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Extrapolation theorem on L^p spaces over infinite measure space: An approach $p \searrow p_0$

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Abstract

After S.Yano's classical work [10], several extrapolation theorems on L^p spaces have been proved to approach to the "singular" spaces, L^1 and L^∞. We sometimes find such "singularity" on L^{p_0}, $1 < p_0 < \infty$. In this paper, we shall prove some extrapolation theorem in the case $p \searrow p_0$.

1. Extrapolation theory on quasi-normed L^p-spaces: An approach to L^1

In [3, §2], Σ-extrapolation space was defined for some suitable family of Banach spaces as follows:

Definition. Let $\{A_\theta\}_{0<\theta<1}$ be a family of Banach spaces which satisfies

(1) (strongly compatible) there exist two Banach spaces Δ and Σ such that

$$\Delta \subset A_\theta \subset \Sigma \qquad \text{for any} \quad 0 < \theta < 1$$

with continuous embeddings, and

(2) (Σ-condition)

$$\sup_\theta \sup_{a \in A_\theta} \frac{\|a\|_\Sigma}{\|a\|_{A_\theta}} < \infty.$$

Then, for any $1 \le r < \infty$, we define $\Sigma_r\{A_\theta : 0 < \theta < 1\}$ as the set of all $a \in \Sigma$ which have a representation $a = \sum_{i=1}^{\infty} a_i$ (with absolute convergence in Σ), where $a_i \in A_{\theta_i}$ for some $0 < \theta_i < 1$ such that

$$\|a\|_{\Sigma_r\{A_\theta:0<\theta<1\}} = \inf \left[\sum_{i=1}^{\infty} \|a_i\|_{A_{\theta_i}}^r\right]^{1/r} < \infty.$$

Here the infimum is taken over all representations $a = \sum_{i=1}^{\infty} a_i,\ a_i \in A_{\theta_i}$. We sometimes write $\Sigma_1\{A_\theta : 0 < \theta < 1\}$ as $\Sigma\{A_\theta : 0 < \theta < 1\}$ simply.

Moreover, by using it, the following result has been shown.

Theorem A

Let (Ω, μ) be a probability space and let α be any non-negative real number. Then, the following conditions are equivalent:
(1) $f \in \sum\{(p-1)^{-\alpha} L^p(\Omega, \mu) : 1 < p < \infty\}$
(2) $\int_\Omega |f(x)|(1 + \log^+ |f(x)|)^\alpha d\mu(x) < \infty$.
Here, $(p-1)^{-\alpha} L^p(\Omega, \mu)$ is the space $L^p(\Omega, \mu)$ with norm

$$\|f\|_{(p-1)^{-\alpha} L^p(\Omega)} = (p-1)^{-\alpha} \left[\int_\Omega |f(x)|^p d\mu(x)\right]^{1/p}. \tag{1.1}$$

This theorem yields S. Yano's classical extrapolation theorem ([10]).

On the other hand, the author tried to prove Yano's type extrapolation theorem on L^p-spaces over σ-finite measure space by using Yano's original idea, but did not get sharp results ([5], [6]).

Of course, even if (Ω, μ) is an infinite measure space, it is possible to define the Banach space $\sum\{(p-1)^{-\alpha} L^p(\Omega, \mu) : 1 < p < \infty\}$. But the author had failed to characterize it as a function space explicitly. Now, on each $L^p(\Omega, \mu)$, we shall consider non-homogeneous quasi-norms

$$\|f\|_p^\gamma = \left[\int_\Omega |f(x)|^p d\mu(x)\right]^{\gamma/p}. \qquad (\gamma > 0)$$

We denote such quasi-normed L^p-space by $(L^p)^\gamma$. And we shall also investigate the following quasi-normed function spaces:

DEFINITION. Let $\alpha \geq 0$ and $1 < q < \infty$. We say that $f \in \mathcal{Z}_{q,\alpha}^\dagger$ if and only if

$$\|f\|_{\mathcal{Z}_{q,\alpha}^\dagger} \equiv \left[\int\limits_{|f|\leq 1} |f(x)|^q d\mu(x) + \int\limits_{|f|>1} |f(x)|(1 + \log|f(x)|)^\alpha d\mu(x)\right]^{1/q} < \infty \tag{1.2}$$

(see [5]).

