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# Isometric stability property of Banach spaces

Pei-Kee Lin

*Department of Mathematics, University of Memphis, Memphis, TN 38152*

*E-mail address:* linpk@mathsci.msci.memst.edu

Abstract

Let $X$ be a separable $L_1$ or a separable $C(K)$-space, and let $Y$ be any Banach space. $I(X,Y)$ denotes the set of all isometries from $X$ to $Y$. showed that for any finite measure space $(\Omega,\mu)$ and any $1 < p < \infty$, every isometry $T : X \to L_p(\Omega, Y)$ has the form

$$T\,x(t) = h(t)\,U(t)x\,,$$

where $h \in L_p$ with $\|h\|_p = 1$ and $U$ is a strongly measurable function from $\Omega$ into $I(X,Y)$. In this article, we extend this result to the Köthe-Bochner function spaces $E(Y)$ when $E$ is strictly convex. We also show that every isometry from $\ell_\infty^n$ into $E(Y)$ has the above form if $n \geq 3$ and $E$ is a strictly monotone Köthe function space.

Let $X$ be a Banach space and let $E$ be a Köthe function space on a finite measure space $(\Omega,\mu)$. The Köthe-Bochner function space $E(X)$ is the set of all measurable functions $f : \Omega \to X$ such that $\|f(\cdot)\|_X \in E$. The norm of $f$ is defined by

$$\|f\| = \big\|\|f(\cdot)\|_X\big\|_E.$$

For any two Banach spaces $X, Y$, let $I(X,Y)$ denote the set of all isometries from $X$ into $Y$. A mapping $U : \Omega \to I(X,Y)$ is called *strongly measurable* if for each $x$, the function $U(\cdot)x$ is measurable. It is easy to see that if $U$ is a strongly measurable mapping from $\Omega$ into $I(X,Y)$ and if $h \in E$ with $\|h\|_E = 1$, then the mapping $T : X \to E(Y)$ defined by

$$T\,x(t) = h(t) \cdot U(t)x \tag{1}$$

is an isometry. In [2], Koldobsky showed that if $X$ is either a separable $L_1$-space or a separable $C(K)$ space, then every isometry $T$ from $X$ into $L_p(Y)$, $1 < p < \infty$, has the form (1). Recall a Banach space is said to be *strictly convex* if $\|x\| = 1 = \|y\| = \frac{1}{2}\|x+y\|$ implies $x = y$. A Köthe function space is said to be *strictly monotone* if $x \geq y \geq 0$ and $\|x\| = \|y\|$ imply $x = y$. In this article, we prove the following two Theorems.

**Theorem 1**

*Let $X$ be a real (respectively, complex) Banach space such that there are two subsets $A$ and $B$ of $X$ which satisfy the following conditions.*

(i) *$A$ is a subset of the unit sphere of $X$ and for any $a_1, a_2 \in A$ there are a unit vectors $x$ and two scalars $\alpha_1, \alpha_2$ with $|\alpha_1| = 1 = |\alpha_2|$ such that*

$$\big\|a_1 + \alpha_1 x\big\| = 2 = \big\|a_2 + \alpha_2 x\big\| .$$

(ii) *$B$ is countable dense subset of $X$.*

(iii) *For any $\alpha \in \mathbb{Q}$ (respectively, $\alpha \in \mathbb{Q} + i\mathbb{Q}$) and any $a_1, a_2 \in B$, $a_1 + \alpha a_2 \in B$.*

(iv) *For any $b \in B$ there are an $a \in A$, a unit vector $x$, and a real number $\alpha$, $0 \leq \alpha \leq 1$ such that*

$$\|a + x\| = 2$$
$$b = \|b\|_X \cdot \big(\alpha a + (1-\alpha)x\big) .$$

*If $E$ is a strictly convex Köthe function space, then every isometry $T : X \to E(Y)$ has the form*

$$T\,x(t) = h(t) \cdot \big(U(t)\big)(x) \tag{2}$$

*where $h \in E$ with $\|h\|_E = 1$ and $U$ is a strongly measurable function from $\Omega$ into $I(X,Y)$.*

**Theorem 2**

*Let $X$ be a real (respectively, complex) Banach space. Suppose there are subsets $A$ and $B$ of $X$ which satisfy the following conditions.*

