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On smoothing conditions of multivariate splines

Jerzy Stankiewicz

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland

Abstract

Let T be a k-simplex in $\mathbb{R}^s$, where $0 \le k < n$, and let S_a and S_b be two adjacent s-simplices with $T = S_a \cap S_b$. Suppose that $F(x) \in C(S_a \cup S_b)$ with

$$F(x)|_{S_a} = P_n(x),$$
$$F(x)|_{S_b} = Q_n(x),$$

where P_n i Q_n are Bezier polynomials in $\mathbb{R}^s$ with total degree n. The conditions, which must be required to function F be in class C^r across T, are introduced by C.K. Chui and M. Lai ([3], [4]). In the present note the improvement of those conditions is obtained. As an application, algorithm for computation of polynomial coefficients is shown.

Introduction

Let $x^0, x^1, x^2, \dots, x^s$ be linear independent points in $\mathbb{R}^s$. Then its convex hull

(1)
$$\langle x^0, \dots, x^s \rangle = \Big\{ \sum_{i=0}^{s} \lambda_i x^i : \sum_{i=0}^{s} \lambda_i = 1, \lambda_0, \dots, \lambda_s \ge 0 \Big\}$$

is a simplex. It is called an s-simplex. Then any $x \in \mathbb{R}^s$ can be identified with an $(s+1)$-tuple $(\lambda_0, \dots, \lambda_s)$ where

(2)
$$\lambda_i = \lambda_i(x) = \frac{\mathrm{vol}_s \langle x^0, \dots, x^{i-1}, x, x^{i+1}, \dots, x^s \rangle}{\mathrm{vol}_s \langle x^0, \dots, x^s \rangle},$$

and

$$\text{vol}_s\langle x^0,\ldots,x^s\rangle = \frac{1}{s!}\begin{vmatrix} 1 & x_1^0 & \ldots & x_s^0 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & x_1^s & \ldots & x_s^s \end{vmatrix}, \tag{3}$$

where $x^i = (x_1^i,\ldots,x_s^i)$ and $x = (x_1,\ldots,x_s)$. The numbers $\lambda_0,\ldots,\lambda_s$ are called the barycentric coordinate of x relative to the simplex $\langle x^0,\ldots,x^s\rangle$. Since each $\lambda_i = \lambda_i(x)$ and by using the notation $\lambda^\beta = \lambda_0^{\beta_0}\ldots\lambda_s^{\beta_s}$ and $\beta! = \beta_0!\ldots\beta_s!$ for any $\beta = (\beta_0,\ldots,\beta_s) \in \mathbb{Z}_+^{s+1}$, where $\mathbb{Z}_+ = \{0,1,\ldots\}$, the Bezier polynomial is defined by

$$P_n(x) = \sum_{|\beta|=n} a_\beta \Phi_\beta^n(\lambda), \tag{4}$$

where $\Phi_\beta^n(\lambda)$ are the Bernstein polynomials of degree n

$$\Phi_\beta^n(\lambda) = \frac{n!}{\beta!}\lambda^\beta, \quad |\beta| = \beta_0 + \ldots + \beta_s = n\,. \tag{5}$$

The set $\left\{\left(\frac{\beta_0}{n},\cdots,\frac{\beta_s}{n},a_\beta\right) : \ |\beta| = n\right\}$ is called the Bezier net of $P_n(x)$. Let $T = \langle x^0,\ldots,x^k\rangle$ be a k-simplex in $\mathbb{R}^s$ where $0 \le k < s$, and let

$$\begin{aligned} S_a &= \langle x^0,\ldots,x^k,x^{k+1},\ldots,x^s\rangle, \\ S_b &= \langle x^0,\ldots,x^k,y^{k+1},\ldots,y^s\rangle, \end{aligned} \tag{6}$$

be two adjacent s-simplices with $T = S_a \cap S_b$. Suppose that $F(x) \in C(S_a \cup S_b)$ with

$$\begin{aligned} F(x)|_{S_a} &= P_n(x) = \sum_{|\alpha|=n} a_\alpha \Phi_\alpha^n\big(\lambda_0(x),\ldots,\lambda_s(x)\big), \\ F(x)|_{S_b} &= Q_n(x) = \sum_{|\beta|=n} a_\beta \Phi_\beta^n\big(\eta_0(x),\ldots,\eta_s(x)\big), \end{aligned} \tag{7}$$

where $\lambda = (\lambda_0,\ldots,\lambda_s)$ and $\eta = (\eta_0,\ldots,\eta_s)$ are the barycentric coordinate of x relative to S_a and S_b respectively.