By using the results from of [1 §4], we shall investigate the extrapolation theory on the family of quasi-normed L^p-spaces $\{(L^p)^{p/q}\}$, and get the following results:

Theorem 1.1 ([8])

Let (Ω, μ) be a σ-finite measure space, $1 < q < \infty$ and $\alpha \geq 0$. Then,

$$\sum \left\{ \left((p-1)^{-\alpha} L^p(\Omega)\right)^{p/q} : 1 < p < q \right\} = \mathcal{Z}^{\dagger}_{q,\alpha}(\Omega)$$

and their quasi-norms are equivalent to each others.

From the definition of Extrapolation spaces, we can, as a corollary of this theorem, get the following extrapolation theorem.

Corollary 1.2 ([8])

Let (Ω, μ) be a σ-finite measure space and $1 < q < \infty$. And let T be a subadditive operator on $L^p(\Omega, \mu)$ for $1 < \forall p < q$, i.e. $|T(f+g)| \leq |Tf| + |Tg|$ a.e. for any $f, g \in L^p(\Omega, \mu)$. Suppose

$$\left[\int_\Omega |Tf(x)|^p d\mu(x)\right]^{1/p} \leq \frac{A}{(p-1)^\alpha} \left[\int_\Omega |f(x)|^p d\mu(x)\right]^{1/p} \tag{1.3}$$

for all $f \in L^p(\Omega, \mu)$. Here, positive constants A and α are independent of p and f. If $f \in \mathcal{Z}^{\dagger}_{q,\alpha}$ then $Tf \in \mathcal{Z}^{\dagger}_{q,0}$. Moreover,

$$\begin{aligned} &\int_{|Tf|\leq 1} |Tf(x)|^q d\mu(x) + \int_{|Tf|>1} |Tf(x)| d\mu(x) \\ \leq& C_{q,\alpha,A} \left[\int_{|f|\leq 1} |f(x)|^q d\mu(x) + \int_{|f|>1} |f(x)|(1+\log|f(x)|)^\alpha d\mu(x)\right]. \end{aligned} \tag{1.4}$$

Here, the constant $C_{q,\alpha,a}$ depends on only q, α and A.

From this corollary, we can get Yano's theorem immediately. However, the original purpose of these studies is the approach to $L^1(\Omega)$. As is known, we can never get (1.4) for $q = 1$. For example, we shall consider the Hilbert transform H. Let $\chi(x) = 1$ for $0 \leq x \leq 1$, and $\chi(x) = 0$ elsewhere. Then it is easy to show $H\chi \in L^p(\mathbf{R})$ for $p > 1$ but $\notin L^1(\mathbb{R})$. So, instead of the case $q = 1$, we shall investigate the following function classes.

Definition (*c.f.* [5]). Let $\alpha, \beta \geq 0$. $f \in \mathcal{Z}^*_{\beta,\alpha}$ if and only if f is measurable function on Ω such that

$$\int_{|f|\leq 1} \frac{|f(x)|}{(1-\log|f(x)|)^\beta} d\mu(x) + \int_{|f|>1} |f(x)|(1+\log|f(x)|)^\alpha d\mu(x) < \infty.$$

Once, the author tried to get some estimation on these classes ([3, Theorem 2]). From Theorem 1, we can get the following result, which is the sharpened one.

