Weighted inequalities for monotone functions[1]

L. Maligranda

Department of Mathematics, Luleå University,

S-971 87 Luleå, Sweden

E-mail: lech@sm.luth.se

Abstract

We give characterizations of weights for which reverse inequalities of the Hölder type for monotone functions are satisfied. Our inequalities with general weights and with sharp constants complement the results of [2], [6], [7] and [14], [15] for the values of parameters $0 < p \leq q < \infty$.

1. Introduction

We consider positive *monotone* functions f on $(0, \infty)$ is the sense that, for some real number α, $x^{-\alpha} f(x)$ is either a decreasing or an increasing function. More precisely, we write $f \in Q_\alpha$ when $x^{-\alpha} f(x)$ is decreasing and $f \in Q^\alpha$ when $x^{-\alpha} f(x)$ is increasing.

The purpose of this paper is to find conditions on weight functions u and v such that the inequality

$$(*) \qquad \left(\int_0^\infty f(x)^q u(x) dx \right)^{1/q} \leq C \left(\int_0^\infty f(x)^p v(x) dx \right)^{1/p}$$

holds, for any positive function f from one of the classes Q^α, Q_α or $Q^{\alpha_0} \cap Q_{\alpha_1}$. Our main here is to prove such inequalities with the best constants. Surprisingly enough,

---

this was possible for all parameters $0 < p \leq q < \infty$. Some results of this type were proved earlier by Lorentz-Hunt (cf. [13]), Bergh [1], Bergh-Burenkov-Persson [2], Stepanov [14], Heinig-Stepanov [7], Pečarić-Persson [11], Heinig-Maligranda [6] and recently by Gol'dman-Heinig-Stepanov [5].

All functions considered throughout the paper are assumed measurable and positive (positive $\equiv$ non-negative and not identically zero) on $(0, \infty)$. We shall write $0 \leq f \downarrow$ to mean that the function f is positive and decreasing (decreasing $\equiv$ non-increasing); for such f we define $f^{-1}(t) = \inf\{s > 0 : f(s) < t\}$, $\inf \emptyset = \infty$. Similarly, $0 \leq f \uparrow$ means that f is positive and increasing (increasing $\equiv$ non-decreasing); for such f we define $f^{-1}(t) = \inf\{s > 0 : f(s) > t\}$, $\inf \emptyset = \infty$.

Weight functions are locally integrable positive functions on $(0, \infty)$ which we usually denote by u, v, w. 1_E denotes the characteristic function of the set E.

The $L_{p,w}$-norm is the functional $\|f\|_{p,w} = \left(\int_0^\infty f(x)^p w(x) dx \right)^{1/p}$.

Inequalities, such as in (∗), are interpreted to mean that if the right side is finite, so is the left, and the inequality holds.

2. The main results

We first prove sharp results for functions from the classes Q_{α_1} and Q^{α_0}. The results for the classes Q_0 and Q^0 were proved by Sawyer [12], Stepanov [14] and Heinig-Stepanov [7]. Different proofs were also given by Heinig-Maligranda [6]. In fact, it is enough to prove such results for classes Q_0 and Q^0 and then by change of weights to get them for classes Q_{α_1} and Q^{α_0}, but we are here giving two new proofs of these results.

The following lemma will be useful in the proofs (see also [9]).

Lemma 1

Let w be a weight function and $0 < \gamma \leq 1$, $0 < r < \infty$.
If $0 \leq \varphi \downarrow$, then

$$(1) \qquad \left(\int_0^\infty \varphi^{-1}(x)^{1/\gamma} w(x) dx \right)^\gamma \leq \int_0^\infty \left[\int_0^{\varphi(y)} w(x) dx \right]^\gamma dy$$

and

$$(2) \qquad \int_{\varphi^{-1}(x)}^\infty \varphi(y)^r dy = \int_0^x \varphi^{-1}(s) d(s^r) - x^r \varphi^{-1}(x).$$

If $0 \leq \varphi \uparrow$, then

$$(3) \qquad \left(\int_0^\infty \varphi^{-1}(x)^{1/\gamma} w(x) dx \right)^\gamma \leq \int_0^\infty \left[\int_{\varphi(y)}^\infty w(x) dx \right]^\gamma dy.$$

For $\gamma = 1$ we have equalities in (1) and (3).

