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Onesided approximation and real interpolation[1]

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Abstract

It is proved that the reiteration theorem is not valid for the spaces $A_p^{\theta,q}$ defined by V. Popov by means of onesided approximation. It is also proved that a class of cones, defined by onesided approximation of piecewise linear functions on the interval $[0,1]$, is stable for the real interpolation method.

1. Introduction

The spaces $A_p^{\theta,q}$, $1 \le p, q \le \infty$, $k > \theta > 0$, were introduced by V. Popov in [5]. It is known (see, for example, [1], [6], [10], [11]) that they are equivalent to the spaces defined by onesided trigonometrical or spline approximation.

The first interpolation result for $A_p^{\theta,q}$ was obtained by V. Popov in [7]; he proved that the average modulus of continuity $\tau_k(f,t)_p$ is equivalent to the onesided K-functional for the Banach couple (L_p, W_p^k). The interpolation properties of A-spaces were also studied in [3]. There the author posed the problem if the A-spaces are stable for the real interpolation method.

It is possible to prove, using the technique of [2], that the embedding

$$\left(A_p^{\theta_0,q_0}, A_p^{\theta_1,q_1}\right)_{\lambda,q} \subset A_p^{(1-\lambda)\theta_0+\lambda\theta_1,q}, 1 \le p,q \le \infty\,, 0 < \lambda < 1\,,$$

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holds. The inverse embedding is not valid, and we present here a counterexample due to N. Krugljak.

We also prove the reiteration theorem for a family of cones of nonnegative functions defined by onesided approximation. For that we modify the sequence of piecewise–linear functions f_n of the best onesided approximation and construct a sequence $\{f_n^+\}$ such that $f_1^+ \le f_2^+ \le \ldots \le f$. It is essential that the degree of the onesided approximation by the sequences $\{f_n\}$ and $\{f_n^+\}$ are equal. The problem of constructing such a sequence is due to S. Stechkin ([9]) and has its own interest (cf. [4] or [8]).

2. An equivalent norm for the space $A_1^{\theta,\infty}$

$A_p^{\theta,q}$ $(1 \le p, q \le \infty$ and $k > \theta > 0)$ is the space of all bounded measurable functions such that

$$\|f\|_{A_p^{\theta,q}} = \left(\int_0^\infty \left(\frac{\tau_k(f,t)_p}{t^\theta} \right)^q \frac{dt}{t} \right)^{1/q} < \infty\,.$$

Here $\tau_k(f,t)_p$ is the average modulus of continuity $\tau_k(f,t)_p = \left(\int_0^1 \omega_k(f,x,t)^p dx \right)^{1/p}$, with

$$\omega_k(f,x,t) = \sup\left\{ \left|\Delta_h^k f(y)\right|;\ y, y+kh \in \left[x - \frac{kt}{2}, x + \frac{kt}{2}\right] \cap [0,1] \right\},$$

and Δ_h^k is the k-difference operator with step h. The seminorm of $A_1^{\theta,\infty}$ is defined by

$$\|f\|_{A_1^{\theta,\infty}} = \sup_{0<t<\infty} \frac{\tau(f,t)_1}{t^\theta}\,.$$

We write ω and τ for ω_1 and τ_1.

For every interval Q (Q can be closed, open, half-open) we denote $\operatorname{osc} \mathrm{f(Q)} = \sup_{\mathrm{x \in Q \cap [0,1]}} \mathrm{f(x)} - \inf_{\mathrm{x \in Q \cap [0,1]}} \mathrm{f(x)}$ and we need the following simple properties of the oscillation:

1. Let Q, Q_1, Q_2 be intervals such that $Q \subset Q_1 \cup Q_2$. Then, for the continuous function f,

$$\operatorname{osc} \mathrm{f(Q)} \le \operatorname{osc} \mathrm{f(Q_1)} + \operatorname{osc} \mathrm{f(Q_2)}\,. \tag{1}$$

If $Q_1 \cap Q_2 \ne \emptyset$, this inequality is true for any function.

2. If f, g are two functions on Q, then

$$\mathrm{osc}\,(\mathrm{f}+\mathrm{g})(\mathrm{Q}) \leq \mathrm{osc}\,\mathrm{f}(\mathrm{Q}) + \mathrm{osc}\,\mathrm{f}(\mathrm{Q})\,. \tag{2}$$

The finite family of intervals $Q_i = [(i-1)t, it] \cap [0,1] \neq \emptyset$ $(0 < t < 1)$ is a partition of $[0,1]$ denoted by π_t. The oscillation of f on π_t is

$$\mathrm{osc}_{\pi_t}\mathrm{f} = \sum_{\mathrm{Q}\in\pi_t} \mathrm{osc}\,\mathrm{f}(\mathrm{Q})\,.$$

We denote by π_{2^n} the partition of $[0,1]$ into 2^n equal intervals, $Q_i = [(i-1)/2^n, i/2^n]$, and then $\mathrm{osc}_{2^n}\mathrm{f} = \sum_{\mathrm{i}=1}^{2^\mathrm{n}} \mathrm{osc}\,\mathrm{f}(\mathrm{Q_i})$.

Proposition 2.1

If f is a measurable function on $[0,1]$, then

$$c_1 \sup_{n\geq 0} \frac{\mathrm{osc}_{2^n}\mathrm{f}}{2^{(1-\theta)\mathrm{n}}} \leq \|f\|_{A_1^{\theta,\infty}} \leq c_2 \sup_{n\geq 0} \frac{\mathrm{osc}_{2^n}\mathrm{f}}{2^{(1-\theta)\mathrm{n}}}\,,$$

where c_1 and c_2 are two constants independent on f and n.