Let $s_i(\alpha) = (\alpha_0,\ldots,\alpha_{i-1},\alpha_i+1,\alpha_{i+1},\ldots,\alpha_s)$ and $\Delta_{ij}a_\alpha = a_{s_i\alpha} - a_{s_j\alpha}$. Using above notation, one can formulate the following smoothing conditions:

Theorem 1 (C.K. Chui and M.J. Lai)

Let

$$c_{ji} = \frac{\mathrm{vol}_s\langle x^0, \dots, x^{i-1}, y^j, x^{i+1}, \dots, x^s\rangle}{\mathrm{vol}_s\langle x^0, \dots, x^s\rangle}. \tag{8}$$

Then for any $r \in \mathbb{Z}_+,\ F(x) \in C^r(S_a \cup S_b)$ *if and only if*

$$\begin{aligned} &\Delta_{k+1,0}^{\gamma_{k+1}} \dots \Delta_{s,0}^{\gamma_s} b_{\alpha_0 \dots \alpha_k 0 \dots 0} \\ &= \Big(\sum_{i=1}^{s} c_{k+1,i}\ \Delta_{i0}\Big)^{\gamma_{k+1}} \cdots \Big(\sum_{i=1}^{s} c_{s,i}\ \Delta_{i0}\Big)^{\gamma_s} a_{\alpha_0 \dots \alpha_k 0 \dots 0} \end{aligned} \tag{9}$$

for all $\gamma_{k+1} + \ldots + \gamma_s = l,\ \alpha_0 + \ldots + \alpha_k = n - l$, *and* $l = 0, \dots, r$.

Results

Let $E_i \ \ (0 \le i < s)$ be the partial shift operator defined by $E_i\, a_\alpha = a_{s_i \alpha}$. Thus the Bezier polynomial (4) can be expressed in terms of these operators by

$$P_n(x) = \Big(\sum_{i=0}^{s} \lambda_i E_i\Big)^n a_{0,\dots,0}, \tag{10}$$

where $\lambda_0, \lambda_1, \dots, \lambda_s$ are barycentric coordinates of x (see [1] and [2]). The right side of equality (10) may be evaluated by recursion formula:

$$\Big(\sum_{i=0}^{s} \lambda_i E_i\Big)^n a_{0,\dots,0} = \sum_{j=0}^{s} \left(\lambda_j \Big(\sum_{i=0}^{s} \lambda_i\, E_i\Big)^{n-1} E_j\, a_{0,\dots,0}\right). \tag{11}$$

Applying above notation, the smoothing conditions (9) can be formulated as:

Theorem 2

Under the same assumptions as in Theorem 1 the condition (9) *may be replaced by the following formula*

$$b_{\alpha_0,\dots,\alpha_k,\gamma_{k+1},\dots,\gamma_s} = \Big(\sum_{i=0}^{s} c_{k+1,i}\, E_i\Big)^{\gamma_{k+1}} \cdots \Big(\sum_{i=0}^{s} c_{s,i}\, E_i\Big)^{\gamma_s} a_{\alpha_0,\dots,\alpha_k,0,\dots,0}, \tag{12}$$

for all $\gamma_{k+1} + \cdots + \gamma_s = l,\ \alpha_0 + \ldots + \alpha_k = n - l$, *and* $l = 0, \dots, r$.

Proof. The proof is based on the following observation: the equalities (9) form a consistent system of linear equations. Hence, it is sufficient to show that the equalities (12) are solution of this system.

