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# Extrapolation of Sobolev imbeddings

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Abstract

We survey recent results on limiting imbeddings of Sobolev spaces, particularly, those concerning weakening of assumptions on integrability of derivatives, considering spaces with dominating mixed derivatives and the case of weighted spaces.

## 1. A quintessence

This survey deals with some of the recent results on limiting imbeddings of Sobolev spaces and their generalizations. As is well known the limiting imbedding theorem for Sobolev spaces states ([42], [35]) that $W^{m,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary, $1 < p < \infty$, and $mp = N$, is imbedded into an Orlicz space $L_\Phi(\Omega)$ with $\Phi(t) = \exp t^{N/(N-m)} - 1$. Since $W^{m,p} \hookrightarrow L_q(\Omega)$ if $mp = N$ and $q$ is an arbitrary finite positive number the limiting imbedding can be considered as an improvement or a refinement of these imbeddings in the framework of a finer scale of target spaces, namely of Orlicz spaces. We recall that the limiting imbedding is also sharp in terms of usual ordering of Young functions ($\Phi_1 \prec \Phi_2$ if $\lim_{t\to\infty} \Phi_1(\lambda t)/\Phi_2(t) = 0$ for all $\lambda > 0$). Different techniques of proofs have usually a common underlying idea, namely, $u \in L_\Phi(\Omega)$, with $\Phi$ as above if

$$\|u|L_q\| \le cq^{1-m/N} \tag{1.1}$$

for $\|u|W^{m,p}\| \leq 1$, where $c$ is independent of $u$ and $q$, that is, equivalently, if

$$\sup_{1<q<\infty} q^{-(1-m/N)}\|u|L_q\| < \infty. \tag{1.2}$$

Similarly, the condition

$$\lim_{p\to\infty} \frac{1}{p}\biggl(\int\limits_\Omega |u(x)|^{Np/(N-m)}\,dx\biggr)^{1/p} < \infty$$

(see [16]) can be employed instead. This follows easily by expanding $\Phi$ into its Taylor series and using Stirling's formula. (Let us observe that the same argument can be used to show that the quantity in (1.2) is equivalent to the (Luxemburg) norm in $L_\Phi(\Omega)$.)

The desired estimates of type (1.1) have been obtained for various spaces by different methods in particular cases, including analysis of the asymptotic behavior of convolution integrals with Riesz kernels ([42], [35], [19], [1]) or making use of the Fourier analysis approach ([34]).

On a more abstract level one can use a language of extrapolation spaces as in [26]. There are certain constraints, however, independent of the kind of approach – likely no reasonable systematic treatment exists for spaces characterized by more general control of the balance of speed at which $\|u|L_q\|$ tends to infinity as $q \to \infty$ than powers of $q$ as in (1.2). This seems to be a rather difficult problem, linking the classical analysis with contemporary efficient tools and techniques for dealing with function spaces.

We also observe that there are recent papers on a "dual" problem to weakening of integrability conditions, namely, a slight strengthening of the integrability of derivatives, leading to a modified sort of exponential Orlicz spaces – see, e.g. [11].

We shall pursue three directions here, weakening of the standard limiting assumptions, namely, weaker integrability conditions (logarithmic Sobolev spaces), neglecting some of the highest derivatives (Sobolev spaces with dominating mixed derivatives), and finally, plugging weight functions into limiting considerations.

The effect of the assumption

$$\int\limits_\Omega |\nabla u(x)|^N \log^{-\sigma}(e + |\nabla u(x)|)\,dx < \infty \tag{1.3}$$

for some $\sigma > 0$ was considered by Fusco, Lions and Sbordone [16]. They have shown that $u \in L_1(\Omega)$, satisfying (1.3), belongs to $L_{\Phi_\sigma}(\Omega)$ as a target space, where

$\Phi_\sigma(t) \sim \exp t^{N/(N-1+\sigma)}$. This has been generalized to potential logarithmic Sobolev spaces of arbitrary order by Edmunds and Krbec [12]; at the same time a natural link has been found in [12] between the limiting theorems of type we have been discussing up to now and extrapolation behavior of imbeddings of potential Sobolev spaces $H^{\ell,r}(\Omega)$ when $\ell r \nearrow p+N$; the space $H^{k,s}(\Omega)$ is imbedded into every space $\mathcal{H}^\lambda(\Omega)$ of Hölder continuous functions provided $ks = p+N$, but a function in $W^{k,s}(\Omega)$ need not be generally Lipschitz continuous. Hence there is a certain affinity and one could naturally ask about more precise description of differentiation properties. Brézis and Wainger [9] have proved the very interesting estimate