Proof. Let $Q_i = [x_i, y_i]$ be any interval from the partition π_t. It follows from (1) that

$$\omega(f,x,t) \leq \begin{cases} \mathrm{osc}\,\mathrm{f}(\mathrm{Q_i}) + \mathrm{osc}\,\mathrm{f}(\mathrm{Q_{i+1}}), & \text{if } (x_i+y_i)/2 \leq x < y_i \\ \mathrm{osc}\,\mathrm{f}(\mathrm{Q_{i-1}}) + \mathrm{osc}\,\mathrm{f}(\mathrm{Q_i}), & \text{if } x_i < x < (x_i+y_i)/2\,. \end{cases}$$

Then

$$\tau(f,t)_1 = \int_0^1 \omega(f,x,t)dx \leq 2t\,\mathrm{osc}_{\pi_t}\mathrm{f}\,. \tag{3}$$

On the other hand, if $x \in Q$, where Q is an interval from the partition $\pi_{t/2}$, then $\omega(f,x,t) \geq \mathrm{osc}\,\mathrm{f}(\mathrm{Q})$ and

$$\tau(f,t)_1 \geq \frac{t}{2}\,\mathrm{osc}_{\pi_{t/2}}\mathrm{f}\,. \tag{4}$$

From (3) and (4) we obtain $2^{-\theta}(t/2)^{1-\theta}\,\mathrm{osc}_{\pi_{t/2}}\mathrm{f} \leq \tau(\mathrm{f},\mathrm{t})_1/\mathrm{t}^\theta \leq 2\mathrm{t}^{1-\theta}\mathrm{osc}_{\pi_t}\mathrm{f}$. As $\tau(f,t)_1 = \tau(f,2)_1$ for $t > 2$, then

$$2^{-\theta} \sup_{0<t<2} t^{1-\theta}\,\mathrm{osc}_{\pi_t}\mathrm{f} \leq \|\mathrm{f}\|_{\mathrm{A}_1^{\theta,\infty}} \leq 2 \sup_{0<\mathrm{t}<2} \mathrm{t}^{1-\theta}\,\mathrm{osc}_{\pi_t}\mathrm{f}\,. \tag{5}$$

If $0 < t < 1$, there exists $n \geq 0$ such that $1/2^n \leq t < 1/2^{n-1}$, it follows from (1) that $2\,\mathrm{osc}_{2^n}\mathrm{f} \leq \mathrm{osc}_{\pi_t}\mathrm{f} \leq 3\,\mathrm{osc}_{2^{n-1}}\mathrm{f}$, and (5) and the last inequality finishes the proof. □

3. The embedding $(A_1^{\theta_0,\infty}, A_1^{\theta_1,\infty})_{\lambda,q} \subset A_1^{(1-\lambda)\theta_0+\lambda\theta_1,q}$ is strict

Let $\vec{Y} = (Y_0, Y_1)$ be a couple of Banach spaces. The K–functional is defined by

$$K\big(t, f, \vec{Y}\big) = \inf_{f=f_0+f_1} \big(\|f_0\|_{Y_0} + t\|f_1\|_{Y_1}\big)\,.$$

The interpolation space $Y_{\lambda,q}$ $(0 < \lambda < 1, 1 \le q \le \infty)$ is the space of all the elements $f \in Y_0 + Y_1$ such that $\|f\|_{Y_{\lambda,q}} = \Big(\int_0^\infty (t^{-\lambda} K(t, f, \vec{Y}))^q \frac{dt}{t}\Big)^{1/q} < \infty$.

We shall prove that the embedding $(A_1^{\theta_0,\infty}, A_1^{\theta_1,\infty})_{\lambda,\infty} \supset A_1^{(1-\lambda)\theta_0+\lambda\theta_1,\infty}$ is not valid.

Let us divide $[0,1]$ into 2^n equal intervals and the last interval into 2^m equal intervals, $m, n \in \mathbb{N}$. We define the function $f_{n,m}$ by

$$f_{n,m} = \begin{cases} 0, & \text{if } \ 0 \le x \le 1 - \frac{1}{2^n} \\ 0, & \text{if } \ x = 1 - \frac{1}{2^n} + \frac{2}{2^{n+m}}, 1 - \frac{1}{2^n} + \frac{4}{2^{n+m}}, \dots, 1 \\ 1, & \text{if } \ x = 1 - \frac{1}{2^n} + \frac{1}{2^{n+m}}, 1 - \frac{1}{2^n} + \frac{3}{2^{n+m}}, \dots, 1 - \frac{1}{2^{n+m}} \\ \text{linear on every interval } \Big[\frac{i}{2^{n+m}}, \frac{i+1}{2^{n+m}}\Big], & 2^{n+m} - 2^m \le i \le 2^{n+m} - 1\,. \end{cases}$$

It is not difficult to check that

$$\mathrm{osc}_{\,2^k} \mathrm{f_{n,m}} = \begin{cases} 1, & \text{if } \ 0 \le k \le n \\ 2^m, & \text{if } \ k \ge m+n \\ 2^{k-n}, & \text{if } \ n < k \le m+n\,, \end{cases}$$

and it follows from Proposition 2.1 that

$$\begin{aligned} \|f_{n,m}\|_{A_1^{\theta,\infty}} &\ge c_1 \sup_{k\ge 0} \frac{\mathrm{osc}_{\,2^k} \mathrm{f_{n,m}}}{2^{(1-\theta)\mathrm{k}}} \\ &= c_1 \max\Big(\sup_{0\le k\le n} \frac{\mathrm{osc}_{\,2^k} \mathrm{f_{n,m}}}{2^{(1-\theta)\mathrm{k}}}, \sup_{k\ge n} \frac{\mathrm{osc}_{\,2^k} \mathrm{f_{n,m}}}{2^{(1-\theta)\mathrm{k}}} \Big) = c_1 \max\Big(1, \frac{2^m}{2^{(1-\theta)(n+m)}}\Big). \end{aligned}$$

We denote $q_\theta = 2^{1-\theta}$, hence $\|f_{n,m}\|_{A_1^{\theta,\infty}} \ge c_1 \max(1, 2^m/q_\theta^{n+m})$.

