Collect. Math. **48**, 4-6 (1997), 725–732

The differences of inclusion map operators between rearrangement invariant spaces on finite and σ-finite measure spaces

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Abstract

Let X be a quasi-Banach RIS (QBRIS) on $[0,1]$. Then the following inclusions are valid: $L_\infty \subset X \subset L_p$, where $p = p(X) > 0$. In classical Banach case $p = 1$ and for canonical injection operators $I : L_\infty \to X$; $I : X \to L_1$ it's known conditions for such properties as strict singularity (SS), disjoint strict singularity (DSS), (p,q)-absolutely summing, etc. We prove some similar facts in quasi-Banach case. If X is a QBRIS on $[0,\infty]$, then it is γ-normed for some $0 < \gamma \le 1$ and $L_\infty \cap L_\gamma \subset X \subset L_p + L_\infty$, for some $p = p(X) > 0$. On the contrary to the finite measure case, when $I(L_\infty, X) \in SS$ for any $X \ne L_\infty$, there are many examples of spaces on $[0,\infty)$ such that $I \notin DSS(L_1 \cap L_\infty, X)$. Another deep difference is : on $[0,1]$: $I(X, L_1) \in DSS$ for any Banach $X \ne L_1$; but on $[0,\infty)$: $I(X, L_p + L_\infty) \notin DSS$ for X such that $L_{r,\infty} \subset X$ for some $r > p$.

1. Definitions and basic notations

Let us start with some definitions. We shall use the term operator to mean a bounded linear operator; subspaces are assumed infinite and closed. We shall consider rearrangement invariant spaces *(RIS)* of functions, both Banach and quasi-Banach. A quasi-Banach RIS is a complete quasinormed vector space $(X, \|\cdot\|)$ of measurable functions on $(0,1)$ or $(0,\infty)$ such that $\|\kappa_A\| = 1$ if meas $A = 1$, and if g is in X, then f is in X and $\|f\| \le \|g\|$ if f is a measurable function satisfying $f^* \le g^*$, where h^* denotes the decreasing rearrangement of the function $|h|$ (cf. [1]). A quasinorm is a function which satisfies the axioms for a norm except that the triangle inequality is replaced by

$$\|x + y\| \le K\big(\|x\| + \|y\|\big)$$

with some $K > 1$.

An operator T between two quasi-Banach spaces X and Y is called strictly singular *(S.S.)* (or Kato) if it fails to be an isomorphism on any (infinite dimensional) subspace. This class is a closed operator ideal (cf. [2]).

An operator T between a quasi-Banach lattice X and a quasi-Banach space Y is called disjointly strictly singular *(D.S.S.)* if there is no disjoint sequence of non-null vector (x_n) in X such that the restriction of T to the subspace $[x_n]$ spanned by the vectors (x_n) is an isomorphism. *D.S.S.* operators have been introduced in [3].

Clearly, every *S.S.* operator is *D.S.S.*. However, the converse is not true in general (e.g. the inclusion map $L^p(0,1) \hookrightarrow L^q(0,1)$, for $0 < q < p < \infty$, is *D.S.S.* but it is not *S.S.*). The class of *D.S.S.* operators is not an operator ideal in general, it fails to be stable with respect to the composition on the right (cf. [4]).

2. Inclusion map operators between Banach RIS on $[0,1]$

If X is a Banach *RIS* on $[0,1]$, then $L^\infty \subset X \subset L^1$. The inclusion map from L^∞ to X is S.S. for every Banach *RIS* different from X (the proof was given in [5]). This fact is the generalization of the A. Grothendick's theorem about S.S. of inclusion $L^\infty \hookrightarrow L^p$ $1 \le p < \infty$ ([6], ch. 5). By the way, this theorem together with the A. Pietsch's theorem about factorization of p-absolutely summing operators through the restriction of inclusion $L^\infty \hookrightarrow L^p$ gives an easy way to prove the *S.S.* of every p-absolutely summing operator (compare with the proof of Proposition 4.6.14 in [2]).

The inclusion map $X \hookrightarrow L^1$ is not *S.S.* in general (see examples of tunnel subspaces below). Nevertheless, it is *D.S.S.* for every Banach *RIS* $X \neq L^1$.

Theorem 2.1

Let X be a Banach RIS on $[0,1]$ different from L^1. Then the inclusion map $X \hookrightarrow L^1$ is D.S.S.