First, we note that

$$\Delta^n_{i,j} = E_i\Delta^{n-1}_{i,j} - E_j\Delta^{n-1}_{i,j} \quad \text{where} \quad \Delta_{i,j} = E_i - E_j. \tag{13}$$

Applying (13) to the left side of (9) we obtain:

$$\begin{aligned} &\Delta^{\gamma_{k+1}}_{k+1,0}\cdots\Delta^{\gamma_s}_{s,0}\, B_{\alpha_0\ldots\alpha_k 0\ldots 0} \\ &= (E_{k+1}-E_0)^{\gamma_{k+1}}\cdots(E_s-E_0)^{\gamma_s}\, b_{\alpha_0\ldots\alpha_k 0\ldots 0} \\ &= \Big(\sum_{i_{k+1}=0}^{\gamma_{k+1}}\binom{\gamma_{k+1}}{i_{k+1}}E^{i_{k+1}}_{k+1}(-E_0)^{\gamma_{k+1}-i_{k+1}}\Big) \\ &\qquad\cdots\Big(\sum_{i_s=0}^{\gamma_s}\binom{\gamma_s}{i_s}E^{i_s}_s(-E_0)^{\gamma_s-i_s}\Big)b_{\alpha_0\ldots\alpha_k 0\ldots 0}. \end{aligned} \tag{14}$$

Changing order of summation and using definition of E_i, expression (14) may be written as:

$$\begin{aligned} &\sum_{i_{k+1}=0}^{\gamma_{k+1}}\cdots\sum_{i_s=0}^{\gamma_s}\binom{\gamma_{k+1}}{i_{k+1}}\cdots\binom{\gamma_s}{i_s}(E_{k+1})^{i_{k+1}}\cdots(E_s)^{i_s} \\ &\qquad\cdot(-E_0)^{\gamma_{k+1}+\cdots+\gamma_s-i_{k+1}-\cdots-i_s}b_{\alpha_0\ldots\alpha_k 0\ldots 0} \\ &= \sum_{i_{k+1}=0}^{\gamma_{k+1}}\cdots\sum_{i_s=0}^{\gamma_s}\binom{\gamma_{k+1}}{i_{k+1}} \\ &\qquad\cdots\binom{\gamma_s}{i_s}b_{\alpha_0+(\gamma_{k+1}+\cdots+\gamma_s-i_{k+1}-\cdots-i_s),\alpha_1,\ldots,\alpha_k,i_{k+1},\ldots,i s_s}. \end{aligned} \tag{15}$$

Replacing $b_{\alpha_0+(\gamma_{k+1}+\cdots+\gamma_2-i_{k+1}-\cdots-i_s),\alpha_1,\ldots,\alpha_k,i_{k+1},\ldots,i_s}$ by right side of (12), formula (15) may be written as:

$$\begin{aligned} &\sum_{i_{k+1}=0}^{\gamma_{k+1}}\cdots\sum_{i_s=0}^{\gamma_s}\binom{\gamma_{k+1}}{i_{k+1}}\cdots\binom{\gamma_s}{i_s}\Big(\sum_{i=0}^{s}c_{k+1,i}\ E_{k+1}\Big)^{i_{k+1}} \\ &\qquad\cdots\Big(\sum_{i=0}^{s}c_{s,i}\ E_s\Big)^{i_s}\cdot(-E_0)^{\gamma_{k+1}+\cdots+\gamma_s-i_{k+1}-\cdots-i_s}\ a_{\alpha_0\ldots\alpha_k 0\ldots 0} \\ &= \Big(\sum_{i=0}^{s}c_{k+1,i}\ E_i - E_0\Big)^{\gamma_{k+1}}\cdots\Big(\sum_{i=0}^{s}c_{s,i}\ E_i - E_0\Big)^{\gamma_s}\ a_{\alpha_0\ldots\alpha_k 0\ldots 0}. \end{aligned} \tag{16}$$

It suffices to note that if $c_{j,i}$ are barycentric coordinate then $1 = \sum_{i=0}^{s} c_{j,i}$. Applying this identity to (16) we obtain right side of the equality (9):

$$\begin{aligned}(17)\qquad &\Big(\sum_{i=0}^{s} c_{k+1,i}\ (E_i - E_0)\Big)^{\gamma_{k+1}} \cdots \Big(\sum_{i=0}^{s} c_{s,i}(E_i - E_0)\Big)^{\gamma_s} a_{\alpha_0\ldots\alpha_k 0\ldots 0} \\ &= \Big(\sum_{i=0}^{s} c_{k+1,i}\ \Delta_{i0}\Big)^{\gamma_{k+1}} \cdots \Big(\sum_{i=0}^{s} c_{s,i}\ \Delta_{i0}\Big)^{\gamma_s} a_{\alpha_0\ldots\alpha_k 0\ldots 0}. \ \Box\end{aligned}$$