Let us suppose that $(A_1^{\theta_0,\infty}, A_1^{\theta_1,\infty})_{\lambda,\infty} \supset A_1^{(1-\lambda)\theta_0+\lambda\theta_1,\infty}$. Then there exists a constant $C > 0$ independent on m and n such that

$$\|f_{n,m}\|_{A_1^{(1-\lambda)\theta_0+\lambda\theta_1,\infty}} \ge C \sup_{t\ge 0} \frac{K(t, f_{n,m}, \vec{A})}{t^\lambda},$$

where $\vec{A} = (A_1^{\theta_0,\infty}, A_1^{\theta_1,\infty})$. From now on we shall suppose without any lose of generality that $\theta_0 > \theta_\lambda > \theta_1$, with $\theta_\lambda = (1-\lambda)\theta_0 + \lambda\theta_1$. In particular, for $t = (q_{\theta_1}/q_{\theta_0})^n$ we have

$$\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n \|f_{n,m}\|_{A_1^{(1-\lambda)\theta_0+\lambda\theta_1,\infty}} \geq CK\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n, f_{n,m}, \vec{A}\right).$$

It follows from Proposition 2.1 that

$$\|f_{n,m}\|_{A_1^{(1-\lambda)\theta_0+\lambda\theta_1,\infty}} \leq c_2 \max\left(1, \frac{2^m}{q_{\theta_0}^{(1-\lambda)(n+m)} q_{\theta_1}^{\lambda(n+m)}}\right). \tag{6}$$

Then multiplying both sides of (6) by $(q_{\theta_1}/q_{\theta_0})^{n\lambda}$, we rewrite it as

$$\max\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^{n\lambda}, \frac{2^m}{q_{\theta_0}^{(1-\lambda)(n+m)} q_{\theta_1}^{\lambda(n+m)}}\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^{n\lambda}\right) \geq CK\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n, f_{n,m}, \vec{A}\right)$$

and, if we denote $c_{n,m} = K((q_{\theta_1}/q_{\theta_0})^n, f_{n,m}, \vec{A})$,

$$\max\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^\lambda, \frac{2^{m/n}}{q_{\theta_0} q_{\theta_\lambda}^{m/n}}\right)^n \geq Cc_{n,m}\,.$$

Then there exist $f_{n,m}^0 \in A_1^{\theta_0,\infty}$ and $f_{n,m}^1 \in A_1^{\theta_1,\infty}$ such that $f_{n,m} = f_{n,m}^0 + f_{n,m}^1$ and

$$\|f_{n,m}^0\|_{A_1^{\theta_0,\infty}} + \left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n \|f_{n,m}^1\|_{A_1^{\theta_1,\infty}} \leq 2c_{n,m}\,.$$

We consider two cases: (a) $2c_{n,m} < c_1/4\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n$, and (b) $2c_{n,m} \geq c_1/4\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n$, where c_1 is the constant from Proposition 2.1.

If (a) takes place, then $\|f_{n,m}^1\|_{A_1^{\theta_1,\infty}} < c_1/4$; as $c_1 \operatorname{osc}_{2^0} f_{n,m}^1 < c_1 \sup_{k\geq 0} \frac{\operatorname{osc}_{2^k} f_{n,m}^1}{2^{(1-\theta)k}} \leq \|f_{n,m}^1\|_{A_1^{\theta_1,\infty}}$, then $\operatorname{osc}_{2^0} f_{n,m}^1 < 1/4$. As $f_{n,m}^0 = f_{n,m} - f_{n,m}^1$ and $\operatorname{osc}_{2^0} f_{n,m} = 1$; then it follows from (2) that $\operatorname{osc}_{2^0} f_{n,m}^0 \geq \operatorname{osc}_{2^0} f_{n,m} - \operatorname{osc}_{2^0} f_{n,m}^1 \geq 3/4$. Analogously, from the definition of $f_{n,m}$ we obtain that $\operatorname{osc}_{2^{n+m}} f_{n,m}^0 \geq (3/4)2^m$.

Again from Proposition 2.1 it follows that

$$\|f_{n,m}^0\|_{A_1^{\theta_0,\infty}} \geq c_1 \frac{\operatorname{osc}_{2^{n+m}} f_{n,m}^0}{2^{(1-\theta_0)(n+m)}} \geq \frac{3}{4} c_1 \frac{2^m}{q_{\theta_0}^{n+m}}\,.$$

Consequently $c_{n,m} \geq 3/8c_1\frac{2^m}{q_{\theta_0}^{n+m}}$; that is why we have the last inequality or (b). Then

$$K\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n f_{n,m}, \vec{A}\right) \geq \frac{1}{8}c_1 \min\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^n, \frac{2^m}{q_{\theta_0}^{n+m}}\right).$$

Then for all subsequences such that $n_s, m_s \to \infty$ for $s \to \infty$

$$\max\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^\lambda, \frac{2^{m_s/n_s}}{q_{\theta_0} q_{\theta_\lambda}^{m_s/n_s}}\right) \geq \left(\frac{C}{8}c_1\right)^{1/n_s} \min\left(\frac{q_{\theta_1}}{q_{\theta_0}}, \frac{2^{m_s/n_s}}{q_{\theta_0} q_{\theta_0}^{m_s/n_s}}\right).$$

Let us take n_s, m_s, such that $m_s/n_s = \gamma$; we shall show that, if $\theta_0 > \theta_1$, there exists γ rational such that

$$\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^\lambda, \frac{2^\gamma}{q_{\theta_0} q_{\theta_\lambda}^\gamma}\right) \geq \left(\frac{C}{8}\right)^{1/n_s} \min\left(\frac{q_{\theta_1}}{q_{\theta_0}}, \frac{2^\gamma}{q_{\theta_0} q_{\theta_0}^\gamma}\right)$$

doesn't take place. If this is not the case, it follows that for $n_s, m_s \to \infty$ we have

$$\max\left(\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^\lambda, \frac{2^\gamma}{q_{\theta_0} q_{\theta_\lambda}^\gamma}\right) \geq \min\left(\frac{q_{\theta_1}}{q_{\theta_0}}, \frac{2^\gamma}{q_{\theta_0} q_{\theta_0}^\gamma}\right).$$

We will show that the inequalities

$$\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^\lambda \geq \frac{q_{\theta_1}}{q_{\theta_0}}, \frac{2^\gamma}{q_{\theta_0} q_{\theta_\lambda}^\gamma} \geq \frac{2^\gamma}{q_{\theta_0} q_{\theta_0}^\gamma} \tag{7}$$

and

$$\left(\frac{q_{\theta_1}}{q_{\theta_0}}\right)^\lambda \geq \frac{2^\gamma}{q_{\theta_0} q_{\theta_0}^\gamma}, \frac{2^\gamma}{q_{\theta_0} q_{\theta_\lambda}^\gamma} \geq \frac{q_{\theta_1}}{q_{\theta_0}} \tag{8}$$

are not valid.

Let us start with (7). From $1-\theta_0 < 1-\theta_1$ (we recall that $\theta_0 > \theta_1$) we have that $2^{1-\theta_0} < 2^{1-\theta_1}$; hence $q_{\theta_1} > q_{\theta_0}$ and $q_{\theta_1}/q_{\theta_0} > 1$. Then for $0 < \lambda < 1$ we have $(q_{\theta_1}/q_{\theta_0})^\lambda < q_{\theta_1}/q_{\theta_0}$.