Proof. First we prove that for the decreasing function φ we can write formally the inequality

$$(**) \qquad\qquad \varphi^{-1}(x) \leq \int_0^\infty 1_{[0,\varphi(y)]}(x) dy.$$

In fact, let $\varphi^{-1}(0^+) = \infty$ and $\varphi^{-1}(\infty) = 0$. If $0 < y < \varphi^{-1}(x)$, then $x \leq \varphi(y)$ and we have

$$\varphi^{-1}(x) = \int_0^{\varphi^{-1}(x)} dy = \int_0^\infty 1_{[0,\varphi^{-1}(x))}(y) dy \leq \int_0^\infty 1_{[0,\varphi(y)]}(x) dy \,.$$

In the case when either $\varphi^{-1}(0^+) < \infty$ or $\varphi^{-1}(\infty) > 0$ we can make modifications of this argument to see that inequality $(**)$ is true also in these cases. Then by the Minkowski inequality with the $L_{1/\gamma,w}$-norm to $(**)$, we find that

$$\left(\int_0^\infty \varphi^{-1}(x)^{1/\gamma} w(x) dx \right)^\gamma = \| \varphi^{-1}(x) \|_{1/\gamma,w}$$

$$\leq \left\| \int_0^\infty 1_{[0,\varphi(y)]}(x) dy \right\|_{1/\gamma,w} \leq \int_0^\infty \| 1_{[0,\varphi(y)]}(x) \|_{1/\gamma,w} dy$$

$$= \int_0^\infty \left[\int_0^{\varphi(y)} w(x) dx \right]^\gamma dy$$

and the inequality (1) is proved.

The equality (2) follows from the obvious equality

$$\int_0^{x^r} \varphi^{-1}(t^{1/r}) dt = \int_0^x \varphi^{-1}(s) d(s^r)$$

and geometrically clear equality

$$\int_0^{x^r} \varphi^{-1}(t^{1/r}) dt = x^r \varphi^{-1}(x) + \int_{\varphi^{-1}(x)}^\infty \varphi(y)^r dy \,.$$

To prove the inequality (3) for increasing φ we first observe that

$$(***) \qquad\qquad \varphi^{-1}(x) \leq \int_0^\infty 1_{[\varphi(y),\infty)}(x) dy.$$

In fact, if $\varphi^{-1}(0^+) = 0$, $\varphi^{-1}(\infty) = \infty$ and $0 < y < \varphi^{-1}(x)$, then $\varphi(y) \leq x$ and

$$\varphi^{-1}(x) = \int_0^{\varphi^{-1}(x)} dy = \int_0^\infty 1_{[0,\varphi^{-1}(x))}(y) dy \leq \int_0^\infty 1_{[\varphi(y),\infty)}(x) dy \,.$$

The cases when either $\varphi^{-1}(0^+) > 0$ or $\varphi^{-1}(\infty) < \infty$ need again minor modifications. Then, similarly as above, by the Minkowski inequality with the $L_{1/\gamma,w}$-norm used to $(***)$, we find that

$$
\left(\int_0^\infty \varphi^{-1}(x)^{1/\gamma} w(x)dx\right)^\gamma = \|\varphi^{-1}(x)\|_{1/\gamma,w}
$$

$$
\leq \left\|\int_0^\infty 1_{[\varphi(y),\infty)}(x)dy\right\|_{1/\gamma,w} \leq \int_0^\infty \|1_{[\varphi(y),\infty)}(x)\|_{1/\gamma,w} dy
$$

$$
= \int_0^\infty \left[\int_{\varphi(y)}^\infty w(x)dx\right]^\gamma dy,
$$

and the inequality (3) is proved.

If $\gamma = 1$, then the set $\{(x,y): y = \varphi(x) \text{ or } x = \varphi^{-1}(y)\}$ as the sum of the graphs of $\varphi(x)$ and $\varphi^{-1}(y)$ has two-dimensional Lebesgue measure zero (cf. [16]), and by the Fubini theorem we have equalities. $\square$

Remark 1. If φ is an increasing function and $\varphi(0^+) > 0$, then inequality (3) follows from inequality (1). We should only use inequality (1) to the decreasing function $\psi(x) = 1/\varphi(x)$ with the weight $w_1(x) = w(1/x)/x^2$ and change variable x to $1/x$.

Theorem 1

Let $0 < p \leq q < \infty$ and $-\infty < \alpha_0 < \alpha_1 < \infty$.

(a) *The inequality*

$$
(4) \qquad \left(\int_0^\infty f(x)^q u(x)dx\right)^{1/q} \leq A\left(\int_0^\infty f(x)^p v(x)dx\right)^{1/p}
$$

holds for all $0 \leq f \in Q_{\alpha_1}$ *if and only if*

$$
(5) \qquad A_{\alpha_1} := \sup_{t>0}\left(\int_0^t x^{q\alpha_1} u(x)dx\right)^{1/q}\left(\int_0^t x^{p\alpha_1} v(x)dx\right)^{-1/p} < \infty.
$$

Moreover $A = A_{\alpha_1}$ *is the best constant.*

(b) *The inequality*

$$
(6) \qquad \left(\int_0^\infty f(x)^q u(x)dx\right)^{1/q} \leq B\left(\int_0^\infty f(x)^p v(x)dx\right)^{1/p}
$$

holds for all $0 \leq f \in Q^{\alpha_0}$ *if and only if*

$$
(7) \qquad B_{\alpha_0} := \sup_{t>0}\left(\int_t^\infty x^{q\alpha_0} u(x)dx\right)^{1/q}\left(\int_t^\infty x^{p\alpha_0} v(x)dx\right)^{-1/p} < \infty.
$$

Moreover $B = B_{\alpha_0}$ *is the best constant.*

We give two proofs of the theorem.