Application

We now turn to the case $k = s - 1$, then conditions (11) can be expressed as:

$$\begin{aligned} b_{\alpha_0,\ldots,\alpha_{s-1},\gamma} &= \Big(\sum_{i=0}^{s} c_{s,i}\ E_i\Big)^{\gamma} a_{\alpha_0,\ldots,\alpha_{s-1},0} \\ &= \sum_{j=0}^{s}\Big(c_{s,j}\Big(\sum_{i=0}^{s} c_{s,i}\ E_i\Big)^{\gamma_1} E_j\ a_{\alpha_0,\ldots,\alpha_{s-1},0}\Big),\end{aligned}$$

for all $\alpha_0 + \ldots + \alpha_{s-1} = n - \gamma$, and $\gamma = 0, \ldots, r$. The above formula has following interpretation: $b_{\alpha_0,\ldots,\alpha_{s-1},\gamma}$ may be obtained by evaluation of a Bezier polynomial of degree γ with some coefficients of P_n at y_s. On the other hand, recursion formula (11) provides de Casteljeu algorithm for the evaluation of $P_n(x)$ at given point x. Let $(\lambda_0, \ldots, \lambda_s)$ be barycentric coordinates of x. Algorithm de Casteljeu may be written as:

$$\begin{aligned}(19)\qquad a^0_{\alpha_0,\ldots,\alpha_s} &= a_{\alpha_0,\ldots,\alpha_s}, \\ a^{\gamma}_{\alpha_0,\ldots,\alpha_s} &= \sum_{i=0}^{s} \lambda_i\ E_i\ a^{\gamma-1}_{\alpha_0,\ldots,\alpha_s}, \quad \text{for } \ \gamma = 1, \ldots, n,\end{aligned}$$

where $a^{\gamma}_{\alpha_0,\ldots,\alpha_s}$ $(\gamma = 0, \ldots, n)$ are auxiliary points with $\sum_{i=0}^{s} \alpha_i = n - \gamma$. It is easy to show that $P_n(x) = a^n_{0,\ldots,0}$ (see [5]). Applying above algorithm to point y_s we obtain the following formula:

$$\begin{aligned}(20)\qquad a^0_{\alpha_0,\ldots,\alpha_s} &= a_{\alpha_0,\ldots,\alpha_s}, \\ a^{\gamma}_{\alpha_0,\ldots,\alpha_s} &= \sum_{i=0}^{s} c_{s,i}\ E_i\ a^{\gamma-1}_{\alpha_0,\ldots,\alpha_s}, \quad \text{for } \ \gamma = 1, \ldots, n,\end{aligned}$$

where $c_{s,i}$ $(i = 0, \dots, s)$ are barycentric coordinates of y_s. Now, it is sufficient to note that $b_{\alpha_0,\dots,\alpha_{s-1},\gamma} = a^{\gamma}_{\alpha_0,\dots,\alpha_{s-1},0}$ $(\gamma = 0, \dots, r)$. Since $r \le n$, the algorithm (20) can be expressed as:

$$
\begin{aligned}
a^0_{\alpha_0,\dots,\alpha_s} &= a_{\alpha_0,\dots,\alpha_s}, \\
a^{\gamma}_{\alpha_0,\dots,\alpha_s} &= \sum_{i=0}^{s} c_{s,i}\; E_i\; a^{\gamma-1}_{\alpha_0,\dots,\alpha_s}, \quad \text{for} \;\; \gamma = 1, \dots, r,
\end{aligned}
\tag{21}
$$

where $c_{s,i}$ $(i = 0, \dots, s)$ are barycentric coordinates of y_s.

Discussion of the algorithm for evaluation of the coefficients in general case will be given in other paper.

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