In the same way from $1-\theta_0 < 1-\theta_\lambda < 1-\theta_1$, where

$$1-\theta_\lambda = (1-\lambda)(1-\theta_0) + \lambda(1-\theta_1),$$

we have that $2^{1-\theta_0} < 2^{\theta_\lambda}$; hence $q_{\theta_\lambda} > q_{\theta_0}$ and $(q_{\theta_0}/q_{\theta_\lambda})^\lambda < 1$. Then

$$\left(\frac{1}{q_{\theta_\lambda}}\right)^\gamma < \left(\frac{1}{q_{\theta_0}}\right)^\gamma \text{ and } \frac{2^\gamma}{q_{\theta_0} q_{\theta_\lambda}^\gamma} < \frac{2^\gamma}{q_{\theta_0} q_{\theta_0}^\gamma}.$$

To show that (8) is not valid we will find γ rational such that both inequalities are not true. This is equivalent to

$$q_{\theta_\lambda}^{1/\gamma} q_{\theta_0} < 2 < q_{\theta_1}^{1/\gamma} q_{\theta_\lambda} \,. \tag{9}$$

Let us denote $\phi_1(\gamma) = q_{\theta_\lambda}^{1/\gamma} q_{\theta_0}$ and$\phi_2(\gamma) = q_{\theta_1}^{1/\gamma} q_{\theta_\lambda}$. It is not difficult to see that both functions ϕ_1 and ϕ_2 are continuous, monotone and such that

$$\lim_{\gamma\to 0} \phi_1(\gamma) = \lim_{\gamma\to 0} \phi_2(\gamma) = \infty$$

and

$$\lim_{\gamma\to\infty} \phi_1(\gamma) = q_{\theta_0} < 2 \text{ and } \lim_{\gamma\to\infty} \phi_2(\gamma) = q_{\theta_\lambda} < 2.$$

As $q_{\theta_1} > q_{\theta_0}$ and $q_{\theta_\lambda} > q_{\theta_0}$, then $\phi_2(\gamma) > \phi_1(\gamma)$ for all γ rational. From this it follows that we can find γ such that (9) is valid.

4. Real interpolation of cones defined by the piecewise–linear onesided approximation

Let us denote by S_{2^n}, $n \in \mathbb{N}$, the subspace of $L_p(0,1)$, $1 \le p \le \infty$, consisting of all the piecewise–linear functions on the interval $[0,1]$ with the knots in the points $i/2^n$, $(0 \le i \le 2^n)$. If $f \in S_{2^n}$, then f is linear on every interval $[(i-1)/2^n, i/2^n]$, $1 \le i \le 2^n$. An essential fact is that $S_{2^n} \subset S_{2^{n+1}}$, $n \in \mathbb{N}$.

Let us define for every measurable function f on $[0,1]$ the sequence of the best onesided approximation by functions of the family S_{2^n}, $n \in \mathbb{N}$, as

$$e_{2^n}^+(f)_p = \inf_{f_{2^n}\in S_{2^n},\, f\ge f_{2^n}} \left(\int_0^1 \big(f(x) - f_{2^n}(x)\big)^p dx \right)^{1/p}.$$

Then we define the cone $A_p^{+\theta,q}$ $(0 < \theta < 2,\ 1 \le p,q \le \infty)$ as the set of all real measurable functions on $[0,1]$ with finite seminorm

$$\|f\|_{\theta,q} = \begin{cases} \left(\sum_{n=0}^\infty \big(2^{\theta n} e_{2^n}^+(f)_p\big)^q\right)^{1/q}, & \text{if } 1 \le q < \infty\,; \\ \sup_{n\ge 0} 2^{\theta n} e_{2^n}^+(f)_p, & \text{if } q = \infty\,. \end{cases}$$

For the couple of the cones $\vec{A}_+ = (A_p^{+\theta_0,q_0}, A_p^{+\theta_1,q_1})$ we define the K-functional as

$$K(t,f,\vec{A}_+) = \inf_{f=f_0+f_1} \big(\|f_0\|_{\theta_0,q_0} + t\|f_1\|_{\theta_1,q_1}\big).$$

The interpolation cone $(A_p^{+\theta_0,q_0}, A_p^{+\theta_1,q_1})_{\lambda,q}$ $(0 < \lambda < 1$ and $1 \le q \le \infty)$ is defined in the usual way.

Theorem 4.1

If $0 < \theta_0, \theta_1 < 1 + 1/p$, *then*

$$\left(A_p^{+\theta_0,q_0}, A_p^{+\theta_1,q_1}\right)_{\lambda,q} = A_p^{+(1-\lambda)\theta_0+\lambda\theta_1,q},$$

where $0 < \lambda < 1$ *and* $1 \le p, q \le \infty$.

The proof of Theorem 4.1 is analogous to the proof of the interpolation theorem in [2]. We only need to show that for every $f \in A_p^{+\theta,\infty}$, $(0 < \theta < 1/p$ and $1 \le p \le \infty)$ there exists a sequence of piecewise–linear functions $f_{2^n}^+ \in S_{2^n}$, $n \in \mathbb{N}$, such that

(a) $f_{2^0}^+ \le f_{2^1}^+ \le \ldots f_{2^n}^+ \le f$, and

(b) if $e_{2^n}^+(f)_p = O(2^{-\theta n})$ for some $0 < \theta < 1/p$, then $\|f - f_{2^n}^+\|_p = O(2^{-\theta n})$, $n \in \mathbb{N}$.

We shall organize the construction of the required sequence in two steps. In the first one we prove a theorem that plays the main role in our construction. In the second step we construct the algorithm to obtain the sequence of the piecewise–linear functions with properties (a) and (b).

4.1 Main construction

Let us suppose that there exists a linear function f_{2^0} on the interval $Q = (\alpha, \beta)$ satisfying the inequality $f \ge f_{2^0}$. We also suppose that there exists a piecewise–linear function f_{2^1}, linear on each interval $[\alpha, (\alpha+\beta)/2]$ and $[(\alpha+\beta)/2, \beta]$, satisfying the inequality $f \ge f_{2^1}$.