$$\frac{|u(x)-u(y)|}{|x-y|} \leq c\|u|H^{k,s}\|\,|\log|x-y|\,|^{1-1/s} \tag{1.4}$$

for $u \in H^{k,s}(\mathbb{R}^N)$, $ks = p+N$. Invoking the lifting properties of Riesz potentials and the Donaldson and Trudinger imbedding theorem for Orlicz-Sobolev spaces it has been shown in [12] that (1.4) and its more general variants are a consequence of limiting theorems for $W^{p/s+N/s-1,s}(\Omega)$, throwing thus some light into connection between extrapolation in different scales of function spaces.

The story about weakening of the differentiability properties starts with Adams' paper [3] and spaces considered by Besov, Il'in and Nikol'skii [7]; it has been shown that neglecting of some highest order derivatives does not affect the target space in analogs of Sobolev imbedding theorems. For instance, if $m \in \mathbb{N}$, $mp < N$, we put

$$M(N,m) = \{\alpha;\ |\alpha| = m,\ \alpha_i = 0 \text{ or } \alpha_i = 1 \text{ for } 1 \leq i \leq N\},$$

and if $W_p^{M(N,m)}(\mathbb{R}^N)$ is the space of all $f \in L_p(\mathbb{R}^N)$ such that $D^\beta \in M(N,m)$ for every $\beta \leq \alpha$, $\alpha \in M(N,m)$ with the norm

$$\|u|W_p^{M(N,m)}\| = \sum_{\substack{|\beta|\leq|\alpha| \\ \alpha\in M(N,m)}} \|D^\alpha u|L_p(\mathbb{R}^N)\|,$$

then $W_p^{M(N,m)}(\mathbb{R}^N) \hookrightarrow L_q(\mathbb{R}^N)$ for $1/q = 1/p - m/N$ (see [3]). If $p \nearrow N/m$, then the norms of these imbeddings behave in a different way than in the case of $W^{m,p}(\mathbb{R}^N)$, giving $W_p^{M(N,m)}(\mathbb{R}^N) \hookrightarrow L_\Phi(\mathbb{R}^N)$ with Young's function equivalent to $\exp t^{N/m(N-m)}$ and this imbedding is sharp in the given scale. The Sobolev spaces dealt with in Section 3 are much more general than those just described, including use of mixed norms, but such setting turns out to be essential and quite natural, in particular, the "classical" interpretation of the spaces involved, defined via Fourier analysis approach ($F$-spaces), puts them into standard scales of function spaces. The key estimate of type (1.1) relies heavily on Nikol'skii's inequality and interpolation technique developed for the spaces with dominating mixed smoothness in Schmeisser [30], [32], Schmeisser and Triebel [33].

The concluding Section 4 deals with weighted limiting imbeddings in a bounded domain, specifically, in a unit ball $B \subset \mathbb{R}^N$. The weights involved are products of powers of $|x|$ and powers of $|\log|x||$, covering thus the particularly important spaces where weights are powers of distance to the boundary of the domain in question. A curious difference between imbeddings from Section 3 and 4 is that here the exponential Young's function for the target space has the same growth at infinity as without weights. This does not seem to be straightforward since $W_0^{1,p}(B) \hookrightarrow L_q(B)$ for $p < N$, $q = Np/(N-p)$, and, for $\gamma < 0$, Hölder's inequality gives $W_0^{1,p}(B) \hookrightarrow L_q(B, |x|^\gamma)$, where $p < r < (N+\gamma)p/(N-p)$ (that is, $\gamma > -N - r + Nr/p$). Nevertheless, the extrapolation squares this difference. The analysis of weighted counterparts of (1.1) is based on weighted version of Schur's lemma for integral operators with non-negative kernels, this is applied to Riesz potentials (of first order). Together with a careful study of behavior of constants in inequalities resulting from requirements on the boundedness of potentials in corresponding weighted spaces one gets weighted analogs of (1.1). Besides that, necessary conditions for our limiting imbeddings can be found, too, based essentially on necessary conditions for boundedness of Hardy's operator in weighted spaces (see, e.g. [25]), making the picture complete in the framework of weighted spaces considered. An interested reader finds all the details in [24].

## 2. Limiting imbeddings for logarithmic Sobolev spaces

In this section $\Omega$ will denote a bounded domain with the extension property with respect to Orlicz-Sobolev spaces.