(v) *$A$ is a subset of the unit sphere of $X$ and for any $a_1, a_2 \in A$ there is $x$ in the unit sphere of $X$ such that*

$$\big\|a_1 + x\big\| = \big\|a_1 - x\big\| = 1 = \big\|a_2 + x\big\| = \big\|a_2 - x\big\|.$$

(vi) *$B$ satisfies the conditions (ii) and (iii) of Theorem 1.*

(vii) *For $b \in B$, there are $e_1 \in A$ and two unit vectors $e_2, e_3$ such that $\{e_1, e_2, e_3\}$ is an $\ell^3_\infty$ basis and $b \in span\,\{e_1, e_2, e_3\}$.*

*If $E$ is a strictly monotone Köthe function space, then every isometry $T$ from $X$ into $E(Y)$ has form (2).*

First we need the following two lemmas. We only give a proof of the second lemma and we leave the proof of the first lemma to the readers.

**Lemma 3**

*Let $Y$ be a Banach space and $E$ be a strictly convex Köthe function space. If $f, g$ are two unit vectors in $E(Y)$ such that $\|f+g\|_{E(Y)} = 2$, then for any $0 \le \alpha \le 1$,*

$$\|f(\cdot)\|_Y = \alpha\|f(\cdot)\|_Y + (1-\alpha)\|g(\cdot)\|_Y \,.$$

*Particularly, we have $\|f(\cdot)\|_Y = \|g(\cdot)\|_Y$.*

**Lemma 4**

*Let $Y$ be a Banach space and $E$ be a strictly monotone Köthe function space. If $f, g$ are two nonzero elements in $E(X)$ and if $\|f+g\|_{E(Y)} = \|f\|_{E(Y)} + \|g\|_{E(Y)}$, then for any $0 \le \alpha \le 1$,*

$$\|(\alpha f + (1-\alpha)g)(\cdot)\|_Y = \|\alpha f(\cdot)\|_Y + \|(1-\alpha)g(\cdot)\|_Y \,.$$

*Proof.* Exchange $f$ and $g$ if necessary. We may assume that $\alpha \le \frac{1}{2}$. So

$$\begin{aligned}\|\alpha f + (1-\alpha)g\|_{E(Y)} &\ge (1-\alpha)\|f+g\|_{E(Y)} - (1-2\alpha)\|f\|_{E(Y)} \\ &= (1-\alpha)\|g\|_{E(Y)} + \alpha\|f\|_{E(Y)} \,.\end{aligned}$$

Note: $0 \le \alpha \le 1$, $\|(\alpha f + (1-\alpha)g)(\cdot)\|_Y \le (1-\alpha)\|g(\cdot)\|_Y + \alpha\|f(\cdot)\|_Y$. But $E$ is strictly monotone. We have

$$\big\|(\alpha f + (1-\alpha)g)(\cdot)\big\|_Y = \big\|\alpha f(\cdot)\big\|_Y + \big\|(1-\alpha)g(\cdot)\big\|_Y)$$

for $0 \le \alpha \le 1$. $\square$

*Proof of Theorem 1.* Let $a$ be any vector in $A$ and let $h(\cdot) = \|(T(a))(\cdot)\|_Y$. We claim that for any non-zero vector $b \in B$, $h(\cdot) = \frac{\|(T(b))(\cdot)\|_Y}{\|b\|_X}$. Note: $T$ is an isometry. Suppose that claim were proved. By (ii), $B$ is a countable set and there exists a measurable set $D \subseteq \Omega$ such that $\mu(\Omega \setminus D) = 0$ and for every $t \in D$ and every $b_1, b_2 \in B$ and $\alpha \in \mathbb{Q}$ (respectively, $\alpha \in \mathbb{Q} + i\mathbb{Q}$),

$$\begin{gathered}\big\|T(b_1)(t)\big\|_Y = h(t) \cdot \big\|b_1\big\|_X \\ \big(T(b_1 + \alpha b_2)\big)(t) = T(b_1)(t) + T(\alpha b_2)(t) \,.\end{gathered}$$