Theorem 1.3 ([9])

Let (Ω, μ) be a σ-finite measure space and let T be an operator satisfying the assumption above. If $f \in \mathcal{Z}^*_{\varepsilon,\alpha}$, then $Tf \in \mathcal{Z}^*_{\alpha+\varepsilon,0}$ for arbitrary $\varepsilon > 0$. Moreover,

(1.5)
$$\int_{|Tf|\le 1} \frac{|Tf(x)|}{(1-\log|Tf(x)|)^{\alpha+\varepsilon}} d\mu(x) + \int_{|Tf|>1} |Tf(x)| d\mu(x)$$
$$\le C_{q,\alpha,\varepsilon,A} \left[\int_{|f|\le 1} \frac{|f(x)|}{(1-\log|f(x)|)^{\varepsilon}} d\mu(x) + \int_{|f|>1} |f(x)|(1+\log|f(x)|)^{\alpha} d\mu(x) \right].$$

Here, the positive constant $C_{q,\alpha,\varepsilon,A}$ depends only on q, α, ε and A.

Remark. T. Miyamoto ([4]) has proved the following:

Let $q > 1$. If T is bounded on L^q and satisfies (1.5) or (1.4), then T satisfies (1.3).

So, we may regard the space $\mathcal{Z}^*_{\beta,\alpha}$ as an "extreme space" of the family $\{(p-1)^{-\alpha}L^q\}$.

2. An approach to L^p, $p \searrow p_0$ $(p_0 \ge 1)$

M. Milman has given some comment to our results (in our private communications). He asserted we *can* characterize the Σ-space explicitly:

$$\sum\nolimits_{p_0} \left\{(p-p_0)^{-\alpha} L^p(\Omega) : p_0 < p < p_1\right\} = L^{p_0}\log^{\alpha} L + L^{p_1}(\Omega)$$

for $1 \le p_0 < p_1 \le \infty$ and any measure space Ω. But we cannot get the result similar to Corollary 1.2 when $p_0 > 1$ from it. Moreover, in his method, it has not been succeeded to characterize the space

$$\sum\nolimits_{1} \left\{(p-p_0)^{-\alpha} L^p : p_0 < p < p_1\right\}$$

for $p_0 > 1$, explicitly. However, in our method, we can grow up Theorem 1.1 and Corollary 1.2 as follows:

Theorem 2.1

Let (Ω, μ) be a σ-finite measure space, $1 \le p_0 < p_1 < \infty$ and $\alpha \ge 0$. Then,

$$\sum \left\{ \left((p-p_0)^{-\alpha} L^p(\Omega) \right)^{p/p_1} : p_0 < p < p_1 \right\}$$

is the space $L^{p_0} \log^{p_0 \alpha} L + L^{p_1}$ with quasi-norm

$$\left[\int_{|f| \le 1} |f(x)|^{p_1} d\mu(x) + \int_{|f|>1} |f(x)|^{p_0} (1 + \log |f(x)|)^{p_0 \alpha} d\mu(x) \right]^{1/p_1}. \tag{2.1}$$

Corollary 2.2

Let (Ω, μ) be a σ-finite measure space and $1 \le p_0 < p_1 < \infty$. And let T be a sub-additive operator on $L^p(\Omega, \mu)$ for $p_0 < p < p_1$. Suppose

$$\left[\int_\Omega |Tf(x)|^p d\mu(x) \right]^{1/p} \le \frac{A}{(p-p_0)^\alpha} \left[\int_\Omega |f(x)|^p d\mu(x) \right]^{1/p} \tag{2.2}$$

for all $f \in L^p(\Omega, \mu)$. Then, T satisfies

$$\begin{aligned} &\int_{|Tf| \le 1} |Tf(x)|^{p_1} d\mu(x) + \int_{|Tf|>1} |Tf(x)|^{p_0} d\mu(x) \\ \le &C \left[\int_{|f| \le 1} |f(x)|^{p_1} d\mu(x) + \int_{|f|>1} |f(x)|^{p_0} (1 + \log |f(x)|)^{p_0 \alpha} d\mu(x) \right]. \end{aligned} \tag{2.3}$$

Proof of Theorem 2.1. Modifying the proof of Theorem 1.1 ([8]), we can prove Theorem 2.1. First, we prepare some lemmas.