We shall consider the (potential) Orlicz-Sobolev spaces $H^s_\Phi(\mathbb{R}^N)$ as isomorphic copies of $L_\Phi(\mathbb{R}^N)$ under the mapping $g \mapsto J_s * g$, where $J_s$ is Bessel's potentials of order $s$, $s \in \mathbb{R}^1$, and $\Phi$ is Young's function. The space $H^s_\Phi(\Omega)$ is defined as the restriction of $H^s_\Phi(\mathbb{R}^N)$ to $\Omega$ with a factor norm. The symbol $H^s_\Phi$ will be used instead of $H^s_\Phi(\Omega)$. The classical interpretation of these spaces is discussed in [12]; if the smoothness is a positive integer and $\Omega$ has the extension property mentioned above, they coincide with Orlicz–Sobolev spaces defined in the "classical" way via generalized derivatives. For $1 < p < \infty$ and $\sigma \geq 0$, we denote $M_\sigma(t) = t^p \log^{-\sigma}(e+t)$, $t \geq 0$.

The effect of enlarging the Sobolev space in the limiting situation by adding a logarithmic term with a negative power to the function $t^{N/p}$ is described in the following theorem, see [12].

**Theorem 2.1**

*Let $u \in H^{N/p}_{M_\sigma}$. Then $u \in L_\Phi$ with $\Phi(t) = \exp t^\alpha - 1$ where $\alpha = p/(p-1+\sigma)$ and there exists $c > 0$ independent of $u$ such that*

$$\|u|E_\Phi\| \le c\|u|H^{N/p}_{M_\sigma}\|.$$

For the proof one has first to cope with the unpleasant fact that the convolution with the Bessel kernels does not preserve supports and our spaces are defined as restrictions of spaces on $\mathbb{R}^N$. Nevertheless, this can be solved (details in [12]); one can assume without loss of generality that $u = J_{N/p} * g$ for some $g \in L_{M_\sigma}(\mathbb{R}^N)$. Next the membership of $u$ in the target space is established once we are able to show that

$$\lim_{\varepsilon\to 0_+} \varepsilon^{1/\alpha}\|u|L_{1/\varepsilon}\| = 0 \tag{2.1}$$

(see Section 1). To this end, we use the estimate due to Strichartz [35],

$$\|u|L_q\| \le c\left(1 + \frac{q}{r'}\right)^{1/q+1/r'} \|g|L_r\| \tag{2.2}$$

where $q \ge r > 1$, $r' = r/(r-1)$. The estimate 2.2 yields, for small $\varepsilon$,

$$\varepsilon^{1/\alpha}\|u|L_{1/\varepsilon}\| \le c\left(1 + \frac{p-\varepsilon-1}{\varepsilon(p-\varepsilon)}\right)^{1+\varepsilon-1/p-\varepsilon} \varepsilon^{1/\alpha}\|g|L_{p-\varepsilon}\|$$

Estimating the behavior of the right hand side as $\varepsilon \to 0$ gives (2.1). We get also the boundedness of the inclusion into $L_{M_\sigma}$ and as a by-product that $u$ actually belongs to the closure of bounded measurable functions in $L_\Phi$.

This imbedding implies the celebrated theorem due to Brézis and Wainger [9] on imbeddings into spaces of “almost Lipschitz continuous” functions.

Indeed, if $u \in H^{1+N/p}_{M_\sigma}$, then by virtue of Theorem 2.1, $\nabla u \in L_{\Phi_\alpha}$ with $\Phi_\alpha(t) = \exp t^\alpha - 1$, $\alpha = p/(p-1+\sigma)$, and $u \in H^1_{\Phi_\alpha}$. According to known theorems due to Donaldson and Trudinger for Orlicz–Sobolev spaces on imbeddings into generalized Hölder spaces, see e.g. [2], Theorem 8.36, we have, for $z = |x-y|^{-N}$, $z$ large,

$$\begin{aligned}\int_z^\infty \frac{\log^{1/\alpha}(1+t)}{t^{1+1/N}}\,dt &= -N\left[t^{-1/N}\log^{1/\alpha}(1+t)\right]_z^\infty + \frac{N}{\alpha}\int_z^\infty \frac{\log^{-1+1/\alpha}(1+t)}{t^{1/N}(1+t)}\,dt \\ &\le Nz^{-1/N}\log^{1/\alpha}(1+z) + \frac{N}{\alpha}\int_z^\infty \frac{\log^{1/\alpha}(1+t)}{t^{1/N}(1+t)}\,dt.\end{aligned}$$

Estimating of the integral on the right hand side leads to

**Theorem 2.2** ([12])

*Let $u \in H^{1+N/p}_{M_\sigma}$ for some $\sigma \geq 0$. Then there is $c > 0$ independent of $u$ such that*

$$|u(x) - u(y)| \leq c\|u|H^{1+N/p}_{M_\sigma}\| \, |x-y| \, |\log|x-y|\,|^{1/\alpha},$$

*for all $x, y \in \Omega$, where $\alpha = p/(p-1+\sigma)$.*

Theorem 3.2 can be extended to spaces on $\mathbb{R}^N$.