Let $t$ be any element in $D$ such that $h(t) \neq 0$. Define a mapping $U(t) : B \to Y$ by

$$U(t)(b) = T(b)(t)/h(t).$$

Since $B$ is dense in $X$, $U(t)$ can be uniquely extended to an isometry on $X$ and we still denote it by $U(t)$. Clearly, $U(t)$ is linear. So the set $I(X, Y)$ is non-empty. Let $S$ be any element in $I(X, Y)$. We define $U(t) = S$ if $h(t) = 0$. Now we only need to show that for any $x \in X$,

$$(T\,x)(\cdot) = h(\cdot)\,U(\cdot)(x) \quad a.e.$$

For any $x \in X$, there is a sequence $\{b_n\} \subseteq B$ such that $\lim_{n\to\infty} b_n = x$. Then

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|b_k - x\|_X \\ &= \lim_{k\to\infty} \|T(b_k) - T(x)\|_{E(Y)} \\ &\geq \lim_{k\to\infty} \max\{\|T(x)(\cdot)\cdot 1_{\Omega\setminus \mathrm{supp\,(h)}}(\cdot)\|_{E(Y)}, \|T(x)(\cdot)\cdot 1_{\mathrm{supp\,(h)}}(\cdot) - h(\cdot)\,U(\cdot)\,b_k\|_{E(Y)}\} \\ &= \max\{\|T(x)(\cdot)\cdot 1_{\Omega\setminus \mathrm{supp}\ (h)}(\cdot)\|_{E(Y)}, \|T(x)(\cdot)\cdot 1_{\mathrm{supp\,(h)}}(\cdot) - h(\cdot)\,U(\cdot)\,x\|_{E(Y)}\}\,. \end{aligned}$$

This implies $T\,x(\cdot) = h(\cdot)\,U(\cdot)(x)$.

We claim that for any $a, a' \in A$, $\|T(a)(\cdot)\|_Y = \|T(a')(\cdot)\|_Y$. By (i), there are a unit vector $x$ and two numbers $\alpha_1, \alpha_2$ with $|\alpha_1| = 1 = |\alpha_2|$ such that

$$\left\|a + \alpha_1 x\right\|_X = 2 = \left\|a' + \alpha_2 x\right\|_X .$$

Since $T$ is an isometry and $E$ is strictly convex, by Lemma 3, we have

$$\left\|T(a)(\cdot)\right\|_Y = \left\|\alpha_1 T(x)(\cdot)\right\|_Y = \|T(x)(\cdot)\|_Y = h(\cdot) = \left\|T(a')(\cdot)\right\|_Y .$$

We proved our claim. By (iii), for any non-zero $b \in B$, there are $a' \in A$ and a unit vector $x \in X$ with $\|x + a'\| = 2$ such that

$$\frac{b}{\|b\|_X} = \alpha a' + (1 - \alpha)x\,.$$

By Lemma 3 again, we have $\frac{\left\|T(b)(\cdot)\right\|_Y}{\|b\|_X} = h(\cdot)$. The proof is complete. □

*Proof of Theorem 2.* Let $a$ be any vector in $A$, and let

$$h(\cdot) = \left\|T(a)(\cdot)\right\|_Y .$$

As the proof of Theorem 1, we only need to show for any non-zero vector $b \in B$,

$$\left\|T(b_1)(\cdot)\right\|_Y = \left\|b_1\right\|_X \cdot h(\cdot)\,.$$

For any other vector $a' \in A$, there is a unit vector $x$ such that

$$\left\|a + x\right\|_X = \left\|a - x\right\|_X = 1 = \left\|a' + x\right\|_X = \left\|a' - x\right\|_X\,.$$

So

$$\begin{aligned}2 = \|2a\|_X &= \|(a+x) + (a-x)\|_X \\ &= 2\|x\|_X = \|(a+x) + (x-a)\|_X \\ &= \|(a'+x) - (a'-x)\|_X \\ &= 2\|a'\|_X = \|(a'+x) + (a'-x)\|_X\,.\end{aligned}$$

Since $T$ is an isometry and $E$ is strictly monotone, by Lemma 4, we have

$$\begin{aligned}2\|T(a)(\cdot)\|_Y &= \|T(a+x)(\cdot)\|_Y + \|T(a-x)(\cdot)\|_Y \\ &= 2\|T(x)(\cdot)\|_Y \\ &= \|T(a'+x)(\cdot)\|_Y + \|T(a'-x)(\cdot)\|_Y \\ &= \|T(a')(\cdot)\|_Y\,.\end{aligned}$$