Proof. The fundamental function ϕ_X of a Banach *RIS* X is defined by

$$\phi_X(t) = \|\kappa_{[0,t]}\|_X.$$

Fix $\epsilon > 0$ and for $f \in X$ we define

$$M_{f,\epsilon} = \{t :| f(t) |\geq \epsilon \|f\|_X\}$$

and

$$M_\epsilon^X = \{f \in X :| M_{f,\epsilon} |> \epsilon\}.$$

Such classes appeared first in the classical paper [7] for $X = L_p$. Suppose that a subset $K \subset X$ is contained in M_ϵ^X for some $\epsilon > 0$. Then

$$\|f\|_1 = \int_0^1 |f(t)|dt \geq \int\limits_{M_{f,\epsilon}} |f(t)|dt \geq \epsilon \|f\|_X |M_{f,\epsilon}| \geq \epsilon^2 \|f\|_X$$

for every $f \in K$.

Conversely, suppose that the norms of X and L^1 are equivalent on K: $\delta\|f\|_X \leq \|f\|_1$ for every $f \in K$ and for some $\delta > 0$. Suppose that for every $\epsilon > 0$ there is a function $f_\epsilon \in K$ with $f_\epsilon \not\subset M_\epsilon^X$. Since $X \neq L^1$, the associate space X' of X is an *RIS* on $[0,1]$ different from L^∞, so that $\lim\limits_{t\to 0} \phi_{X'}(t) = 0$ (cf. [8], lemma 3). Let's define $\epsilon > 0$ such that $\phi_{X'}(\epsilon) + \epsilon \leq \delta/2$. Then we have

$$\begin{aligned}\delta\|f_\epsilon\|_X \leq \|f_\epsilon\|_1 = \int\limits_{M_{f_\epsilon,\epsilon}} |f_\epsilon(t)|dt + \int\limits_{[0,1]\setminus M_{f_\epsilon,\epsilon}} |f_\epsilon(t)|dt \\ \leq \|f_\epsilon\|_X \phi_{X'}(\epsilon) + \epsilon |f_\epsilon\|_X \leq (\delta/2)\|f\|_X,\end{aligned}$$

which leads to a contradiction.

So we have proved that the norms of X and L^1 are equivalent on a subset K of X if and only if K is contained in M_ϵ^X for some $\epsilon > 0$. If we assume that the inclusion map $I : X \hookrightarrow L^1$ is not *D.S.S.*, then there exists a disjoint sequence of non-null functions (f_n) in X such that the $I|_{[f_n]}$ is an isomorfism. So there exists $\epsilon > 0$ with $[f_n] \subseteq M_\epsilon^X$. Hence we get that

$$1 = |[0,1]| \geq |\cup M_{f_n,\epsilon}| = \Sigma |M_{f_n,\epsilon}| = \infty;$$

note that the sets $M_{f_n,\epsilon}$ are disjoint. $\square$

Remark. The essential part of this proof was published in [9], but the notion of *D.S.S.* operator was not introduced at that time.

Let's return to an example, connected with the A. Grothendick's theorem. Are there the *RIS* $X \neq L^\infty$ such that the inclusion map $X \hookrightarrow L^p$ is *S.S.* for every $p < \infty$? The affirmative answer to this question is given by.

Example 2.1: Let X_0 be the closure of the space $C[0,1]$ in the Orlicz space L_{M_q} with $M_q^{(u)} \sim \exp u^q$ for some $q > 2$. Then $X_0 \subset \cap_{p<\infty} L_p$, but $X_0 \neq L^\infty$. We can prove that the inclusion map $X_0 \hookrightarrow L_p$ is *S.S.* for every finite p. For this purpose we define

$$\eta_X(K) = \lim_{\tau\to 0} \sup_{x\in K, x\neq 0} \frac{\|x^*\kappa_{[0,\tau]}\|_X}{\|x\|_X}, \quad K \subset X, \quad X - RIS.$$

If $\eta_X(K) < 1$, then $K \subset M_\epsilon^X$ for some $\epsilon > 0$ ([9]). Suppose that there exists subspace $H \subset X_0$ closed in L_p. Then $\eta_{L_p}(H) = 0$ and it means that H is closed in L_1 (see the proof of the Theorem 2.1). So the inclusion map $X_0 \hookrightarrow L_1$ is not *S.S.* But this fact contradicts Theorem 2 from [5].

3. Inclusion map operators between quasi-Banach *RIS* on $[0,1]$. Tunnel subspaces

Every quasi-Banach *RIS* (*QBRIS*) X can be equivalently re-normed as an γ-Banach space for some $\gamma \in (0,1)$, i.e. in addition to axioms we can add inequality

$$\|f+g\|^\gamma \le \|f\|^\gamma + \|g\|^\gamma, \quad f, g \in X.$$

It was proved in [10] that for γ-Banach space X the following continuous inclusions are valid (on $[0,1]$):

$$L^\infty \subset X \subset L^{\gamma,\infty} \tag{3.1}$$

where $L^{\gamma,\infty} = \{f : \|f\|_{\gamma,\infty} = \sup_{t>0} t^{1/\gamma} f^*(t) < \infty\}$. In the same paper it was shown also that $L^{\gamma,\infty}$ is a γ-normed space. The questions about *S.S.* of two inclusion map operators: $L^\infty \hookrightarrow X$ and $X \hookrightarrow L^{\gamma,\infty}$ in general case are opened.

