*}{+} C = \overline{\overline{\text{conv}}\,(A \cup B) + C} = \overline{\text{conv}\,(A \cup B) + C}.$$

Hence, from Lemma 1

$$A \vee B \overset{*}{+} C = (A \overset{*}{+} C) \vee (B \overset{*}{+} C). \ \square$$

Lemma 3

If $A, B \in \mathcal{B}(X)$, then $A \cup B$ is convex if and only if

$$A \overset{*}{+} B = A \vee B \overset{*}{+} A \cap B, \quad A \cap B \neq \emptyset.$$

Proof. Sufficiency. Let $A \cup B$ is convex. Then $A \cap B \neq \emptyset$ separates the sets A and B. Given any $a \in A$ and $b \in B$, there exist $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ such that $\alpha \cdot a + \beta \cdot b \in A \cap B$. It follows that

$$a + b = \beta \cdot a + \alpha \cdot a + \alpha \cdot a + \beta \cdot b \in A \vee B \overset{*}{+} A \cap B.$$

Hence $A + B \subset A \vee B \overset{*}{+} A \cap B$. But from Lemma 2

$$A \vee B \overset{*}{+} A \cap B = (A \overset{*}{+} A \cap B) \vee (B \overset{*}{+} A \cap B) \subset A \overset{*}{+} B.$$

Hence

$$A \overset{*}{+} B = A \vee B \overset{*}{+} A \cap B.$$

Necessity. By Theorem 1 and Proposition 1 it follows that $A \cup B$ is convex. □

We call a set $A \in \mathcal{B}(X)$ is a *summand* of $B \in \mathcal{B}(X)$ if there exists a set $C \in \mathcal{B}(X)$ such that $A \overset{*}{+} C = B$.

Theorem 2

If $A, B \in \mathcal{B}(X)$*, then* $A \vee B$ *is a summand of* $A \overset{*}{+} B$ *if and only if* $A \cup B$ *is convex.*

Proof. Let $A \overset{*}{+} B = A \vee B \overset{*}{+} S$ for some $S \in \mathcal{B}(X)$. Then $A \overset{*}{+} B \supset A \overset{*}{+} S$ and $A \overset{*}{+} B \supset B \overset{*}{+} S$. By the order cancellation law, we have $S \subset A \cap B$. Now, from Theorem 1 and Proposition 1 it follows that $A \cup B$ is convex.

If $A \cup B$ is convex, then by Lemma 3 we have $A \overset{*}{+} B = A \vee B \overset{*}{+} A \cap B$. Hence $A \vee B$ is summand of $A \overset{*}{+} B$. □

Theorem 3

Let $A, B, C \in \mathcal{B}(X)$*. Then* $(A \overset{*}{+} C) \cup (B \overset{*}{+} C)$ *is a convex set if and only if the pair* $(A \vee B, A \overset{*}{+} B)$ *is equivalent to* (C, D) *for some* $D \in \mathcal{B}(X)$*.*

Proof. Necessity. Suppose that $(A \overset{*}{+} C) \cup (B \overset{*}{+} C)$ is convex. Then by Theorem 2 we have

$$A \overset{*}{+} C \overset{*}{+} B \overset{*}{+} C = (A \overset{*}{+} C) \cup (B \overset{*}{+} C) \overset{*}{+} (A \overset{*}{+} C) \cap (B \overset{*}{+} C).$$

But

$$(A \overset{*}{+} C) \cup (B \overset{*}{+} C) = (A \overset{*}{+} C) \vee (B \overset{*}{+} C) = A \vee B \overset{*}{+} C.$$

Hence

$$A \overset{*}{+} B \overset{*}{+} 2 \cdot C = A \vee B \overset{*}{+} C \overset{*}{+} (A \overset{*}{+} C) \cap (B \overset{*}{+} C)$$

and so

$$A \overset{*}{+} B \overset{*}{+} C = A \vee B \overset{*}{+} (A \overset{*}{+} C) \cap (B \overset{*}{+} C).$$