The very same idea can be used then to prove more general statements, when the the spaces involved are based for instance on Lorentz spaces instead of Lebesgue spaces (a survey of the corresponding Sobolev (sublimiting) imbeddings can be found e.g. in [4]). Thus if $\nabla u^2$ is for instance in the the Marcinkiewicz space $L_{N,q}$, $1 \leq q \leq \infty$, and if $u$ is supported in $\Omega$, then

$$|u(x) - u(y)| \leq c\|\nabla^2 u|L_{N,q}\| \, |x-y| \, |\log|x-y|\,|^{1-1/q}.$$

## 3. Limiting imbeddings of Sobolev spaces with dominating mixed derivatives

We introduce more general spaces than in Section 1. First, let $n_1 + \ldots + n_m = N$ and

$$\mathcal{S}(N, n_1, \ldots, n_m) = \{\alpha = (\alpha^1, \ldots, \alpha^m);\ \alpha^j = (\alpha^j_1, \ldots, \alpha^j_{n_j}),\ |\alpha^j| = 1,\ 1 \leq j \leq m\}.$$

Put $\mathcal{S}(N, m) = \mathcal{S}(N, n_1, \ldots, n_m)$ when $n_1 = \ldots = n_m = n$, that is, $N = m \cdot n$.

If $\mathcal{M} \subset M(N, m)$, we define the *reduced Sobolev space*

$$W^{\mathcal{M}}_p(\mathbb{R}^N) = \{f \in L_p;\ D^\beta f \in L_p \text{ for all } \beta \leq \alpha \text{ and all } \alpha \in \mathcal{M}\}.$$

In some cases the spaces $W^{\mathcal{M}}_p(\mathbb{R}^N)$ can be considered as Sobolev spaces with dominating mixed derivatives, studied in [30], [32], [33].

Let $\mathbb{R}^N = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$. A point $x \in \mathbb{R}^N$ will be written as $x = (x^1, \ldots, x^m)$ with $x^j \in \mathbb{R}^{n_j}$, $1 \leq j \leq m$. The symbol $L_{\bar{p}}(\mathbb{R}^N)$ will stand for the mixed norm Lebesgue space (the integration in order $x^1, \ldots, x^m$), similarly for the sequence spaces $\ell_{\bar{p}}$ whose elements are $(a_\kappa)$, $\kappa = (\kappa_1, \ldots, \kappa_m)$.

Let $\{\phi_k^j(\xi^j)\}$ be the standard resolution of unity associated to dyadic balls in $\mathbb{R}^{n_j}$, $1 \le j \le m$, that is, a sequence of functions $\{\phi_k^j\}$ such that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \phi_k^j(\xi^j) \equiv 1 \qquad \text{in } \mathbb{R}^{n_j},\\
&\operatorname{supp} \phi_0^j \subset \{\xi^j \in \mathbb{R}^{n_j};\ |\xi^j| \le 2\},\\
&\operatorname{supp} \phi_k^j \subset \{\xi^j \in \mathbb{R}^{n_j};\ 2^{k-1} \le |\xi^j| \le 2^{k+1}\},
\end{aligned}$$

with $\phi_k^j(\xi^j) = \phi^j(2^{-k}\xi^j)$, $k = 1, 2, \ldots$, where $\phi$ is a $C^\infty$ function whose support is contained in $\{1/2 \le |\xi^j| \le 2\}$.

If $f$ is a tempered distribution in $\mathbb{R}^N$, we put

$$f_k(x) = f_{k_1,\ldots,k_m}(x^1, \ldots, x^m) = \mathcal{F}^{-1}[(\phi_{k_1}^1 \otimes \ldots \otimes \phi_{k_m}^m)\mathcal{F}f](x^1, \ldots, x^m).$$