If we assume in addition that a γ-normed space X is γ-convex, i.e. for any $f_1, \dots, f_n$ in X

$$\left\| \left(\sum_{i=1}^n |f_i|^\gamma \right)^{1/\gamma} \right\| \le \left(\sum_{i=1}^n \|f_i\|^\gamma \right)^{1/\gamma},$$

the situation becomes similar to Banach case.

In this case inclusions (3.1) transform to

$$L^\infty \subset X \subset L^\gamma \quad \text{(cf. [10])}$$

and the following theorem takes place (cf. [11]).

Theorem 3.1

Let X be an arbitrary RI γ-convex function space $(0 < \gamma < 1)$, $X \neq L^\gamma$, then the inclusion $L^\infty \hookrightarrow X$ is S.S. and the inclusion $X \hookrightarrow L^\gamma$ is D.S.S.

Proof. We will use a standard construction of γ-convexification of X (cf. [1], [11]). Let us consider the space $X_{(\gamma)} = \{f : |f(t)|^{1/\gamma} \in X\}$ with $\|f\|_{(\gamma)} = \||f|^{1/\gamma}\|^{\gamma}$. It is known that X_γ is a Banach *RIS* and $X_{(\gamma)} \hookrightarrow X$. Besides it's not difficult to show that the inclusion map $X \hookrightarrow Y$ is *D.S.S.* if and only if the inclusion map $X_{(\gamma)} \hookrightarrow Y_{(\gamma)}$ is *D.S.S.* Now the first assertion of the theorem is deduced from the factorization $L^\infty \hookrightarrow X_{(\gamma)} \hookrightarrow X$, and the second one is deduced from the *D.S.S.* of inclusion $X_{(\gamma)} \hookrightarrow (L^\gamma)_\gamma = L_1$. □

Quite a natural way to show that inclusion are not *S.S.* is to find the so-called tunnel subspaces. Such name is given to the subspace H such that such that $H \subset X_1 \subset X_2$ and H is closed in X_2. On such a subspace (quasi-) norms of *RIS* X_1 and X_2 are equivalent. The spaces X_1 and X_2 may be called the tunnel's bounds.

The most well-known example of a tunnel subspace is $R_2 = \{f : f \overset{L_0}{=} \Sigma c_k r_k, (c_k) \in l_2\}$, r_k - Rademacher functions; L_0 is the space of all Lebesgue-measurable functions with measure convergence. It was shown in [8], that R_2 is a tunnel subspace for Banach *RIS* X such that $G \subset X \subset L_1$, where G is the closure of $C[0,1]$ in L_{M_2}, $M_2(u) \sim \exp u^2$. Moreover, we can state that the subspace R_2 is a tunnel subspace for a wider class of *RIS* X such that $G \subset X \subset L_0$.

We will represent below some other examples of tunnel subspaces. The common feature of these examples is that they are spanned by sequences of independent identically distributed (*i.i.d.*) symmetric random variables (*r.v.*).

A real function (or *r.v.*) $s^{(r)}(\omega)$, $0 \le \omega \le 1$, $0 < r \le 2$ is called r-stable, if $\int_R e^{its^{(r)}(\omega)} d\omega = e^{-c|t|^r}$ for some $c > 0$ and $\forall t \in R$. 2-stable *r.v.* is a gaussian *r.v.* Together with r-stable *r.v.* we will consider the next *r.v.*

$$g^{(r)}(\omega) = \omega^{-1/r}, \quad 0 < r < \infty, \quad 0 < \omega \le 1.$$

If $(s_n^{(r)})$, $(g_n^{(r)})$ are *i.i.d.* copies of functions $s^{(r)}(\cdot)$, $g^{(r)}(\cdot)$, then we have the following: for *RIS* X such that $L^0 \supset X \supset L^{r,\infty}$,

$$\|\Sigma a_n s_n^{(r)}\|_X \approx \|\Sigma a_n g_n^{(r)}\|_X \approx (\Sigma |a_n|^r)^{1/r},$$

for $0 < r < 2$.

It means that there is the tunnel subspace isomorphic l_r in the bounds marked above.

