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The completely continuous property in Orlicz spaces

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Abstract

We show that in Orlicz spaces equipped with Luxemburg norm and Orlicz norm, the RNP, CCP, PCP and CPCP are equivalent.

Let X be a Banach space, and let $B(X)$ and $S(X)$ be the unit ball and unit sphere of X respectively. For a set A in X, let $\alpha(A)$ be the Kuratowski index of non-compactness of A, i.e,

$$\alpha(A) = \inf\left\{\varepsilon > 0 : A \text{ is covered by a finite family of sets of diamete less than } \varepsilon\right\}.$$

X is said to possess the *complete continuity property (CCP)* if every bounded linear operator from $L_1[0,1]$ into X is completely continuous (i.e., maps weakly convergent sequences to norm convergent sequences). X is said to possess the *point of continuity property (PCP)* if every non-empty bounded closed set $C \subset B(X)$ has an element $x \in C$ so that the weak and norm topologies (restricted to C) coincide at x. X is said to possess the *convex point of continuity property (CPCP)* if every non-empty bounded closed convex set $C \subset B(X)$ has an element $x \in C$ so that the weak and norm topologies (restricted to C) coincide at x. X is said to possess the Radon-Nikodym property (RNP) if every non-empty bounded closed convex set has a denting point. It is known [2] that

$$RNP \Rightarrow PCP \Rightarrow CPCP \Rightarrow CCP$$

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In general, the converse are false. For some recent results on CCP, see [1] and [3]. In this paper we show that in Orlicz spaces $L_{(M)}$ or L_M, $RNP, PCP, CPCP$ and CCP are equivalent and $L_{(M)}$ or L_M to have these properties if and only if M satisfies the $\triangle_2$-condition.

A function $M : \mathbb{R} \to \mathbb{R}_+$ is called an N-function if M is convex, even, $M(0) = 0$ and $M(\infty) = \infty$. A complemented function N of M in the sense of *Young* is defined by

$$N(v) = \sup_{u>0} \{uv - M(u)\} .$$

It is known that if M is a N-function, then its complemented function N is also a N-function. M is said to satisfy the $\triangle_2$-condition if for some K and $u_0 > 0$, $M(2u) < KM(u)$ for all $|u| \geq u_0$. Let G be a bounded set in R^n and let (μ, Σ, G) be a finite non-atomic measure space. For a real-valued measurable function $x(t)$ on (μ, Σ, G), let $\rho_M(x) = \int_G M\Big(x(t)\Big) d\mu$. The Orlicz space $L_{(M)}$ and L_M generated by M is the Banach space

$$\{x(t) : \rho_M(\lambda x) < \infty \text{ for some } \lambda\} .$$

equipped with the Luxemburg norm $\|x\|_{(M)}$ on $L_{(M)}$ and the Orlicz norm $\|x\|_M$ on L_M, respectively, where

$$\| x \|_{(M)} = \inf \left\{ \lambda > 0 : \rho_M\left(\frac{x}{\lambda}\right) \leq 1 \right\} .$$

and

$$\| x \|_M = \inf_{k>0} \frac{1}{k} \big\{1 + \rho_M(kx)\big\} .$$

Theorem

An Orlicz function space $L_{(M)}$ possesses the CCP if and only if M satisfies $\triangle_2$-condition.

Proof. Suppose that M satisfies $\triangle_2$-condition. By [7] it follows that $L_{(M)}$ possesses the RNP. Hence $L_{(M)}$ possesses the CCP.

Suppose that $L_{(M)}$ possesses the CCP. If $M \notin \triangle_2$, then there exist $u_n \nearrow \infty$ such that

$$M\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^n M(u_n) , \qquad (n = 1, 2, ...) .$$

Take disjoint subsets $G_n \subset G$ so that

$$\mu G_n \leq \frac{1}{2^n} , \qquad (n = 1, 2, ...) .$$

For each n, choose a subsequence $\{u_k^n\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and disjoint subsets $G_{n,k} \subset G_n$ such that

$$M(u_k^n)\mu G_{n,k} = \frac{1}{2^{n+k}}\,, \qquad (k = 1, 2, ...)\,.$$

Define

$$x_n(t) = \sum_{k=1}^{\infty} u_k^n \chi_{G_{n,k}}(t)\,, \qquad (n = 1, 2, ...)\,.$$

Then

$$\rho_M(x_n) = \sum_{k=1}^{\infty} M(u_k^n)\mu G_{n,k} = \sum_{k=1}^{\infty} \frac{1}{2^{n+k}} = \frac{1}{2^n} < 1\,.$$