For $\bar{p} = (p_1, \ldots, p_m)$, $\bar{q} = (q_1, \ldots, q_m)$, $\bar{r} = (r_1, \ldots, r_m)$ with $0 < p_j < \infty$, $0 < q_j \le \infty$, and $-\infty < r_j < \infty$, $j = 1, \ldots, m$, we define

$$\begin{aligned}
S_{\bar{p},\bar{q}}^{\bar{r}} F(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}) = \Big\{ &f \in \mathcal{S}'(\mathbb{R}^N);\\
&\|f|S_{\bar{p},\bar{q}}^{\bar{r}}F\| = \|2^{k_1 r_1 + \ldots + k_m r_m} f_{k_1,\ldots,k_m}(x)|\ell_{\bar{p}}|L_{\bar{q}}\| < \infty \Big\}.
\end{aligned}$$

We refer to [30] for details. The model case $N = 2$, $n_1 = n_2 = 1$ can be found in [33], Chapter 2. Analogously, one can define spaces of Besov type $S_{\bar{p},\bar{q}}^{\bar{r}} B(\mathbb{R}^N)$; for $n_1 = \ldots = n_m = 1$, $r_j \ge 0$, $p_j \ge 1$, $q_j \ge 1$ they have been studied in Amanov's book [5] (see also the literature quoted there). In the case $0 < p_1 = \ldots = p_m < \infty$, $0 < q_1 = \ldots = q_m \le \infty$, $n_1 = \ldots = n_m = 1$, the spaces $S_{\bar{p},\bar{q}}^{\bar{r}} B(\mathbb{R}^N)$ and $S_{\bar{p},\bar{q}}^{\bar{r}} F(\mathbb{R}^N)$ can be described as special cases of non-homogeneous general function spaces in the sense of [38–40].

The spaces introduced include scales of known spaces of functions with generalized derivatives. If $-\infty < r_j < \infty$, $j = 1, \ldots, m$, then

$$S_{\bar{p},\bar{2}}^{\bar{r}} F(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}) = S_{\bar{p}}^{\bar{r}} H(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m})$$

(equivalent norms), where

$$\begin{aligned}
S_{\bar{p}}^{\bar{r}} H(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}) = \Big\{ &f \in \mathcal{S}'(\mathbb{R}^N);\\
&\|f|S_{\bar{p}}^{\bar{r}}H\| = \|\mathcal{F}^{-1}[(1 + |\xi^1|^2)^{r_1/2} \ldots (1 + |\xi^m|^2)^{r_m/2}\mathcal{F}f]|L_p\| < \infty \Big\};
\end{aligned}$$

if $r_j$ $(j = 1, \ldots, m)$ are non-negative integers, then

$$S^{\bar r}_{\bar p,\bar 2}F(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}) = S^{\bar r}_{\bar p}W(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m})$$

(equivalent norms), where

$$\begin{aligned} S^{\bar r}_{\bar p}W(\mathbb{R}^{n_1}\times \ldots \times \mathbb{R}^{n_m}) = \Big\{ & f \in L_{\bar p}(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}); \\ & \|f|S^{\bar r}_{\bar p}W\| = \sum_{\substack{\theta_j=0,1;\ |\alpha^j|=r_j \\ j=1,\ldots,m}} \|D^{(\theta_1\alpha^1,\ldots,\theta_m\alpha^m)} f|L_{\bar p}\| < \infty \Big\}. \end{aligned}$$

(See [30], Theorem A.2, [31], Thm. 3 and [33], Thm. 2.3.1.)

The basic simplified setting of the problem of extrapolation of imbeddings into $L_{\bar q}(\mathbb{R}^N)$ is subject of the following theorem:

**Theorem 3.1**

*Let $1 < p_j < q_j < \infty$, $r_j = n_j/p_j$ $(j = 1, \ldots, m)$. If $2 \le p_1 \le p_2 \le \ldots \le p_m$, then there exists a constant $c$ independent of $\bar q$ and $f$ such that*

$$\|f|L_{\bar q}(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m})\| \le c \prod_{j=1}^{m} q_j^{1-1/p_j} \|f|S^{\bar r}_{\bar p,\bar 2}F(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m})\|$$

*for all $f \in S^{\bar r}_{\bar p,\bar 2}F(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m})$.*