For any $b \in B$, there are three unit vectors $\{e_1, e_2, e_3\}$ of $X$ such that $e_1 \in A$, $b \in \operatorname{span}\{e_1, e_2, e_3\}$ and for any $\alpha_j$, $1 \le j \le 3$

$$\left\|\sum_{j=1}^{3} \alpha_j e_j\right\| = \max\left\{|\alpha_j| : 1 \le j \le 3\right\}.$$

Without loss of generality, we may assume that there are $\beta_2, \beta_3$ such that $|\beta_2| \le 1$, $|\beta_3| \le 1$ and

$$\frac{b}{\|b\|_X} = e_1 + \beta_2 e_2 + \beta_3 e_3\,.$$

The above proof shows that for any $1 \le j < k \le 3$,

$$\left\|T(\alpha_j e_j + \alpha_k e_k)(\cdot)\right\|_Y = \max\left\{|\alpha_j|, |\alpha_k|\right\} \cdot h(\cdot)\,.$$

Hence if $\max\{|\alpha_2|, |\alpha_3|\} \le 1$, then

$$\left\|T(e_1 + \alpha_2 e_2 + \alpha_3 e_3)(\cdot)\right\|_Y + \left\|T(e_1 - \alpha_2 e_2)(\cdot)\right\|_Y = \left\|T(2e_1 + \alpha_3 e_3)(\cdot)\right\|_Y = 2h(\cdot)\,.$$

Note: $\|T(e_1 - \alpha_2 e_2)(\cdot)\|Y = h(\cdot)$. We have

$$\left\|T(e_1 + \alpha_2 e_2 + \alpha_3 e_3)(\cdot)\right\|_Y = h(\cdot)\,.$$

So we proved that for any $b \in B$, $\|T(b)(\cdot)\|_Y = h(t) \cdot \|b\|_X$. The proof is complete. □

EXAMPLE 1: Let $X = L_1[0,1]$ and

$$A = \left\{\alpha n 1_{\left(0,\frac{1}{n}\right)} : n \geq 2 \text{ and } |\alpha| = 1\right\}.$$

Let $B$ be a countable dense subset of the set

$$\left\{f \in X : f \text{ is constant on } \left(0, \frac{1}{n}\right) \text{ for some } n \in \mathbb{N}\right\}$$

such that $B$ satisfies (ii) and (iii) of Theorem 1. For any $a \in A$, $\|a + 21_{(\frac{1}{2},1)}\| = 2$. So $A$ satisfies (i) of Theorem 1. Let $b$ be any element of $B$. Then there is $n \geq 2$ such that $b$ is constant on $(0, \frac{1}{n})$.

**Case 1.** $1_{(\frac{1}{n},1)} \cdot b = 0$. In this case, there is $\alpha$ such that $b = \alpha 1_{(0,\frac{1}{n})}$. Let $a = \frac{n\alpha}{|\alpha|} 1_{(0,\frac{1}{n})}$ and $x = 21_{(\frac{1}{2},1)}$. Then

$$b = \frac{|\alpha|}{n}(a + 0x)\,.$$

**Case 2.** $1_{(\frac{1}{n},1)} \cdot b \neq 0$. There is $\alpha$ such that $1_{(0,\frac{1}{n})} \cdot b = \alpha \cdot 1_{(0,\frac{1}{n})}$. Without loss of generality, we assume that $\alpha \geq 0$. Let

$$x = \frac{1_{(\frac{1}{n},1)} \cdot b}{\|1_{(\frac{1}{n},1)} \cdot b\|_X} \text{ and } a = n 1_{(0,\frac{1}{n})}\,.$$

Then $\|x + a\| = 2$ and

$$b = \|b\|_X \cdot \big((1 - \alpha_1)a + \alpha_1 x\big)$$

where $\alpha_1 = \|1_{(\frac{1}{n},1)} \cdot b\|_X / \|b\|_X$ .

Hence if $E$ is a strictly convex Köthe function space and if $T$ is an isometry from $X$ into the vector valued Köthe function space $E(Y)$, then $T$ has the form (2).

EXAMPLE 2: Let $X$ be a Banach space. Suppose that there is a unit vector $a$ such that for any $y \in X$ there is $\alpha \neq 0$ such that $\|a + \alpha y\| = \|a\| + |\alpha|\|y\|$. Let $Z$ be any separable subspace of $X$ which contains $a$. Then there is a countable dense subset $B$ of $Z$ such that $B$ satisfies (ii) and (iii) of Theorem 1. Let $A = \{\alpha a : |\alpha| = 1\}$.