Lemma 2.3

Let $p = (1-\theta) p_0 + \theta p_1$ $(0 < \theta < 1)$. Then

$$\left((L^{p_0})^{p_0/p_1}, L^{p_1} \right)_{\theta, p_1; K} = (L^p)^{p/p_1}. \tag{2.4}$$

Moreover, the norms satisfy

$$\left(\theta(1-\theta) \right)^{1/p_1} \|f\|_{\theta, p_1; K} \approx ||f||_p^{p/p_1} \tag{2.5}$$

for any $f \in L^p(\Omega, \mu)$.

Proof. From the definition of K functional, we have

$$2^{(1/p_1)-1}K\big(t^{1/p_1}, f; (L^{p_0})^{p_0/p_1}, L^{p_1}\big) \le K\big(t, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1}\big)^{1/p_1} \le 2K\big(t^{1/p_1}, f; (L^{p_0})^{p_0/p_1}, L^{p_1}\big). \tag{2.6}$$

Hence,

$$\begin{aligned}
&\big(\theta(1-\theta)\big)^{1/p_1}\|f\|_{K_{\theta,p_1}((L^{p_0})^{p_0/p_1},L^{p_1})} \\
&= \left[\theta(1-\theta)\sum_{n=-\infty}^{\infty}\left(\frac{K(2^{n/p_1},\ f\ (L^{p_0})^{p_0/p_1}, L^{p_1})}{2^{n\theta/p_1}}\right)^{p_1}\right]^{1/p_1} \\
&\approx \left[\theta(1-\theta)\sum_{n=-\infty}^{\infty}\frac{K(2^n, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1})}{2^{n\theta}}\right]^{1/p_1}
\end{aligned}$$

and by [1, Theorem 5.2.2], this is equivalent to $\big(\|f\|_p^p\big)^{1/p_1}$. □

From this fact, we can get the following relation.

Lemma 2.4

Let $1 \le p_0 < p < p_1 < \infty$ and $p = (1-\theta)p_0 + \theta p_1$. Then, we have embeddings

$$J_{\theta,1}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big) \hookrightarrow (L^p)^{p/p_1} \hookrightarrow K_{\theta,\infty}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big).$$

Moreover, there exist some $M > 0$ such that all of their operator norms are dominated by M.

Proof. From [1, Theorem 3.11.3, Theorem 3.1.2, Theorem 3.4.1(b)], we have

$$\|f\|_{\theta,1;J} \ge \big(\theta(1-\theta)\big)\|f\|_{\theta,1;K} \ge \big(\theta(1-\theta)\big)^{1/p_1}\|f\|_{\theta,p_1;K} \approx ||f||_p^{p/p_1}$$

and we get the first embedding. By [1, Theorem 3.1.2], we can get the second embedding easily. □

We state one more fact.

Fact ([8, Lemma 3.4])

Let $\bar{A} = (A_0, A_1)$ be a mutually closed pair of quasi-Banach spaces such that $A_0 \cap A_1$ is dense in both A_0 and A_1. For any $f \in J_{\rho_\beta,1}(A_0, A_1)$,

$$\|f\|_{\rho_\beta,1;J} \approx \sum_{\mu=-\infty}^{0}(1-\mu)^{\beta-1}K(2^\mu, f; A_0, A_1). \tag{2.7}$$

Now, we shall prove Theorem 2.1 in several steps.