Setting $D = (A \overset{*}{+} C) \cap (B \overset{*}{+} C)$, we get

$$(A \vee B, A \overset{*}{+} B) \sim (C, D).$$

Sufficiency. Let $A \overset{*}{+} B \overset{*}{+} C = A \vee B \overset{*}{+} D$ for some $D \in K(X)$. Then

$$A \overset{*}{+} C \overset{*}{+} B \overset{*}{+} C = (A \overset{*}{+} C) \vee (B \overset{*}{+} C) \overset{*}{+} D.$$

Denote $A_1 = A \overset{*}{+} C$, $B_1 = B \overset{*}{+} C$. By Theorem 2 it follows that $A_1 \cup B_1$ is a convex set. □

Corollary 1

If $A, B \in \mathcal{B}(X)$, *then the set* $(A \overset{*}{+} A \vee B) \cup (B \overset{*}{+} A \vee B)$ *is convex.*

Proof. Given $C = A \vee B$ and $D = A \overset{*}{+} B$. Then $(A \vee B, A \overset{*}{+} B) \sim (C, D)$ and, from Theorem 3, we have $(A \overset{*}{+} A \vee B) \cap (B \overset{*}{+} A \vee B) = A \overset{*}{+} B$. $\square$

Proposition 2

For every pair $(A, B) \in \mathcal{B}^2(X)$ *there exists an equivalent convex pair.*

Proof. The Proposition follows immediately by Corollary 1 and by the relation $(A, B) \sim (A \overset{*}{+} A \vee B, B \overset{*}{+} A \vee B)$. $\square$

Lemma 4

If $(A, B), (C, D) \in \mathcal{B}^2(X)$ *are two equivalent pairs and* $A \cup B$ *is convex, then*

$$A \overset{*}{+} D = B \overset{*}{+} C = C \vee D \overset{*}{+} A \cap B.$$

Proof. Let $A \overset{*}{+} D = B \overset{*}{+} C$. Then

$$A \vee B \overset{*}{+} C = (A \overset{*}{+} C) \vee (B \overset{*}{+} C) = (A \overset{*}{+} C) \vee (A \overset{*}{+} D) = A \overset{*}{+} C \vee D.$$

But from the convexity of $A \cup B$ and Lemma 3 it follows that

$$A \overset{*}{+} B = A \cup B \overset{*}{+} A \cap B.$$

Now we observe that

$$A \overset{*}{+} B \overset{*}{+} C \vee D = B \overset{*}{+} C \overset{*}{+} A \cup B,$$

and

$$A \cup B \overset{*}{+} A \cap B \overset{*}{+} C \vee D = B \overset{*}{+} C \overset{*}{+} A \cup B.$$

From the law of cancellation, we have

$$A \overset{*}{+} D = B \overset{*}{+} C = C \vee D \overset{*}{+} A \cap B. \ \square$$

Proposition 3

If $(A, B), (C, D) \in \mathcal{B}^2(X)$ *are two equivalent convex pairs, then*

$$A \overset{*}{+} D = B \overset{*}{+} C = A \cup B \overset{*}{+} C \cap D = C \cup D \overset{*}{+} A \cap B.$$

Proof. It follows from Lemma 4 immediately. $\square$

Proposition 4

If $(A, B) \in \mathcal{K}(X) \times (\mathcal{B}(X) \setminus \mathcal{K}(X))$ is a pair equivalent to $(C, D) \in \mathcal{B}^2(X)$, then $D \in \mathcal{B}(X) \setminus \mathcal{K}(X)$.

Proof. Suppose $D \in \mathcal{K}(X)$. Then

$$B \overset{*}{+} C = A + D \in \mathcal{K}(X).$$

It follows that $B + d \subset A + D$ for some $d \in D$. Hence B is a compact set which is a contradiction. □

Corollary 2

For any $(A, B) \in \mathcal{K}(X) \times (\mathcal{B}(X) \setminus \mathcal{K}(X))$ there is no minimal pair $(C, D) \in \mathcal{K}^2(X)$ equivalent to (A, B).

In [4] it has been proved that for any $(A, B) \in \mathcal{K}^2(X)$ there exists a minimal pair $(A_o, B_o) \in \mathcal{K}^2(X)$ equivalent to (A, B). The question arises if it remains true for $\mathcal{B}^2(X)$ in place of $\mathcal{K}^2(X)$.

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