Theorem 4.2

For every interval $Q = (\alpha, \beta)$ *and every* $m \in \mathbb{N}$, *there exists a piecewise–linear function* $f_{2^{m+1}}$, *linear on every interval* $[\alpha + (i-1)(\beta-\alpha)/2^{m+1}, \alpha + i(\beta-\alpha)/2^{m+1}]$, $1 \le i \le 2^{m+1}$, *and satisfying the following two conditions:*

(a) $f \ge f_{2^{m+1}} \ge f_{2^0}$

(b) $\|f - f_{2^{m+1}}\|_{L_p(Q)}$

$$\le c\left(\|f - f_{2^1}\|_{L_p(Q)} + \left(\frac{1}{2^m}\right)^{1+\frac{1}{p}} \|f - f_{2^0}\|_{L_p(Q)}\right), \tag{10}$$

where c *is a constant depending only on* p.

Proof. Without any lose of generality we prove the theorem for the interval $Q = [0,1]$. Then the function f_{2^0} is linear on the interval $[0,1]$ and f_{2^1} is linear on each of the intervals $[0,1/2]$ and $[1/2,1]$.

For the natural number m we divide every interval $[0,1/2]$ and $[1/2,1]$ into 2^m equal intervals. To construct the function $f_{2^{m+1}}$ we define

$$f_{2^{m+1}}(x) = \max\left(f_{2^0}(x)\,,\ f_{2^1}(x)\right) \tag{11}$$

on every interval of the partition, $[(i-1)/2^{m+1}, i/2^{m+1}]$, $1 \le i \le 2^{m+1}$, where the equation $f_{2^0}(x) = f_{2^1}(x)$ has no solution. This means that the graphics of the functions don't cross on such intervals and we have

$$f_{2^0}(x) < f_{2^1}(x) \ \text{or}\ f_{2^0}(x) > f_{2^1}(x)$$

for all x from the interval. For the other intervals we put $f_{2^{m+1}}(x) = f_{2^0}(x)$. We have obtained a piecewise–linear function, linear on every interval of the partition, $[(i-1)/2^{m+1}, i/2^{m+1}]$, $1 \le i \le 2^{m+1}$, which satisfy the inequalities $f \ge f_{2^{m+1}} \ge f_{2^0}$.

Let us prove the estimate (10). It is enough to prove it for the interval $[0,1/2]$, the proof for $[1/2,1]$ being the same.

There are three possible cases for the functions f_{2^0} and f_{2^1} on $[0,1/2]$.

(a) $f_{2^0}(x) < f_{2^1}(x)$ if $x \in [0,1/2]$,
(b) $f_{2^0}(x) > f_{2^1}(x)$ if $x \in [0,1/2]$, and
(c) there exists $a \in [0,1/2]$ such that $f_{2^0}(a) = f_{2^1}(a)$.

In case (a), it follows from (11) that $f_{2^{m+1}}(x) = f_{2^1}(x)$ if $x \in [0,1/2]$, and

$$\|f - f_{2^{m+1}}\|^p_{L_p(0,1/2)} = \|f - f_{2^1}\|^p_{L_p(0,1/2)}\,. \tag{12}$$

In the case (b) it follows from (11) that $f_{2^{m+1}}(x) = f_{2^0}(x)$ if $x \in [0,1/2]$ and

$$\|f - f_{2^{m+1}}\|^p_{L_p(0,1/2)} = \|f - f_{2^0}\|^p_{L_p(0,1/2)} \le \|f - f_{2^1}\|^p_{L_p(0,1/2)}\,. \tag{13}$$

In case (c) there are two possible situations:

(c_1) $f_{2^0}(x) < f_{2^1}(x)$ if $0 < x < a$ and $f_{2^0}(x) > f_{2^1}(x)$ if $a < x < 1/2$, and
(c_2) $f_{2^0}(x) > f_{2^1}(x)$ if $0 < x < a$ and $f_{2^0}(x) < f_{2^1}(x)$ if $a < x < 1/2$.

We consider only the case (c_1) – the case (c_2) is similar.

Again we have two possible situations in this case (c_1):

(c_{11}) The point $a \in [0,1/2]$ satisfies the inequality $0 < a \le 1/2 - 1/2^{m+2}$.
(c_{12}) the point a and satisfies the inequality $1/2 - 1/2^{m+2} < a \le 1/2$.

We start with (c_{11}). Let $a \in \big[(k-1)/2^{m+1}, k/2^{m+1}\big]$ for some $k \in \mathbb{N}$, $1 \le k \le 2^m$. We shall estimate $f - f_{2^{m+1}}$ on every interval $[0, (k-1)/2^{m+1}]$, $[(k-1)/2^{m+1}, a]$ and $[a, 1/2]$.

If $0 \le x \le (k-1)/2^{m+1}$ then

$$(f - f_{2^{m+1}})(x) = (f - f_{2^1})(x). \tag{14}$$

If $a \le x \le 1/2$ then

$$(f - f_{2^{m+1}})(x) < (f - f_{2^1})(x). \tag{15}$$

If $(k-1)/2^{m+1} < x < a$ then

$$(f - f_{2^{m+1}})(x) < (f - f_{2^1})(x) + (f_{2^1} - f_{2^0})(x). \tag{16}$$

Let us estimate the second member in (16). For every $(k-1)/2^{m+1} < x < a$ there exists $x' = 2a - x$, $a < x' < a + (a - (k-1)/2^{m+1})$ (the points x and x' are symmetrical with respect to a) such that $(f_{2^1} - f_{2^0})(x) = (f_{2^0} - f_{2^1})(x')$.

For $a < x' < a + (a - (k-1)/2^{m+1})$, $(f_{2^0} - f_{2^1})(x') < (f - f_{2^1})(x')$, hence, for $(k-1)/2^{m+1} < x < a$, $(f_{2^1} - f_{2^0})(x) < (f - f_{2^1})(x')$.