Proof 1. (a) (4) $\Rightarrow$ (5). Take $f(x) = x^{\alpha_1} 1_{[0,t]}(x)$, $t > 0$, in (4). This also gives $A_{\alpha_1} \leq A$.

(5) $\Rightarrow$ (4). Let $A_{\alpha_1} < \infty$, i.e.

$$\left(\int_0^t x^{q\alpha_1} u(x)dx \right)^{1/q} \leq A_{\alpha_1} \left(\int_0^t x^{p\alpha_1} v(x)dx \right)^{1/p} \quad \forall t > 0.$$

Then, by putting both sides to the power p and $t = \varphi(y)$ with decreasing φ, and integrating from 0 to ∞ with respect to y, we obtain

$$\int_0^\infty \left(\int_0^{\varphi(y)} x^{q\alpha_1} u(x)dx \right)^{p/q} dy \leq A_{\alpha_1}^p \int_0^\infty \left(\int_0^{\varphi(y)} x^{p\alpha_1} v(x)dx \right) dy.$$

Now by using the inequality (1) with $\gamma = p/q \leq 1$ and the equality in (1) when $\gamma = 1$, we obtain

$$\left(\int_0^\infty \varphi^{-1}(x)^{q/p} x^{q\alpha_1} u(x)dx \right)^{p/q} \leq A_{\alpha_1}^p \int_0^\infty \varphi^{-1}(x) x^{p\alpha_1} v(x)dx.$$

Taking $\varphi^{-1}(x) = (x^{-\alpha_1} f(x))^p$, which is a decreasing function, we have

$$\left(\int_0^\infty f(x)^q u(x)dx \right)^{p/q} \leq A_{\alpha_1}^p \int_0^\infty f(x)^p v(x)dx$$

and so $A \leq A_{\alpha_1}$.

(b) The necessity and inequality $B_{\alpha_0} \leq B$ follows at once by taking $f(x) = x^{\alpha_0} 1_{[t,\infty)}(x)$, $t > 0$, in (6).

(7) $\Rightarrow$ (6). Now, using the assumption

$$\left(\int_t^\infty x^{q\alpha_0} u(x)dx \right)^{1/q} \leq B_{\alpha_0} \left(\int_t^\infty x^{p\alpha_0} v(x)dx \right)^{1/p} \quad \forall t > 0,$$

making the substitution $t = \varphi(y)$ with increasing φ, integrating in y from 0 to ∞ and applying the inequality (3) from Lemma 1, we get that

$$\left(\int_0^\infty \varphi^{-1}(x)^{q/p} x^{q\alpha_0} u(x)dx \right)^{p/q} \leq \int_0^\infty \left(\int_{\varphi(y)}^\infty x^{q\alpha_0} u(x)dx \right)^{p/q} dy$$

$$\leq B_{\alpha_0}^p \int_0^\infty \left(\int_{\varphi(y)}^\infty x^{p\alpha_0} v(x)dx \right)$$

$$= B_{\alpha_0}^p \int_0^\infty \varphi^{-1}(x) x^{p\alpha_0} v(x)dx.$$

Taking $\varphi^{-1}(x) = (x^{-\alpha_0} f(x))^p$, which is an increasing function, we obtain

$$\left(\int_0^\infty f(x)^q u(x)dx \right)^{p/q} \leq B_{\alpha_0}^p \int_0^\infty f(x)^p v(x)dx$$

and so $B \leq B_{\alpha_0}$.

Proof 2. $(5) \Rightarrow (4)$. Let $A_{\alpha_1} < \infty$ and $\int_0^\infty v(x)f(x)^p dx < \infty$. From the assumption $0 \leq f \in Q_{\alpha_1}$ it follows that $t^{-p\alpha_1} f(t)^p \leq x^{-p\alpha_1} f(x)^p$ for $0 < x \leq t$. Multiplying this inequality by $x^{p\alpha_1} v(x)$ and integrate in x from 0 to t we find that

$$t^{-p\alpha_1} f(t)^p \int_0^t x^{p\alpha_1} v(x)dx \leq \int_0^t f(x)^p v(x)dx \,.$$