We shall touch the main steps of the proof. First, let $(f_{k_1,\ldots,k_m}(x^1, \ldots, x^m))$ be the decomposition of $f$, then

$$\begin{aligned} \|f|L_{\bar q}\| &= \Big\| \sum_{k_m=0}^{\infty} \ldots \sum_{k_1=0}^{\infty} f_{k_1,\ldots,k_m}(x^1, \ldots, x^m)|L_{\bar q}\Big\| \\ &\le \|f_{k_1,\ldots,k_m}(x^1, \ldots, x^m)|\ell_{1|k_1}(L_{q_1})| \ldots |\ell_{1|k_m}(L_{q_m})\|, \end{aligned}$$

where $\ell_{\kappa_j|k_j}(L_{q_j})$ refers to integration with respect to $x^j \in \mathbb{R}^{n_j}$ and taking the $\ell_{\kappa_j}$ norm with respect to $k_j$. Applying Nikolskii's inequality to the right hand side we obtain

$$\|f|L_{\bar q}\| \le c\Big\| \prod_{j=1}^{m} 2^{k_j n_j(1/p_j-1/q_j)} f_{k_1,\ldots,k_m}(x^1, \ldots, x^m)|\ell_{1|k_1}(L_{p_1})| \ldots |\ell_{1|k_m}(L_{p_m})\Big\|,$$

where $c$ is independent of $\bar{q}$. Next, repeating use of Hölder inequality shows after some effort that

$$\|f\|_{L_{\bar{q}}} \le c \prod_{j=1}^{m} q_j^{1-1/p_j} \Big\| \prod_{j=1}^{m} 2^{k_j n_j/p_j} f_{k_1,\dots,k_m} |\ell_{p_1}(L_{p_1})| \dots |\ell_{p_m}(L_{p_m})\Big\|.$$

Hence after interchanging the order of summing up and integration in the above mixed norms (this is possible thanks to the assumption about ordering of the exponents $p_1, \dots, p_m$) we arrive at

$$\begin{aligned}\|f|L_{\bar{q}}\| &\le c \prod_{j=1}^{m} q_j^{1-1/p_j} \Big\| \prod_{j=1}^{m} 2^{k_j n_j/p_j} f_{k_1,\dots,k_m} |\ell_{\bar{p}}|L_{\bar{p}}\Big\| \\ &\le c \prod_{j=1}^{m} q_j^{1-1/p_j} \|f|S^{\bar{r}}_{\bar{p},\bar{2}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})\|.\end{aligned}$$

The remaining cases of ordering of the exponents $p_j$ are more difficult to handle. A basic reduction to $m = 2$ can always be done. Then one has to use, besides Nikol'skii's inequality, also a suitable combination of tools from interpolation theory (real and complex interpolation of vector-valued spaces, in particular, the proofs rely on [20], [28], [29] [30], [33], [36]); the crucial point is that the $\|f|L_{\bar{q}}\|$ norms in the remaining cases can be shown to blow up at the same speed as in Theorem 3.1. The reference for full proofs is [23].

Moreover, all these estimates are sharp with respect to the exponents $1 - 1/p_j$. This follows by modification of examples in Edmunds and Triebel [13] or [14], Subsection 2.7.1.

It will be illustrative to formulate at least one special case of these estimates – the connections of $S^{\bar{r}}_{\bar{p},\bar{2}}F$ spaces with Sobolev spaces defined in a standard manner via generalized derivatives were briefly discussed at the beginning of this section.

**Theorem 3.2**

*Let $1 \le p_j < q_j < \infty$, $r_j = n_j/p_j$ $(j = 1, \dots, m)$. Then there exists a constant $c$ independent of $q$ and $f$ such that*

$$\|f|L_q\| \le c q^{m - \sum_{j=1}^{m} 1/p_j} \|f|S^{\bar{r}}_{\bar{p},\bar{2}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})\|$$

*holds for every $q > \max_{j=1,\dots,m} p_j$.*

Some special "classical" spaces with integer derivatives permit use of more elementary methods to get limiting imbeddings. This is also dealt with in [23].

## 4. Limiting imbeddings of weighted Sobolev spaces

As observed in the foregoing section the key is a detailed analysis of what is going on with the norms of sublimiting imbeddings, trying to find an optimal way of control of their blow up.

Let $\Omega \subset \mathbb{R}^N$ be a measurable subset of the $\mathbb{R}^N$, $N \geq 3$. To simplify notation and formulas we shall suppose that all the functions in the sequel will be non-negative. Let $K : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ and put, for $\alpha > 0$,

$$T_\alpha f(x) = \int_\Omega K(x,y)^\alpha f(y)\,dy;$$

denote $T = T_1$. Let $T_\alpha^*$ be the dual operator to $T_\alpha$. If $v$ and $w$ are weights in $\Omega$, then the inequality

$$\left(\int_\Omega |Tf(x)|^q w(x)\,dx\right)^{1/q} \leq c\left(\int_\Omega |f(x)|^p v(x)\,dx\right)^{1/p}, \tag{4.1}$$

holds for each non-negative $f$ iff

$$\left(\int_\Omega |T^* f(x)|^{p'} v(x)^{-p'/p}\,dx\right)^{1/p'} \leq c\left(\int_\Omega |f(x)|^{q'} w(x)^{-q'/q}\,dx\right)^{1/q'}. \tag{4.2}$$

The best constants in (4.1) and (4.2) coincide.