Finally, for $(k-1)/2^{m+1} < x < a$ and $x' = 2a - x$, $a < x' < a + (a - (k-1)/2^{m+1})$,

$$(f - f_{2^{m+1}})(x) < (f - f_{2^1})(x) + (f - f_{2^1})(x'). \tag{17}$$

Then, from (14), (15) and (17), we obtain for $1 \le p < \infty$

$$\begin{aligned}
\int_0^{1/2} (f - f_{2^{m+1}})^p(x)dx &= \Big(\int_0^{(k-1)/2^{m+1}} + \int_{(k-1)/2^{m+1}}^{a} + \int_a^{1/2} \Big) (f - f_{2^{m+1}})^p(x)dx \\
&\le \Big(\int_0^{(k-1)/2^{m+1}} + \int_a^{1/2} \Big) (f - f_{2^{m+1}})^p(x)dx \\
&\quad + \int_{(k-1)/2^{m+1}}^{a} (f - f_{2^1})^p(x)dx + \int_a^{1/2} (f - f_{2^1})^p(x')dx' \\
&\le 2 \int_0^{a} (f - f_{2^1})^p(x)dx.
\end{aligned} \tag{18}$$

If $p = \infty$ then

$$\sup_{0 \le x \le 1/2} (f - f_{2^{m+1}})(x) < 2 \sup_{0 \le x \le 1/2} (f - f_{2^1})(x). \tag{19}$$

The estimate (10) follows from (18) and (19).

In the case (c_{12}) we first prove (10) for $p < \infty$. Let $d = a - (1/2 - 1/2^{m+1})$. Then

$$\int_0^{1/2} (f - f_{2^{m+1}})^p(x)dx$$
$$= \Big(\int_0^{1/2-1/2^{m+1}} + \int_{1/2-1/2^{m+1}}^a + \int_a^{1/2} \Big)(f - f_{2^{m+1}})^p(x)dx \qquad (20)$$

Since $f_{2^{m+1}}(x) = f_{2^1}(x)$ for $0 \le x \le 1/2 - 1/2^{m+1}$ and $(f - f_{2^{m+1}})(x) = (f - f_{2^0})(x) < (f - f_{2^1})(x)$ for $a \le x \le 1/2$, then (20) is not more than

$$\Big(\int_0^{1/2-1/2^{m+1}} + \int_a^{1/2} \Big)(f - f_{2^1})^p(x)dx$$
$$+ \int_{1/2-1/2^{m+1}}^a (f - f_{2^0})^p(x)dx$$
$$\le \int_0^{1/2} (f - f_{2^1})^p(x)dx + \int_{1/2-1/2^{m+1}}^a (f - f_{2^0})^p(x)dx. \qquad (21)$$

To estimate the last member in (21) we use the inequality $(\alpha + \beta)^p \le 2^{p-1}(\alpha^p + \beta^p)$ and (16). Then

$$\int_{1/2-1/2^{m+1}}^a (f - f_{2^0})^p(x)dx$$
$$\le 2^{p-1}\Big(\int_{1/2-1/2^{m+1}}^a (f - f_{2^1})^p(x)dx + \int_{1/2-1/2^{m+1}}^a (f_{2^1} - f_{2^0})^p(x)dx \Big). \qquad (22)$$

Here

$$\int_{1/2-1/2^{m+1}}^a (f - f_{2^1})^p(x)dx \le \int_0^{1/2} (f - f_{2^1})^p(x)dx\,.$$

To estimate the last term of (22) we use the following property of monomials

$$\frac{\int_0^\alpha x^p dx}{\int_0^\beta x^p dx} = \frac{\alpha^{p+1}}{\beta^{p+1}}$$

and we obtain

$$\frac{\int_{1/2-1/2^{m+1}}^a (f_{2^1} - f_{2^0})^p(x)dx}{\int_0^a (f_{2^1} - f_{2^0})^p(x)dx} = \Big(\frac{d}{\frac{1}{2} - \frac{1}{2^{m+1}} + d} \Big)^{p+1}$$
$$= \Big(\frac{1}{\frac{1}{2d} - \frac{1}{2^{m+1}d} + 1} \Big)^{p+1}.$$

Since $0 < d < 1/2^{m+1}$,

$$\left(\frac{1}{\frac{1}{2d} - \frac{1}{2^{m+1}d} + 1}\right)^{p+1} < \left(\frac{1}{2^m}\right)^{p+1},$$

and it follows that

$$\int_{1/2-1/2^{m+1}}^{a} (f_{2^1} - f_{2^0})^p(x)dx < \left(\frac{1}{2^m}\right)^{p+1} \int_0^a (f_{2^1} - f_{2^0})^p(x)dx.$$

Also $(f_{2^1} - f_{2^0})(x) < (f - f_{2^0})(x)$ for $0 \le x \le a$, hence

$$\int_{1/2-1/2^{m+1}}^{a} (f_{2^1} - f_{2^0})^p(x)dx < \left(\frac{1}{2^m}\right)^{p+1} \int_0^{1/2} (f - f_{2^0})^p(x)dx.$$

Summing all the estimates we obtain

$$\int_{1/2-1/2^{m+1}}^{a} (f - f_{2^0})^p(x)dx < 2^{p-1}\left(\int_0^{1/2} (f - f_{2^1})^p(x)dx + \left(\frac{1}{2^m}\right)^{p+1} \int_0^{1/2} (f - f_{2^0})^p(x)dx\right)$$

and

$$\int_0^{1/2} (f - f_{2^{m+1}})^p(x)dx \le (1 + 2^{p-1}) \times \left(\int_0^{1/2} (f - f_{2^1})^p(x)dx + \left(\frac{1}{2^m}\right)^{p+1} \int_0^{1/2} (f - f_{2^0})^p(x)dx\right). \tag{23}$$

The corresponding inequality for the interval $[1/2, 1]$ is

$$\int_{1/2}^{1} (f - f_{2^{m+1}})^p(x)dx \le (1 + 2^{p-1}) \times \left(\int_{1/2}^{1} (f - f_{2^1})^p(x)dx + \left(\frac{1}{2^m}\right)^{p+1} \int_{1/2}^{1} (f - f_{2^0})^p(x)dx\right),$$

which can be summed with (23) to prove the case $1 \le p < \infty$.

In the case $p = \infty$ we estimate $f - f_{2^{m+1}}$ on $[0, (k-1)/2^{m+1}]$, $[(k-1)/2^{m+1}, a]$ and $[a, 1/2]$. The estimate on the intervals $[0, (k-1)/2^{m+1}]$ and $[(k-1)/2^{m+1}, a]$

follows from (14) and (15) for $k = 2^m$, and on the interval $[a, 1/2]$ from (16) for $k = 2^m$.