Since $A$ is contained in a one dimensional subspace, $A$ satisfies (i). The assumption implies $A$ and $B$ satisfies (iv). So if $E$ is a strictly convex Köthe function space and $Y$ is any Banach space, then every isometry from $Z$ into $E(Y)$ has form (2). Examples of Banach spaces with this property.

(1) Let $X = C(K)$ and let $a = 1_K$. Then for any $y \in X$ there is $\alpha \neq 0$ such that $\|a + \alpha y\| = \|a\| + \|\alpha y\|$.

(2) A Banach space $Y$ is said to have the (DE)-*property* if for every weakly compact operator $T : X \to X$, $\|T + I\| = \|T\| + 1$. It is known that $L_1([0,1])$ and $L_\infty([0,1])$ have the (DE)-property (see [1] and its references). For any Banach space $Y$ with (DE)-property, let $X$ be the space generated by the weakly compact operators and the identity. Then for any $T \in X$, $T = S + \alpha I$ for some compact operator and $\alpha$. If $\alpha = 0$, then $\|T + I\| = \|T\| + 1$. If $\alpha \neq 0$, then $\|I + \frac{\overline{\alpha}}{|\alpha|} T\| = \|S\| + (|\alpha| + 1) = \|T\| + 1$.

Example 3: Let $(\ell_1, |\|\cdot\||)$ be the real $\ell_1$ with the equivalent norm

$$\big|\|x\|\big| = \max\left\{\|x^+\|_1, \|x^-\|_1\right\}.$$

Let $\{e_k : k \in \mathbb{N}\}$ be the natural basis. Let

$$\begin{aligned} A &= \left\{\pm e_k : k \in \mathbb{N}\right\} \\ B &= \left\{\sum_{k=1}^{n} a_k e_k : n \in \mathbb{N} \ \text{ and } \ a_k \in \mathbb{Q} \text{ for every } a_k\right\}. \end{aligned}$$

Clear, $B$ satisfies (ii) and (iii) of Theorem 3. For any $k$, let $i, j$ be two distinct natural numbers such that $i \neq k \neq j$. Then

$$\big|\|(e_i - e_j) \pm e_k\|\big| = 2 = 2\big|\|e_i - e_j\|\big|.$$

So $A$ satisfies (i) of Theorem 1.

Let $b = \sum_{k=1}^{n} a_k e_k \neq 0$ be any element of $B$. Then

$$2 = \max\left\{\Big|\Big\|\frac{b}{|\|b\||} + e_{n+1}\Big\|\Big|, \Big|\Big\|\frac{b}{|\|b\||} - e_{n+1}\Big\|\Big|\right\}.$$

Without loss of generality, we assume that $|\|\frac{b}{|\|b\||} + e_{n+1}\||$. Then

$$b = |\|b\|| \cdot \left(\frac{b}{|\|b\||} + 0 \cdot e_{n+1}\right).$$

Hence if $E$ is strictly convex, then every isometry from $(\ell_1, |||\cdot|||)$ into $E(Y)$ has form (2).

EXAMPLE 4: Let $\{e_1, e_2, \cdots\}$ be the natural basis of $\ell_p$, $1 \le p < \infty$ and

$$E_n = \operatorname{span}\{e_1, e_2, \cdots, e_n\}$$
$$F_n = \operatorname{span}\{e_{n+1}, \cdots\}.$$

Let $X$ be the set of all compact operator from $\ell_p$, $1 \le p < \infty$, into itself and let $S_n$ denote the operator

$$S_n\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \alpha_n e_n .$$

Let

$$A = \left\{S_n : n \in \mathbb{N}\right\}.$$

It is known that there is a countable dense subset $B$ of $X$ such that $B$ satisfies (vi) of Theorem 2 and for any $S \in B$ there is $n \in \mathbb{N}$ such that $S(E_n) \subseteq E_n$ and $S|_{F_n} = 0$. Note: if $S(E_n) \subseteq E_n$, $S|_{F_n} = 0$, and $\|S\| = 1$, then

$$\left\|S \pm S_{n+1} \pm S_{n+2}\right\| = 1 .$$

Hence if $E$ is strictly monotone, then every isometry from $X$ into $E(Y)$ has the form (2).

## References


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