Proposition 2.5

Let $1 \le p_0 < p < p_1 < \infty$ and $\alpha > 0$. Then,

$$\begin{aligned}(2.8)\qquad &\sum\Big\{(p-p_0)^{-\alpha p/p_1}(L^p)^{p/p_1} : p_0 < p < p_1\Big\} \\ &\quad = (p_1-p_0)^{-\alpha/p_1} \sum\Big\{\theta^{-\alpha p_0/p_1} J_{\theta,p_1}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big) : 0<\theta<1\Big\}.\end{aligned}$$

Proof. From the extremal property of the functors $J_{\theta,1}$ and $K_{\theta,\infty}$ ([1, Theorem 3.11.4]), we have embeddings

$$J_{\theta,1}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big) \hookrightarrow J_{\theta,p_1}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big) \hookrightarrow K_{\theta,\infty}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big)$$

and all of their norms are uniformly bounded. So, with Lemma 2.4, it is enough to prove

$$\begin{aligned}(2.9)\qquad &\sum\Big\{\theta^{-\alpha p_0/p_1} J_{\theta,1}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big) : 0<\theta<1\Big\} \\ &\quad \hookleftarrow \sum\Big\{\theta^{-\alpha p_0/p_1} K_{\theta,\infty}\big((L^{p_0})^{p_0/p_1}, L^{p_1}\big) : 0<\theta<1\Big\}.\end{aligned}$$

By using (2.7) for $\beta = \frac{\alpha p_0}{p_1}$, we have

$$\begin{aligned}(2.10)\qquad &\|f\|_{\sum\{\theta^{-\alpha p_0/p_1} J_{\theta,1}((L^{p_0})^{p_0/p_1}, L^{p_1})\}} \\ &\le C \sum_{\mu=-\infty}^{0} (1-\mu)^{(p_0\alpha/p_1)-1}\; K\big(2^\mu, f; (L^{p_0})^{p_0/p_1}, L^{p_1}\big) \\ &\le C \sum_{\mu=-\infty}^{0} (1-\mu)^{(p_0\alpha/p_1)-1}\; 2^{\mu\theta}\|f\|_{K_{\theta,\infty}((L^{p_0})^{p_0/p_1}, L^{p_1})} \\ &\le C'\theta^{-\alpha p_0/p_1}\|f\|_{K_{\theta,\infty}((L^{p_0})^{p_0/p_1}, L^{p_1})}\end{aligned}$$

and we get (2.9) easily. □

Proposition 2.6

Let $\bar{A} = (A_0, A_1)$ be a pair of quasi-Banach spaces and let $1 \le r < \infty$ and $\beta > 0$. Then,

$$(2.11)\qquad \sum\big\{\theta^{-\beta} J_{\theta,r}(A_0, A_1) : 0<\theta<1\big\} = J_{\rho_\beta,r}(A_0, A_1)$$

where

$$\rho_\beta(t) = \sup_{0<\theta<1} t^\theta \theta^\beta \sim \begin{cases} (1-\log t)^{-\beta} & (0<t\le 1) \\ t & (t>1). \end{cases}$$

Moreover, these quasi-norms are equivalent.

Proof. This can be proved similarly to [8 Lemma 3.3] and we omit it (*c.f.* [3 Theorem 3.1 and (5.1.1)]). □

Proposition 2.7

For any $f \in \bar{A}_{\rho_{(p_0\alpha/p_1)},p_1;J}$,

(2.12)

$$\|f\|_{J_{\rho_{(p_0\alpha p_1)},p_1}((L^{p_0})^{p_0/p_1},L^{p_1})} \approx \left[\sum_{\mu=-\infty}^{0} (1-\mu)^{p_0\alpha-1} K(2^\mu, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1})\right]^{1/p_1}.$$

Proof. From the definition of J functional, we can get

$$\begin{aligned} J\big(t^{1/p_1}, f; (L^{p_0})^{p_0/p_1}, L^{p_1}\big) &\le J\big(t, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1}\big)^{1/p_1} \\ &\le 2J\big(t^{1/p_1}, f; (L^{p_0})^{p_0/p_1}, L^{p_1}\big). \end{aligned}$$