Let us estimate the second member in (16). Since $f_{2^0}(a) = f_{2^1}(a)$ and $f_{2^0}(x) < f_{2^1}(x)$ for $1/2 - 1/2^{m+1} < x < a$, the difference $f_{2^1} - f_{2^0}$ is a monotone decreasing function on $\big[1/2 - 1/2^{m+1}, a\big]$. Thus, for $1/2 - 1/2^{m+1} < x < a$,

$$(f_{2^1} - f_{2^0})(x) < (f_{2^1} - f_{2^0})\Big(\frac{1}{2} - \frac{1}{2^{m+1}}\Big).$$

From the homotety between the triangles with the vertices in the points

$$\Big(a, f_{2^1}(a)\Big),\ \left(\frac{1}{2} - \frac{1}{2^{m+1}}, f_{2^1}\Big(\frac{1}{2} - \frac{1}{2^{m+1}}\Big)\right),\ \left(\frac{1}{2} - \frac{1}{2^{m+1}}, f_{2^0}\Big(\frac{1}{2} - \frac{1}{2^{m+1}}\Big)\right)$$

and

$$\big(a, f_{2^1}(a)\big),\ \big(0, f_{2^1}(0)\big), \big(0, f_{2^0}(0)\big)$$

we obtain

$$\begin{aligned}(f_{2^1} - f_{2^0})\Big(\frac{1}{2} - \frac{1}{2^{m+1}}\Big) &= \frac{d}{\frac{1}{2} - \frac{1}{2^{m+1}} + d}(f_{2^1} - f_{2^0})(0) \\ &= \left(\frac{1}{\frac{1}{2d} - \frac{1}{2^{m+1}d} + 1}\right)(f_{2^1} - f_{2^0})(0).\end{aligned}$$

Let $d = a - \big(1/2 - 1/2^{m+1}\big)$. Then $0 < d < 1/2^{m+1}$, $1/2^{m+1}d > 1$ and $\frac{1}{\frac{1}{2d} - \frac{1}{2^{m+1}d} + 1} < 1/2^m$. Hence, for $1/2 - 1/2^{m+1} < x < a$, we have $(f_{2^1} - f_{2^0})(x) < (1/2^m)(f_{2^1} - f_{2^0})(0)$ and, since $(f_{2^1} - f_{2^0})(0) < (f - f_{2^0})(0)$, for $1/2 - 1/2^{m+1} < x < a$,

$$(f_{2^1} - f_{2^0})(x) < \frac{1}{2^m}(f - f_{2^0})(0) < \frac{1}{2^m} \sup_{0 \le x \le 1/2} (f - f_{2^0})(x). \tag{24}$$

Taking together the inequalities (14)–(16) and (24) we obtain

$$\sup_{0 \le x \le 1/2} (f - f_{2^{m+1}})(x) < \sup_{0 \le x \le 1/2} (f - f_{2^1})(x) + \frac{1}{2^m} \sup_{0 \le x \le 1/2} (f - f_{2^0})(x). \tag{25}$$

The corresponding inequality for the interval $[1/2, 1]$ is

$$\sup_{1/2 \le x \le 1} (f - f_{2^{m+1}})(x) < \sup_{1/2 \le x \le 1} (f - f_{2^1})(x) + \frac{1}{2^m} \sup_{1/2 \le x \le 1} (f - f_{2^0})(x). \tag{26}$$

Since $\sup_{0 \le x \le 1} f(x) = \max\big(\sup_{0 \le x \le 1/2} f(x), \sup_{1/2 \le x \le 1} f(x)\big)$, from (25) and (26) we obtain (10). $\square$

4.2 The algorithm and the estimation

For a measurable function f on $[0,1]$, we take a linear function $f^+_{2^0}$ satisfying $f(x) \geq f^+_{2^0}(x)$ if $x \in [0,1]$ and such that $\|f - f^+_{2^0}\|_p \leq 2e^+_{2^0}(f)_p$.

Then we define the piecewise–linear function $f^+_{2^1}$, linear on $[0,1/2]$ and on $[1/2,1]$, satisfying $f(x) \geq f^+_{2^1}(x)$ if $x \in [0,1]$, and such that $\|f - f^+_{2^1}\|_p \leq 2e^+_2(f)_p$.

Let us divide $[0,1/2]$ and $[1/2,1]$ in 2^m equal intervals. Using the method of the Theorem 4.2, with $Q = [0,1]$ and $f_{2^0} = f^+_{2^0}$, $f_{2^1} = f^+_{2^1}$ on $[0,1]$, we construct the piecewise–linear function $f^+_{2^{m+1}}$, linear on $[(i-1)/2^{m+1}, i/2^{m+1}]$ $(1 \leq i \leq 2^{m+1})$ such that $f^+_{2^{m+1}}(x) \geq f^+_{2^0}(x)$ if $x \in [0,1]$, and

$$\|f - f^+_{2^{m+1}}\|_p \leq c\left(\|f - f^+_{2^1}\|_p + \left(\frac{1}{2^m}\right)^{1+1/p}\|f - f^+_{2^0}\|_p\right).$$

Let us further divide every interval of length $1/2^{m+1}$ into two equal intervals and take a piecewise–linear function $f^+_{2^{2(m+1)-m}}$ satisfying

$$\left\|f - f^+_{2^{2(m+1)-m}}\right\|_p \leq 2e^+_{2^{2(m+1)-m}}(f)_p.$$

Then we divide the intervals of length $1/2^{2(m+1)-m}$ in 2^m equal intervals. We consider Theorem 4.2 with $f_{2^0} = f^+_{2^{m+1}}$, $f_{2^1} = f^+_{2^{2(m+1)-m}}$; then for every interval

$$Q_i = \left[\frac{i-1}{2^{2(m+1)-m}}, \frac{i}{2^{2(m+1)-m}}\right],\ 1 \leq i \leq 2^{2(m+1)-m}$$

we construct a piecewise–linear function $f^+_{2^{2(m+1)}}$, linear on $[(i-1)/2^{2(m+1)}, 1/2^{2(m+1)}]$ $(1 \leq i \leq 2^{2(m+1)})$ such that $f \geq f^+_{2^{2(m+1)}} \geq f^+_{2^{m+1}} \geq f^+_{2^0}$ on $[0,1]$ and

$$\left\|f - f^+_{2^{2(m+1)}}\right\|_p \leq c\left(\|f - f^+_{2^{2(m+1)-m}}\|_p + \left(\frac{1}{2^m}\right)^{1+1/p}\|f - f^+_{2^{m+1}}\|_p\right).$$