Using this we therefore obtain

$$\frac{d}{dt}\left[\left(\int_0^t f(x)^p v(x)dx \right)^{q/p} \right] - A_{\alpha_1}^{-1} f(t)^q u(t)$$

$$= \frac{q}{p}\left(\int_0^t f(x)^p v(x)dx \right)^{q/p-1} f(t)^p v(t) - A_{\alpha_1}^{-q} f(t)^q u(t)$$

$$\geq \frac{q}{p}\left[t^{-p\alpha_1} f(t)^p \int_0^t x^{p\alpha_1} v(x)dx \right]^{q/p-1} f(t)^p v(t) - A_{\alpha_1}^{-q} f(t)^q u(t)$$

$$= t^{-q\alpha_1} f(t)^q \frac{d}{dt}\left[\left(\int_0^t x^{p\alpha_1} v(x)dx \right)^{q/p} - A_{\alpha_1}^{-q} \int_0^t x^{q\alpha_1} u(x)dx \right]$$

$$= t^{-q\alpha_1} f(t)^q \frac{d}{dt} k(t) \,,$$

where $k(t) = \left(\int_0^t x^{p\alpha_1} v(x)dx \right)^{q/p} - A_{\alpha_1}^{-q} \int_0^t x^{q\alpha_1} u(x)dx$.

Integrating from 0 to ∞ we see that

$$\left(\int_0^\infty f(x)^p v(x)dx \right)^{q/p} - A_{\alpha_1}^{-q} \int_0^\infty f(t)^q u(t)dt \geq \int_0^\infty t^{-q\alpha_1} f(t)^q k'(t)dt \,.$$

Moreover, by integration by parts, the right hand side can be written

$$= \left[t^{-q\alpha_1} f(t)^q k(t)|_0^\infty \right] - \int_0^\infty k(t) d\left[t^{-q\alpha_1} f(t)^q \right] \geq 0$$

since $k(t) \geq 0$ for $t > 0$, $f \in Q_{\alpha_1}$ and $\lim_{t \to 0^+} t^{-q\alpha_1} f(t)^q k(t) \leq 0$. Thus $A \leq A_{\alpha_1}$. Similarly, we can prove that $B \leq B_{\alpha_0}$, by using almost the same arguments:

1^0. $0 \leq f \in Q^{\alpha_0} \Rightarrow t^{-p\alpha_0} f(t)^p \int_t^\infty x^{p\alpha_0} v(x) dx \leq \int_t^\infty f(x)^p v(x) dx$,

2^0. $\frac{d}{dt}\left[-\left(\int_t^\infty f(x)^p v(x) dx \right)^{q/p} \right] - B_{\alpha_0}^{-q} f(x)^q u(t) \geq t^{-q\alpha_0} f(t)^q \frac{d}{dt} h(t)$,

where $h(t) = -\left(\int_t^\infty x^{p\alpha_0} v(x) dx \right)^{q/p} + B_{\alpha_0}^{-q} \int_t^\infty x^{q\alpha_0} u(x) dx$.

3^0. Integrate from 0 to ∞ and use the integration by parts to obtain

$$\left(\int_0^\infty f(x)^p v(x) dx \right)^{q/p} - B_{\alpha_0}^{-q} \int_0^\infty f(t)^q u(t) dt \geq 0. \qquad \square$$

Remark 2. Theorem 1 can be also written in the following from: If $0 < p \leq q < \infty$, then

$$\sup_{0 \leq f \in Q_{\alpha_1}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} = \sup_{t>0} \left(\int_0^t x^{q\alpha_1} u(x) dx \right)^{1/q} \left(\int_0^t x^{p\alpha_1} v dx \right)^{-1/p}$$

and

$$\sup_{0 \leq f \in Q^{\alpha_0}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} = \sup_{t>0} \left(\int_t^\infty x^{q\alpha_0} u(x) dx \right)^{1/q} \left(\int_t^\infty x^{p\alpha_0} v(x) dx \right)^{-1/p}.$$

EXAMPLE 1: (Embedding of Lorentz spaces).

Let $0 < p \leq q < \infty$ and $u(x) = x^{\alpha q - 1}$, $v(x) = x^{\alpha p - 1}$ with $\alpha > -\alpha_1$. Then $A_{\alpha_1} = p^{1/p} q^{-1/q} (\alpha + \alpha_1)^{1/p - 1/q}$ and, by Theorem 1(a), we conclude that

$$\left(\int_0^\infty \left(x^\alpha f(x) \right)^q \frac{dx}{x} \right)^{1/q} \leq p^{1/p} q^{-1/q} (\alpha + \alpha_1)^{1/p - 1/q} \left(\int_0^\infty \left(x^\alpha f(x) \right)^p \frac{dx}{x} \right)^{1/p},$$

for any $0 \leq f \in Q_{\alpha_1}$. This result, for $\alpha_1 = 0$, is known in the theory of Lorentz spaces (cf. [13, Theorem 3.11]) as the embedding $L(1/\alpha, p) \subset L(1/\alpha, q)$ of Lorentz spaces with the norm equal to 1, where the Lorentz space $L(1/\alpha, p)$ is the space generated by the functional $\|f\|_{L(1/\alpha,p)} = \left[p\alpha \int_0^\infty (x^\alpha f^*(x))^p dx \right]^{1/p}$ and f^* denotes the rearrangement of f. The usual proofs here are using simple functions and the Fatou lemma.