Hence, we have

(2.13)

$$\begin{aligned} \|f\|^{p_1}_{J_{\rho_{(p_0\alpha/p_1)},p_1}((L^{p_0})^{(p_0/p_1)},L^{p_1})} &= \inf \sum_{n=-\infty}^{\infty} \left(\frac{J\big((2^{1/p_1})^n,\ f_n\ (L^{p_0})^{p_0/p_1}, L^{p_1}\big)}{\rho_{p_0\alpha/p_1}\big((2^{1/p_1})^n\big)} \right)^{p_1} \\ &\approx \inf \sum_{n=-\infty}^{\infty} \frac{J(2^n,\ f_n\ (L^{p_0})^{p_0}, (L^{p_1})^{p_1})}{\rho_{p_0\alpha/p_1}\,(2^{n/p_1})^{p_1}} \\ &\approx \inf \sum_{n=-\infty}^{\infty} \frac{J\big(2^n,\ f_n\ (L^{p_0})^{p_0}, (L^{p_1})^{p_1}\big)}{\rho_{p_0\alpha}(2^n)} \\ &= \|f\|_{J_{\rho_{p_0\alpha},1}((L^{p_0})^{p_0},(L^{p_1})^{p_1})}. \end{aligned}$$

Now, applying (2.7), we get our conclusion. □

Proposition 2.8

For any $\beta > 0$,

$$\begin{aligned} &\sum_{\nu=-\infty}^{0} (1-\nu)^{\beta-1} K\big(2^\nu, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1}\big) \\ &\approx \int_{|f|\le 1} |f(x)|^{p_1} d\mu(x) + \int_{|f|>1} |f(x)|^{p_0} \big(1+\log|f(x)|\big)^{\beta} d\mu(x). \end{aligned} \tag{2.14}$$

Proof. First,

$$\begin{aligned} K\big(t, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1}\big) &= \inf_{f_0+f_1=f} \big(\|f_0\|_{p_0}^{p_0} + t\|f_1\|_{p_1}^{p_1}\big) \\ &= \int_\Omega \inf_{f_0(x)+f_1(x)=f(x)} (|f_0(x)|^{p_0} + t|f_1(x)|^{p_1})\, d\mu(x) \\ &= \int_\Omega |f(x)|^{p_0} F(t|f(x)|^{p_1-p_0}) d\mu(x) \end{aligned}$$

where $F(s) = \inf_{y_0+y_1=1}(|y_0|^{p_0} + s|y_1|^{p_1}) \approx \min(1, s)$ (see [1, Theorem 5.2.2]). So, we have

$$\begin{aligned} K\big(t, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1}\big) \approx \int_{|f|>t^{1/p_0-p_1}} |f(x)|^{p_0} d\mu(x) \\ + t\int_{|f|\le t^{1/p_0-p_1}} |f(x)|^{p_1} d\mu(x). \end{aligned} \tag{2.15}$$

Now,

$$\begin{aligned} &\sum_{\nu=-\infty}^{0} (1-\nu)^{\beta-1} K\big(2^\nu, f; (L^{p_0})^{p_0}, (L^{p_1})^{p_1}\big) \\ &= \sum_{\nu=-\infty}^{0} (1-\nu)^{\beta-1} \left[2^\nu \int_{|f|\le 1} |f(x)|^{p_1} d\mu(x) + 2^\nu \int_{1<|f|\le 2^{\nu/p_0-p_1}} |f(x)|^{p_1} d\mu(x) \right. \\ &\quad \left. + \int_{|f|>2^{\nu/p_0-p_1}} |f(x)|^{p_0} d\mu(x)\right] \\ &= \sum_{\nu=-\infty}^{0} (1-\nu)^{\beta-1} 2^\nu \int_{|f|\le 1} |f(x)|^{p_1} d\mu(x) \\ &\quad + \sum_{\nu=-\infty}^{0} (1-\nu)^{\beta-1} \left[2^\nu \int_{1<|f|\le 2^{\nu/p_0-p_1}} |f(x)|^{p_1} d\mu(x) + \int_{|f|>2^{\nu/p_0-p_1}} |f(x)| d\mu(x)\right] \\ &= S_1 + S_2, \quad \text{say.} \end{aligned}$$