By the same method we obtain a sequence of piecewise–linear functions $f^+_{2^{n(m+1)}} \in S_{2^{n(m+1)}}$, $n \in \mathbb{N}$, satisfying $f \geq f^+_{2^{n(m+1)}} \geq f^+_{2^{(n-1)(m+1)}}$ and

$$\|f - f^+_{2^{n(m+1)}}\|_p \leq c\left(\|f - f^+_{2^{n(m+1)-m}}\|_p + \left(\frac{1}{2^m}\right)^{1+1/p}\|f - f^+_{2^{(n-1)(m+1)}}\|_p\right).$$

Let us estimate $\|f - f^+_{2^{n(m+1)}}\|_{L_p(0,1)}$ by $\|f - f^+_{2^0}\|_{L_p(0,1)}$ and $\|f - f^+_{2^{i(m+1)-m}}\|_{L_p((0,1)}$ $(0 \le i \le n)$ using recurrently the last inequality:

$$\begin{aligned}\|f - f^+_{2^{n(m+1)}}\|_p &\le c\|f - f^+_{2^{n(m+1)-m}}\|_p \\ &\quad + c^2\Big(\frac{1}{2^m}\Big)^{1+1/p}\|f - f^+_{2^{(n-1)(m+1)-m}}\|_p \\ &\quad + c^2\Big(\frac{1}{2^m}\Big)^{2(1+1/p)}\|f - f^+_{2^{(n-2)(m+1)}}\|_p \\ &< \cdots \\ &< c^n\Big(\frac{1}{2^m}\Big)^{(1+1/p)n}\|f - f^+_{2^0}\|_p \\ &\quad + c\sum_{i=0}^{n-1}\Big(c\Big(\frac{1}{2^m}\Big)^{1+1/p}\Big)^i\|f - f^+_{2^{(n-i)(m+1)-m}}\|_p.\end{aligned}$$

From the choice of $f^+_{2^{(n-i)(m+1)}}$, such that $\|f-f^+_{2^{(n-i)(m+1)}}\|_{L_p(0,1)} \le 2e^+_{2^{(n-i)(m+1)}}(f)_p$, we have

$$\begin{aligned}\|f - f^+_{2^{n(m+1)}}\|_p &\le c^n\Big(\frac{1}{2^m}\Big)^{(1+1/p)n}\|f - f^+_{2^0}\|_p \\ &\quad + c\sum_{i=0}^{n-1}\Big(c\Big(\frac{1}{2^m}\Big)^{1+1/p}\Big)^i\|f - f^+_{2^{(n-i)(m+1)-m}}\|_p,\end{aligned} \tag{27}$$

where c is a constant depending only on p.

Now we prove that, if $e^+_{2^n}(f)_p = O(2^{-\theta n})$ $(0 < \theta < 1 + 1/p)$, there exists $m \in \mathbb{N}$ such that $\|f - f^+_{2^{n(m+1)}}\|_p = O(2^{-\theta n(m+1)})$. It follows from (27) that

$$\begin{aligned}\|f - f^+_{2^{n(m+1)}}\|_p &\le c^n\Big(\frac{1}{2^m}\Big)^{(1+1/p)n} \\ &\quad + c\sum_{i=0}^{n-1}\Big(c\Big(\frac{1}{2^m}\Big)^{1+1/p}\Big)^i\Big(\frac{1}{2}\Big)^{\theta[(n-i)(m+1)-m]} \\ &= c\Big(\frac{1}{2}\Big)^{\theta n(m+1)}\Big[c^{n-1}2^{\theta n}\Big(\frac{1}{2}\Big)^{(1+1/p-\theta)nm} \\ &\quad + 2^{\theta m}\sum_{i=0}^{n-1}c^i2^{\theta i}\Big(\frac{1}{2}\Big)^{im(1+1/p-\theta)}\Big].\end{aligned}$$

If $c2^\theta/2^{(1+1/p-\theta)m} < 1$ the sum of the series is finite.

Remark 4.1. If $p = \infty$ our algorithm keeps the degree of approximation only for $0 < \theta < 1$, but in this case we can obtain another algorithm saving the degree of approximation for $0 < \theta < 2$. Let $f_{2^n} \in S_{2^n}(0,1)$ be the sequence of the piecewise–linear functions of the best uniform approximation, $\|f - f_{2^n}\|_{L_\infty(0,1)} = e_{2^n}(f)_\infty$.

Then the sequence $f^+_{2^n} = f_{2^n} - 2\sum_{k>n} e_{2^n}(f)_\infty$ satisfies $f^+_{2^0} \leq f^+_{2^1} \leq \cdots \leq f^+_{2^n} \leq f$ and the degree of approximation is saved for $0 < \theta < 2$.

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References

1. A. Andreev, V. Popov and Bl. Sendov, Jackson's type theorems for onesided polynomial and spline approximation, *C. R. Acad. Bulg. Sci.* **30** (1977), 1533–1536.
2. Yu.A. Brudnyi and N. Krugljak, About a family of approximation spaces, *Issled. Teor. Func. Mnog. Vech. Per. - Yaroslavl* (1978), 15–41.
3. L. Dechevski , $\tau-$moduli and interpolation, *Lect. Notes in Math.* **1302** (1988), 177–190.
4. E. Matvejev, About superonesided spline–approximation of functions of several variables, *Izv. Vuzov. Matematica* **6** (1988), 49–54.
5. V. Popov, Function spaces, generated by the average modulus of smoothness, *Pliska Stud. Math. Bulg.* **5** (1983), 132–143.
6. V. Popov and A. Andreev, Stechkin–type theorems for onesided trigonometrical and spline approximation, *C. R. Acad. Bulg. Sci.* **31** (1978), 151–154.
7. V. Popov, Onesided K–functional and its interpolation spaces, *Tr. MIAN USSR* **163** (1984), 196–199.
8. A. Shadrin, About monotone approximation of functions by trigonometrical polynomials, *Mat. Zametki* **34**:3 (1983), 375–386.
9. S. Stechkin, Constructive theory of functions, *Sofia* (1983), 595–598.
10. G. Totkov, Direct and converse theorems for best onesided approximation in spaces $L_p(0,1)$, $1 \leq p \leq \infty$, *Nauchn. Tr. Inst. Xranit. i Bkus. Prom.* **28**:1 (1981), 183–191.
11. G. Totkov, Converse theorems for the onesided spline approximation, *C. R. Acad. Bulg. Sci.* **32**:7 (1979), 875–878.