If $\Phi$ is a “classical” Young function (the $N$-function in the sense of [22], Chapter 1) and if $w_1, w_2$ are weights in $\Omega$, then $L_\Phi(\Omega, w_1, w_2)$ will stand for the weighted Orlicz space generated by the modular

$$m_\Phi(f, w_1, w_2) = \int_\Omega \Phi(w_1(x)f(x))w_2(x)\,dx.$$

We shall simply write $L_\Phi(\Omega)$ if $w_1 \equiv w_2 \equiv 1$.

A sufficient condition for (4.1) is

**Lemma 4.1**

*Let $1 < p \leq q < \infty$ and let $r = r(p,q)$ be defined by*

$$\frac{1}{r} = \frac{1}{p'} + \frac{1}{q}.$$

*Assume that there exist a positive function $\varphi$ and $c > 0$ such that*

$$T_r^*[w(T_r\varphi)^{q/p'}](x) \leq cv(x)^{q/p}\varphi(x)^{q/p'} \qquad \text{a.e. in } \Omega.$$

*Then (4.1) holds with $c_1^* \leq c^{1/q}$.*

For $N = 1$, $\Omega = (0, \infty)$, and $Tf(x) = \int_0^x f(t)\,dt$, we get the condition due to Gurka [18]. The case $p = q$ in Lemma 4.1 can be found in Kerman and Sawyer [21]. In both cases the existence of a positive function $\varphi$ satisfying the condition of Lemma 4.1 is also necessary for (4.1).

**Theorem 4.2**

*Let $r$ be defined as in Lemma 4.1 and suppose that there exist a positive function $\varphi$ and a constant $c > 0$ such that*

$$T_r\left[v^{-p'/p}(T_r^*\varphi)^{p'/q}\right](x) \le cw(x)^{-p'/q}\varphi(x)^{p'/q} \quad \text{a.e. in } \Omega.$$

*Let $c_1^*$ be the best constant in* (4.1). *Then* (4.1) *holds with $c_1^* \le c^{1/p'}$.*

Applying Theorem 4.2 to the Riesz potentials of first order

$$If(x) = \int_B |x-y|^{1-N} f(y)\,dy$$

where $B$ is the unit ball in $\mathbb{R}^N$, $N \ge 3$, we get

**Corollary 4.3**

*Let $r$ be as in Lemma 4.1 and suppose that there is a positive function $\varphi$ and $C > 0$ such that*

$$C = \sup_{x\in B}\left[\frac{w(x)}{\varphi(x)}\right]^{N'/q} \int_B |x-y|^{r(1-N)} v(y)^{-N'/N} \times \\ \times \left(\int_B |y-z|^{r(1-N)}\varphi(z)\,dz\right)^{N'/q} dy < \infty.$$

*Then*

$$\left(\int_B |If(x)|^q w(x)\,dx\right)^{1/q} \le C^{1/N'}\left(\int_B |f(x)|^N v(x)\,dx\right)^{1/N}.$$

We shall consider radial weights, which are combinations of powers of $|x|$ and powers of $|\log|x||$. The first type of a weighted inequality is

$$\left(\int_B \left| |x|^\varepsilon f(x) \right|^q |x|^\eta \, dx\right)^{1/q} \le c_q \left(\int_B |\nabla f(x)|^N |x|^\delta \, dx\right)^{1/N} \tag{4.3}$$

for $f \in C_0^\infty(B)$, $q \ge q_0$, where $\delta$, $\varepsilon$, and $\eta$ are real valued parameters. Plugging in the known conditions for validity of Hardy's inequality (cf. e.g. [25], Subsection 1.3.1) we arrive at necessary conditions for (4.1) to hold with some finite $c_q$, depending on the range for the parameter $\eta$, specifically,

(i) For $\eta > -N$:

$$\varepsilon \ge 0 \text{ if } \delta \le 0 \text{ and } \varepsilon \ge \delta/N \text{ if } \delta > 0. \tag{4.4}$$

(ii) For $\eta = -N$:

$$\varepsilon > 0 \text{ if } \delta \le 0 \text{ and } \varepsilon \ge \delta/N \text{ if } \delta > 0. \tag{4.5}$$