Simply we have

$$S_1 = \left(\sum_{\nu=-\infty}^{0} (1-\nu)^{\beta-1} 2^\nu\right) \int_{|f|\le 1} |f(x)|^{p_1} d\mu(x) = M(\beta) \int_{|f|\le 1} |f(x)|^{p_1} d\mu(x)$$

where $M(\beta)$ is defined in (3.7). Next, we estimate S_2. Put

$$I_\mu = \int_{2^{\mu/p_0-p_1}<|f|\le 2^{\mu-1/p_0-p_1}} |f(x)|^{p_0} d\mu(x).$$

If $2^{\mu/p_0-p_1} < |f(x)| \le 2^{\mu-1/p_0-p_1}$, we have $2^{-\mu} < |f(x)|^{p_1-p_0} \le 2^{-\mu+1}$. So, we have

$$2^{-\mu} I_\mu < \int_{2^{\mu/p_0-p_1}<|f|\le 2^{\mu-1/p_0-p_1}} |f(x)|^{p_1} d\mu(x) \le 2^{-\mu+1} I_\mu.$$

Therefore,

$$\begin{aligned} S_2 &\approx \sum_{\nu=-\infty}^{-1} (1-\nu)^{\beta-1} \left[2^\nu \sum_{\mu=0}^{\nu+1} 2^{-\mu} I_\mu + \sum_{\mu=-\infty}^{\nu} I_\mu \right] + \sum_{\mu=-\infty}^{0} I_\mu \\ &= \sum_{\mu=-\infty}^{0} \left[\sum_{\nu=\mu}^{0} (1-\nu)^{\beta-1} + \sum_{\nu=-\infty}^{\mu-1} (1-\nu)^{\beta-1} 2^{\nu-\mu} \right] I_\mu \\ &= \sum_{\mu=-\infty}^{0} S_\mu^- I_\mu, \end{aligned}$$

where S_μ^- has been defined in (3.6). Now, we can get

$$\begin{aligned} S_2 &\approx M'(\beta) \sum_{\mu=-\infty}^{0} (1-\mu)^\beta I_\mu \\ &= M'(\beta) \sum_{\mu=-\infty}^{0} (1-\mu)^\beta \int_{2^{\mu/p_0-p_1}<|f|\le 2^{\mu-1/p_0-p_1}} |f(x)|^{p_0} d\mu(x) \\ &\approx M'(\beta) \int_{|f|>1} |f(x)|^{p_0} \big(1+\log|f(x)|\big)^\beta d\mu(x) \end{aligned}$$

and complete the proof. □

With these propositions, we conclude Theorem 2.1.

Proof of Corollary 2.2. If T is linear operator satisfying (2.2), we can get (2.3) in consequence of the general result ([3 §2]). Let T satisfy the subadditive property only. Suppose $f \in \sum \left\{ \big((p-p_0)^{-\alpha} L^p(\Omega)\big)^{p/p_1} : p_0 < p < p_1 \right\}$ and it has a representation $f = \sum_n f_n$, $f_n \in L^{r_n}$ $p_0 < r_n < p_1$. Then, we have

$$\begin{aligned} \|Tf\|_{\sum\{(L^p(\Omega))^{p/p_1}:p_0<p<p_1\}} &\le \left\| \left(\sum_n |Tf_n| \right) \right\|_{\sum\{(L^p(\Omega))^{p/p_1}:p_0<p<p_1\}} \\ &\le \sum_n \|Tf_n\|_{r_n}^{r_n/p_1} \\ &\le C \sum_n \frac{1}{(r_n-1)^\alpha} \|f_n\|_{r_n}^{r_n/p_1}. \end{aligned}$$

Take infimum over all representation $f = \sum_n f_n$, we get our conclusion. □

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