We now prove the result where both of the monotonicity conditions Q_{α_1} and Q^{α_0} on a function are assumed. Such class of functions was early studied, e.g. by Chen [4].

Theorem 2

Let $0 < p \leq q < \infty$. Assume that $0 \leq f \in Q^{\alpha_0} \cap Q_{\alpha_1}$ with $-\infty < \alpha_0 < \alpha_1 < \infty$ and weights u, v satisfying the conditions (5) and (7). Then

(8) $$\left(\int_0^\infty f(x)^q u(x) dx \right)^{1/q} \leq C_{p,q} \left(\int_0^\infty f(x)^p v(x) dx \right)^{1/p},$$

where $C_{p,q} = (A^{1/(1/q-1/p)} + B^{1/(1/q-1/p)})^{1/q-1/p}$ for $p \neq q$ and $C_{p,q} = \max(A, B)$.

Proof. For any $M > 0$ we choose $\xi \in (0, \infty)$ such that

$$\int_\xi^\infty f(x)^q u(x) dx = M \int_0^\xi f(x)^q u(x) dx \,.$$

Then

$$\left(\int_0^\infty f(x)^q u(x) dx\right)^{p/q} = \left(\int_0^\xi f(x)^q u(x) dx + \int_\xi^\infty f(x)^q u(x) dx\right)^{p/q}$$

$$= (1 + M)^{p/q} \left(\int_0^\xi f(x)^q u(x) dx\right)^{p/q}$$

$$= \frac{(1 + M)^{p/q}}{1 + \lambda} \left[\left(\int_0^\xi f(x)^q u(x) dx\right)^{p/q}\right.$$

$$\left. + \frac{\lambda}{M^{p/q}} \left(\int_\xi^\infty f(x)^q u(x) dx\right)^{p/q}\right].$$

Therefore, by using Theorem 1(a) and (b) and choosing λ such that $A^p = \frac{\lambda}{M^{p/q}} B^p$, we find that

$$\left(\int_0^\infty f(x)^q u(x) dx\right)^{p/q}$$

$$\leq \frac{(1 + M)^{p/q}}{1 + \lambda} \left[A^p \int_0^\xi f(x)^p v(x) dx + \frac{\lambda}{M^{p/q}} B^p \int_\xi^\infty f(x)^p v(x) dx\right]$$

$$= \frac{(1 + M)^{p/q}}{1 + \lambda} A^p \int_0^\infty f(x)^p v(x) dx = \frac{(1 + M)^{p/q}}{B^p + A^p M^{p/q}} A^p B^p \int_0^\infty f(x)^p v(x) dx \,.$$

The infimum over $M > 0$ is attained at $M = M_0 = (A/B)^{pq/(q-p)}$ and it is equal to $[AB(A^{pq/(q-p)} + A^{pq/(q-p)})^{1/q - 1/p}]^p = C_{p,q}^p$ for $p \neq q$. In the case when $p = q$ the infimum is equal to $\max(A^p, B^p)$. $\square$

EXAMPLE 2: (Embedding of interpolation spaces).

Let $0 < p \leq q < \infty$ and $u(x) = x^{-\alpha q - 1}$, $v(x) = x^{-\alpha p - 1}$ with $\alpha_0 < \alpha < \alpha_1$. Then conditions (5) and (7) holds, respectively with $A = p^{1/p} q^{-1/q} (\alpha_1 - \alpha)^{1/p - 1/q}$ and $B = p^{1/p} q^{-1/q} (\alpha - \alpha_0)^{1/p - 1/q}$. By Theorem 2 we obtain that inequality

$$\left(\int_0^\infty \left(x^{-\alpha} f(x)\right)^q \frac{dx}{x}\right)^{1/q}$$

$$\leq p^{1/p} q^{-1/q} \left(\frac{(\alpha - \alpha_0)(\alpha_1 - \alpha)}{\alpha_1 - \alpha_0}\right)^{1/p - 1/q} \left(\int_0^\infty \left(x^{-\alpha} f(x)\right)^p \frac{dx}{x}\right)^{1/p},$$

holds for any $0 \le f \in Q^{\alpha_0} \cap Q_{\alpha_1}$, i.e., for any f satisfying

$$0 \le f(x) \le \max\left((s/t)^{\alpha_0}, (s/t)^{\alpha_1}\right) f(t) \quad \forall s, t > 0.$$

This result in the case when $\alpha_0 = 0$ and $\alpha_1 = 1$ was stated in [3], as the exact constant for the embedding of real interpolation spaces $(A_0, A_1)_{\alpha,p} \subset (A_0, A_1)_{\alpha,q}$, and was proved by Bergh [1] who used a symmetrization argument. Another proof of this result was given by Bergh-Burenkov-Persson [2]. Our idea of the proof is in fact taken from this paper. Recently, with the same technique, Pečarić-Persson [11] proved the result for functions satisfying $0 \le f(s) \le C \max\left((s/t)^{\alpha_0}, (s/t)^{\alpha_1}\right) f(t)$ for all $s, t > 0$, where $C \ge 1$.