For the case $r \geq 2$ the situation radically changes:

$$\overline{\text{span}}^X\left(s_n^{(2)}\right) \approx l_2 \Longleftrightarrow X \supset G$$

$$\overline{\text{span}}^X\left(g_n^{(2)}\right) \approx l_M \Longleftrightarrow X \supset L^{2,\infty},$$

where l_M is an Orlicz sequence space, generated by the function $M(u) = u^2 \ln(1 + 1/u),\ u > 0$. The proof of the last fact is rather long and will be published separately.

For $r > 2$ the r-stable function can't be defined, but

$$\overline{\text{span}}^X\left(g_n^{(r)}\right) \approx l_2 \Longleftrightarrow X \supset L^{r,\infty},$$

The tunnels for subspaces spanned by disjoint functions are never long. We only represent here two propositions: 1) Inclusions $L^{p,1} \hookrightarrow L^p$ and $L^p \hookrightarrow L^{p,\infty}$ are *D.S.S.* on $[0,1]$ for $p > 1$. 2) For any $p > 1$ there is an Orlicz function space L_M such that the inclusion map $L_M \hookrightarrow L^p$ is not *D.S.S.* (cf. [3]).

4. Inclusion map operators between *RIS* on $[0,\infty)$

Many interesting results on this theme were obtained in [11]. Contrary to the $[0,1]$-situation it is possible to construct the subspaces spanned by disjoint functions with rather long tunnels. It may be noted from the

Theorem 4.1.

Let $f(\omega) = \omega^{-1/r},\ 0 < \omega < \infty;\ 0 < r < \infty;\ \{f_i\}$ are disjoint copies of this function. Then $H_r := \overline{\text{span}}^{L^{r,\infty}}\{f_i^{(r)}\} \approx l_r$ and H_r is closed in $L_p + L_\infty$ for $0 < p < r$. So there is a tunnel subspace H_r, spanned by disjoint functions, with the bounds: $L^{r,\infty} \subset X \subset L_p + L_\infty$.

Proof. The upper estimate for $\|\sum a_i f_i^{(r)}\|_X$ is well-known:

$$\Big\|\sum a_i f_i^{(r)}\Big\|_X \leq C\Big\|\sum a_i f_i^{(r)}\Big\|_{r,\infty} \leq C'\Big(\sum |a_i|^r\Big)^{1/r},$$

since the space $L^{r,\infty}$ satisfies the upper r-estimate (for disjoint functions).

In order to obtain the lower estimate we begin with with the writing the (quasi) norm of $L^r + L^\infty$ in the following view

$$\|f\|^p_{L^p+L^\infty} = \int\limits_0^1 x^*(t)^p dt = \big(x^*(1)\big)^p + p \int\limits_{x^*(1)}^{\infty} d(x,s)s^{p-1}ds,$$

where $x^*(\cdot)$ is a decreasing rearrangement of $|x(\cdot)|$, $d(x;s) = m(t : |x(t)| > s)$.

After that we receive:

$$\Big(\sum a_n f_n^{(r)}\Big)^*(1) = \Big(\sum |a_n|^r\Big)^{1/r}$$

and

$$p \int\limits_{(\sum |a_n|^r)^{1/r}}^{\infty} d\Big(\sum a_n f_n^{(r)}; s\Big) s^{p-1}ds = \frac{p}{r-p}\Big(\sum |a_n|^r\Big)^{p/r}.$$

Hence,

$$\Big\|\sum a_n f_n^{(r)}\Big\|_{L^p+L^\infty} = \Big(\frac{r}{r-p}\Big)^{1/p}\Big(\sum |a_n|^r\Big)^{1/r},$$

and

$$\Big\|\sum a_n f_n^{(r)}\Big\|_X \geq \frac{1}{C}\Big\|\sum a_n f_n^{(r)}\Big\|_{L^p+L^\infty} = \frac{1}{C}\Big(\frac{r}{r-p}\Big)^{1/p}\|(a_n)\|_r . \ \Box$$

Another deep difference between properties of inclusion map operators on $[0,1]$ and $[0,\infty)$ be seen by considering the extreme inclusions. As was shown above, the inclusion of Banach *RIS* $X[0,1] \hookrightarrow L^1$ is always *D.S.S.* On the other hand, it's easy to see that the inclusion $L^\infty[0,\infty) \hookrightarrow L^1 + L^\infty$ is not *D.S.S.* The tunnel subspace here is spanned by functions $\{\kappa_{[n-1,n]}(\cdot)\}$. From this simple fact we deduce that between L^∞ and $L^1 + L^\infty$ there is not a single reflexive Banach *RIS*. The reason of this phenomena is the Dunford-Pettis property of the space L^∞. In the $[0,1]$-case for any Banach *RIS* $X \neq L_1$, L_∞ there are reflexive Banach *RIS* E_1 and E_2 such that $E_1 \subset X \subset E_2$ ("reflexive gates") (cf. [5], proof of th. 1).

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