But for any $\lambda > 1$, choose k_0 with $1 + \frac{1}{k_0} \le \lambda$. Then

$$\begin{aligned}\rho_M(\lambda x_n) &\ge \sum_{k=k_0}^{\infty} M\left(\left(1 + \frac{1}{k}\right)u_k^n\right)\mu G_{n,k}\\ &> \sum_{k=k_0}^{\infty} 2^k M(u_k^n)\frac{1}{2^{n+k}M(u_k^n)} = \infty\,.\end{aligned}$$

Hence $\|x_n\|_{(M)} = 1, (n = 1, 2, ...)$. Let Let

$$X_0 = \left\{x_\xi : x_\xi(t) = \sum_{n=1}^{\infty} \xi_n x_n(t),\ \xi = \{\xi_n\}_{n=1}^\infty \in l_\infty\right\}.$$

Then for any $x_\xi \in X_0$, we have $x_\xi \in L_{(M)}$ and $\|x_\xi\|_{(M)} = \|\xi\|_\infty = \sup|\xi_n|$. Hence $X_0 \subset L_{(M)}$ and X_0 is isometric to l_∞ [6].

We now show that there exist $T \in B\Big(L_1[0,1], L_{(M)}\Big)$ and a weakly convergent sequence $\{r_n\}_{n=1}^\infty \subset L_1[0,1]$, with $\text{sep}\{Tr_n\} = \inf\{\|Tr_n - Tr_m\| : n \ne m\} = 1$. Hence $L_{(M)}$ fails to possesse the CCP.

Let r_n be the Rademacher functions, i.e,

$$r_n(t) = 2^n \begin{cases} 1 & \text{if} \quad t \in \left(\frac{1}{2^n}, \frac{3}{2^{n+1}}\right] \\ -1 & \text{if} \quad t \in \left(\frac{3}{2^{n+1}}, \frac{1}{2^{n-1}}\right] \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\{r_n\}_{n=1}^\infty$ is a weakly convergent sequence in $L_1[0,1]$. For each $f \in L_\infty[0,1] = L_1^*[0,1]$ and an arbitrary $\varepsilon > 0$, by Lusin Theorem, there is $g \in C[0,1]$ such that

$$\int_0^1 |f - g| d\mu < \varepsilon\,.$$

Then for n sufficiently large,

$$\left|\int_0^1 r_n f d\mu\right| \leq \left|\int_0^1 r_n(f-g)d\mu\right| + \left|\int_0^1 r_n g d\mu\right|$$
$$\leq \varepsilon + 2^n \left|\int_{\frac{1}{2^n}}^{\frac{3}{2^{n+1}}} g(t)d\mu - \int_{\frac{3}{2^{n+1}}}^{\frac{1}{2^{n-1}}} g(t)d\mu\right|$$
$$\leq \varepsilon + \varepsilon 2^n \frac{1}{2^n} = 2\varepsilon\,.$$

Hence $r_n \xrightarrow{w} 0$. Define T from $L_1[0,1]$ into X_0 by

$$T(r_n) = x_n,\ n = 1, 2, ...$$

and

$$T\left(\sum_{i=1}^n \lambda_i r_i\right) = \sum_{i=1}^n \lambda_i x_i$$

for $\lambda_i \in \mathbb{R}, i = 1, ..., n, n \in \mathbb{N}$. Then

$$\left\|T\Big(\sum_{i=1}^n \lambda_i r_i\Big)\right\|_{(M)} = \left\|\sum_{i=1}^n \lambda_i x_i\right\|_{(M)} = \max_{1\leq i\leq n} |\lambda_i|\,.$$

Since $\{r_n\}$ have disjoint supports,

$$\left\|\sum_{i=1}^n \lambda_i r_i\right\|_1 = \sum_{i=1}^n |\lambda_i| \geq \max_{1\leq i\leq n} |\lambda_i| = \left\|T\Big(\sum_{i=1}^n \lambda_i r_i\Big)\right\|_{(M)}\,.$$

Hence T is a linear bounded operator from span$\{r_n\}$ into X_0. By the Proposition 2.f.2 of [5], there exists an extension $\bar{T}$ of T with $\bar{T} \in B\Big(L_1[0,1], L_{(M)}\Big)$. But for any $n \neq m$

$$\|Tr_n - Tr_m\|_{(M)} = \|x_n - x_m\|_{(M)} = 1\,.$$

Hence sep$\{Tr_n\} = 1$, which shows that $L_{(M)}$ fails to havge the CCP. $\square$

Remark. Sincce the CCP is an isomorphic invariant, it follows that for an Orlicz function space L_M possesses the CCP if and only if $M \in \triangle_2$. It is known [7] that if $M \in \triangle_2$ then $L_{(M)}$ has the RNP. We conclude that in Orlicz function spaces $L_{(M)}$ or L_M, the $RNP, PCP, CPCP$ and CCP are equivalent and $L_{(M)}$ or L_M has these properties if and only if M satisfies the $\triangle_2$-condition.

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