Theorem 2 motivate us to give necessary and sufficient conditions on the weights u and v that inequality $(*)$ holds for functions f from the classes $Q^{\alpha_0} \cap Q_{\alpha_1}$. We will prove this but also with a good control of the constants (see also [9], Theorem 1).

Theorem 3

Let $0 < p \le q < \infty$ and $-\infty < \alpha_0 < \alpha_1 < \infty$. The inequality

$$(9) \qquad \left(\int_0^\infty f(x)^q u(x) dx\right)^{1/q} \le C \left(\int_0^\infty f(x)^p v(x) dx\right)^{1/p}$$

holds for all $0 \le f \in Q^{\alpha_0} \cap Q_{\alpha_1}$ if and only if (9) is true for the functions $f_t(x) = \min\left(x^{\alpha_1}, t^{\alpha_1 - \alpha_0} x^{\alpha_0}\right)$ with arbitrary $t > 0$, i.e.,

$$D = \sup_{t > 0} \left\{ \left(\int_0^\infty \left[\min(x^{\alpha_1}, t^{\alpha_1 - \alpha_0} x^{\alpha_0})\right]^q u(x) dx\right)^{1/q} \right.$$

$$(10) \qquad\qquad \left. \times \left(\int_0^\infty \left[\min(x^{\alpha_1}, t^{\alpha_1 - \alpha_0} x^{\alpha_0})\right]^p v(x) dx\right)^{-1/p} \right\} < \infty.$$

Moreover $D \le C \le 2D$.

Proof. (We repeat here the proof from [9] for the convenience of the reader).

$(9) \Rightarrow (10)$. This implication is obvious with $D \le C$. Moreover, if $D < \infty$ and $\int_0^\infty x^{p\alpha_1} v(x) dx < \infty$, then

$$t^{p(\alpha_1 - \alpha_0)} \int_t^\infty x^{p\alpha_0} v(x) dx \le \int_t^\infty x^{p\alpha_1} v(x) dx \to 0 \quad \text{as} \quad t \to \infty$$

and thus, according to (10), $\int_0^\infty x^{q\alpha_1} u(x)dx < \infty$. This gives that again

$$t^{q(\alpha_1-\alpha_0)} \int_t^\infty x^{q\alpha_0} u(x)dx \le \int_t^\infty x^{q\alpha_1} v(x)dx \to 0 \quad \text{as} \quad t \to \infty \,.$$

Therefore

$$\left(\int_0^\infty x^{q\alpha_1} u(x)dx\right)^{1/q} \left(\int_0^\infty x^{p\alpha_1} v(x)dx\right)^{-1/p} \le D \,.$$

$(10) \Rightarrow (9)$. If $0 \le f \in Q^{\alpha_0} \cap Q_{\alpha_1}$, then the function h defined by

$$h(x) = x^{-\alpha_0/(\alpha_1-\alpha_0)} f\left(x^{1/(\alpha_1-\alpha_0)}\right)$$

belongs to $Q^0 \cap Q_1$ and its smallest concave majorant $\widehat{h}$, defined by

$$\widehat{h}(x) = \inf_{t>0}\left(1 + \frac{x}{t}\right)h(t), \quad x > 0 \,,$$

satisfy the inequalities $h(x) \le \widehat{h}(x) \le 2h(x)$ for all $x > 0$ (cf., e.g. [3, Lemma 5.4.3] or [8, Lemma 14.1]), and $\widehat{h}$ as a concave function has the representation

$$\widehat{h}(x) = \alpha + \int_0^x m(s)ds, \quad 0 \le m \downarrow \,,$$

where $\alpha = \lim_{x\to 0^+} k(x)$ and $m(s) = \widehat{h}'(s)$, with $\widehat{h}'(s)$ being the right-derivative which is positive right-continuous decreasing function.

The function $g(x) = x^{\alpha_0}\widehat{h}(x^{\alpha_1-\alpha_0})$ belongs to $Q^{\alpha_0} \cap Q_{\alpha_1}$ and has representation

$$(11) \qquad g(x) = c_0 x^{\alpha_0} + x^{\alpha_0} \int_0^{x^{\alpha_1-\alpha_0}} m(s)dx, \quad 0 \le m \downarrow \,,$$

with $c_0 = \lim_{x\to 0^+} x^{-\alpha_0} g(x)$. Since $h \le \widehat{h} \le 2h$ it follows that $f \le g \le 2f$ and it is enough to prove our result for the function g with the representation (11). Thus, let $\int_0^\infty g(x)^p v(x)dx < \infty$ and $D < \infty$, i.e., for every $t > 0$

$$\left\{\int_0^\infty \left[\min(x^{\alpha_1}, t^{\alpha_1-\alpha_0} x^{\alpha_0})\right]^q u(x)dx\right\}^{p/q} \le D^p \int_0^\infty \left[\min(x^{\alpha_1}, t^{\alpha_1-\alpha_0} x^{\alpha_0})\right]^p v(x)dx \,.$$

Putting $t = \varphi(y)$, where φ is a positive decreasing which will be chosen later on, and integrate from 0 to ∞ with respect to y to obtain

$$\int_0^\infty \left\{\int_0^\infty \left[\min(x^{\alpha_1}, \varphi(y)^{\alpha_1-\alpha_0} x^{\alpha_0})\right]^q u(x)dx\right\}^{p/q} dy$$

$$\le D^p \int_0^\infty \left\{\int_0^\infty \left[\min(x^{\alpha_1}, \varphi(y)^{\alpha_1-\alpha_0} x^{\alpha_0})\right]^p v(x)dx\right\} dy \,.$$

By using the Minkowski inequality and the equality (2) in Lemma 1 we find

$$\int_0^\infty \Big[\int_0^\infty \big(\min(x^{\alpha_1}, \varphi(y)^{\alpha_1-\alpha_0}x^{\alpha_0})\big)^q u(x)dx\Big]^{p/q} dy$$

$$\geq \Big\{\int_0^\infty \Big[\int_0^\infty \big(\min(x^{\alpha_1}, \varphi(y)^{\alpha_1-\alpha_0}x^{\alpha_0})\big)^p dy\Big]^{q/p} u(x)dx\Big\}^{p/q}$$

$$= \Big\{\int_0^\infty \Big[\int_0^{\varphi^{-1}(x)} x^{p\alpha_1}dy + \int_{\varphi^{-1}(x)}^\infty \varphi(y)^{p(\alpha_1-\alpha_0)}x^{p\alpha_0}dy\Big]^{q/p} u(x)dx\Big\}^{p/q}$$

$$= \Big\{\int_0^\infty \Big[\varphi^{-1}(x)x^{p\alpha_1} + x^{p\alpha_0}\int_{\varphi^{-1}(x)}^\infty \varphi(y)^{p(\alpha_1-\alpha_0)}dy\Big]^{q/p} u(x)dx\Big\}^{p/q}$$

$$= \Big\{\int_0^\infty \Big[x^{p\alpha_0}\int_0^x \varphi^{-1}(s)d\big(s^{p(\alpha_1-\alpha_0)}\big)\Big]^{q/p} u(x)dx\Big\}^{p/q}.$$

Similarly, by the Fubini theorem and the equality (2) in Lemma 1

$$\int_0^\infty \Big\{\int_0^\infty \big[\min(x^{\alpha_1}, \varphi(y)^{\alpha_1-\alpha_0}x^{\alpha_0})\big]^p v(x)dx\Big\} dy$$

$$= \int_0^\infty \Big[\int_0^\infty \big(\min(x^{\alpha_1}, \varphi(y)^{\alpha_1-\alpha_0}x^{\alpha_0})\big)^p dy\Big] v(x)dx$$

$$= \int_0^\infty \Big[\varphi^{-1}(x)x^{p\alpha_1} + x^{p\alpha_0}\int_{\varphi^{-1}(x)}^\infty \varphi(y)^{p(\alpha_1-\alpha_0)}dy\Big] v(x)dx$$

$$= \int_0^\infty \Big[x^{p\alpha_0}\int_0^x \varphi^{-1}(s)d\big(s^{p(\alpha_1-\alpha_0)}\big)\Big] v(x)dx.$$

From the above inequalities and equalities we obtain the crucial inequality

$$\Big\{\int_0^\infty \Big[x^{p\alpha_0}\int_0^x \varphi^{-1}(x)d\big(s^{p(\alpha_1-\alpha_0)}\big)\Big]^{q/p} u(x)dx\Big\}^{p/q}$$

(12)
$$\leq D^p \int_0^\infty \Big[x^{p\alpha_0}\int_0^x \varphi^{-1}(s)d\big(s^{p(\alpha_1-\alpha_0)}\big)\Big] v(x)dx.$$

Now, if g has the representation (11), then by taking as a decreasing function

$$\varphi^{-1}(x) = m\big(x^{\alpha_1-\alpha_0}\big)\big[x^{-\alpha_1}g(x)\big]^{p-1} = \frac{x^{1-p(\alpha_1-\alpha_0)}}{p(\alpha_1-\alpha_0)}\frac{d}{dx}\big[x^{-p\alpha_0}g(x)^p\big],$$

we obtain

$$\int_0^x \varphi^{-1}(s)d\big(s^{p(\alpha_1-\alpha_0)}\big) = x^{-p\alpha_0}g(x)^p - \lim_{s\to 0^+} s^{-p\alpha_0}g(s)^p = x^{-p\alpha_0}g(x)^p,$$

which we can put to (12) and get

$$\Big(\int_0^\infty g(x)^q u(x)dx\Big)^{p/q} \leq D^p \int_0^\infty g(x)^p v(x)dx.$$

Since $f \leq g \leq 2f$ it follows that (9) holds and $C \leq 2D$. $\square$

Remark 3. The above proof shows that the inequality (9) holds for every f having the representation (12) if and only if (10) holds and in this case $C = D$.

Remark 4. If we take u and v as in Example 2, then

$$D = p^{1/p} q^{-1/q} \left(\frac{(\alpha - \alpha_0)(\alpha_1 - \alpha)}{\alpha_1 - \alpha_0} \right)^{1/p - 1/q}.$$

Special choices of α_0, α_1 and m can give some known results for decreasing functions, which were proved before but not always with the best constant and not in the full range $0 < p \le q < \infty$ of parameters p and q.

Corollary 1

If $0 < p \le q < \infty$, then

$$\sup_{0 \le f \downarrow} \frac{\left\| \int_0^x f \right\|_{q,u}}{\left\| \int_0^x f \right\|_{p,v}} = \sup_{t > 0} \frac{\| \min(x, t) \|_{q,u}}{\| \min(x, t) \|_{p,v}}.$$

Take, in the above proof, $\alpha_0 = 0$, $\alpha_1 = 1$ and $m = f$ with f being positive decreasing function. This result with the best constant but with some additional restrictions on p and q was proved in [6, Theorem 3.7]. Similarly, we can get the result from [6, Theorem 4.3 (a)].

Corollary 2

If $0 < p \le q < \infty$, then

$$\sup_{0 \le f \, \text{concave}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} = \sup_{t > 0} \frac{\| \min(x, t) \|_{q,u}}{\| \min(x, t) \|_{p,v}}.$$

Corollary 3

If $0 < p \le q < \infty$, then

$$\sup_{0 \le f \downarrow} \frac{\left\| \frac{1}{x} \int_0^x f \right\|_{q,u}}{\left\| \frac{1}{x} \int_0^x f \right\|_{p,v}} = \sup_{t > 0} \frac{\| \min(1, t/x) \|_{q,u}}{\| \min(1, t/x) \|_{p,v}}.$$

Take, in the proof of Theorem 3, $\alpha_0 = -1$, $\alpha_1 = 0$ and $m = f$ with f being positive decreasing function. This result with the restriction $q \ge 1$ and not with

the best constant (only equivalence of the above expressions) was quite differently proved by Stepanov [15, Theorem 3.3].

Remark 5. Gol'dmann-Heinig-Stepanov [5] established, using the discretization method, the equivalence of inequality (9) for $f \in Q^0 \cap Q_1$ with inequality (10) in the case $0 < p \leq q < \infty$, but also the equivalence in the case $0 < q < p < \infty$ with some complicated corresponding expression to (10). Our proof here (only the case $0 < p \leq q < \infty$) is different and with a good control of the constants.

Acknowledgment. I would like to thank Prof. L.E. Persson for comments and suggestions, and to Prof. H. Heinig for sending to me in May 1995 a copy of the paper [5].

References

1. J. Bergh, A converse inequality of Hölder type, *Math. Z.* **215** (1994), 205–208.
2. J. Bergh, V. Burenkov and L.E. Persson, Best constants in reversed Hardy's inequalities for quasimonotone functions, *Acta Sci. Math. (Szeged)* **59** (1994), 223–241.
3. J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Berlag, New York, 1976.
4. Y.M. Chen, Theorems of asymptotic approximation, *Math. Ann.* **140** (1960), 360–407.
5. M.L. Gol'dman, H. Heinig and V.D. Stepanov, On the principle of duality in Lorentz spaces, preprint.
6. H. Heinig and L. Maligranda, Weighted inequalities for monotone and concave functions, *Studia Math.* **116** (1995), 133–165.
7. H. Heinig and V.D. Stepanov, Weighted Hardy inequalities for increasing functions, *Canad. J. Math.* **45** (1993), 104–116.
8. L. Maligranda, *Orlicz Spaces and Interpolation*, Univ. of Campinas, Campinas, 1989.
9. L. Maligranda, Weighted inequalities for quasi-monotone functions, *J. London Math. Soc.,* to appear.
10. C.J. Neugebauer, Weighted norm inequalities for averaging operators of monotone functions, *Publ. Mat.* **35** (1991), 429–447.
11. J. Pečarić and L.E. Persson, On Bergh's inequality for quasi-monotone functions, *J. Math. Anal. Appl.* **195** (1995), 393–400.
12. E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.* **96** (1990), 145–158.
13. E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, New Jersey, 1971.
14. V.D. Stepanov, Weighted Hardy's inequality for nonincreasing functions, *Trans. Amer. Math. Soc.* **338** (1993), 173–186.

15. V.D. Stepanov, Integral operators on the cone of monotone functions, *J. London Math. Soc.* **48** (1994), 465–487.
16. A.C. Zaanen, *Linear Analysis*, Amsterdam, 1953.