(iii) For $\eta < -N$:

$$\varepsilon > 0 \text{ if } \delta \le 0 \text{ and } \varepsilon > \delta/N \text{ if } \delta > 0. \tag{4.6}$$

Analogously, one can prove that necessary for

$$\begin{aligned} &\left(\int_B \left| \left(\log \frac{e}{|x|}\right)^\varepsilon f(x) \right|^q \left(\log \frac{e}{|x|}\right)^{\eta-1} |x|^{-N} \, dx\right)^{1/q} \\ &\qquad \le c_q \left(\int_B |\nabla f(x)|^N \left(\log \frac{e}{|x|}\right)^\delta \, dx\right)^{1/N} \end{aligned} \tag{4.7}$$

for $f \in C_0^\infty(B)$, $q \ge q_0$, and real parameters $\delta$, $\varepsilon$, $\eta$, are the conditions

(iv) For $\eta < 0$:

$$\varepsilon \le 0 \text{ if } \delta \ge N-1 \text{ and } \varepsilon \le \delta/N - 1/N' \text{ if } \delta < N-1. \tag{4.8}$$

(v) For $\eta = 0$:

$$\varepsilon < 0 \text{ if } \delta \ge N-1 \text{ and } \varepsilon \le \delta/N - 1/N' \text{ if } \delta < N-1. \tag{4.9}$$

(vi) For $\eta > 0$:

(4.10) $$\varepsilon < 0 \text{ if } \delta \geq N - 1 \text{ and } \varepsilon < \delta/N - 1/N' \text{ if } \delta < N - 1.$$

Let us introduce additional notation. For a non-negative integer $k$ we define Young's function

$$\Phi_k(t) = \sum_{j=k}^{\infty} \frac{t^{N'j}}{j!}, \qquad t \in \mathbb{R}^1.$$

We are now ready to formulate the main results in [24].

**Theorem 4.4**

*Let $\delta$, $\varepsilon$, $\eta \in \mathbb{R}^1$ and $\delta < N(N-1)$. Then there is $q_0 \geq 1$ such that* (4.1) *holds for every $f \in C_0^\infty(B)$ and every $q \geq q_0$ if and only if any of the conditions* (4.4), (4.5), (4.6) *holds.*

*Moreover, if $c_q^*$ denotes the best possible constant in* (4.1)*, then*

(4.11) $$c_q^* \leq c q^{1/N'}, \qquad q \geq q_0.$$

*The constant $c$ in* (4.11) *may depend on $\delta$, $\varepsilon$, $\eta$, $n$ and $q_0$, but it is independent of $q$.*

The previous theorem reads in terms of imbeddings into exponential Orlicz spaces as

**Corollary 4.5**

*Let $\delta$, $\varepsilon$, $\eta \in \mathbb{R}^1$ and $\delta < N(N-1)$. Then there is $k \in \mathbb{N}$ such that the imbedding*

$$W_0^{1,n}(B, |x|^\delta) \hookrightarrow L_{\Phi_k}(B, |x|^\varepsilon, |x|^\eta)$$

*holds if and only if any of the conditions* (4.4), (4.5), (4.6) *holds.*

The conclusions concerning the imbeddings in (4.7) are analogously formulated as

**Theorem 4.6**

*Let $\delta, \varepsilon, \eta \in \mathbb{R}^1$. Then there is $q_0$ such that* (4.7) *holds for every $f \in C_0^\infty(B)$ and every $q \geq q_0$ if and only if any of* (4.8), (4.9), (4.10) *holds.*

*If $c_q^*$ denotes the best constant in* (4.7)*, then*

(4.12) $$c_q^* \leq c q^{1/N'}.$$

*The constant $c$ in* (4.20) *may depend on $\delta$, $\varepsilon$, $\eta$, $n$ and $q_0$, but it is independent of $q$.*

**Corollary 4.8**

*Let $\delta$, $\varepsilon$, $\eta \in \mathbb{R}^1$ and $\delta < N(N-1)$. Then there is $k \in \mathbb{N}$ such that the imbedding*

$$W_0^{1,n}\left(B, \left(\log \frac{e}{|x|}\right)^{\delta}\right) \hookrightarrow L_{\Phi_k}\left(B, \left(\log \frac{e}{|x|}\right)^{\varepsilon}, |x|^{-N}\left(\log \frac{e}{|x|}\right)^{\eta-1}\right)$$

*holds if and only if any of the conditions* (4.8), (4.9), (4.10) *holds.*

The way to Theorems 4.4 and 4.6 is rather long. We shall restrict ourselves only to main points. Of course the estimates for gradients depend on corresponding estimates for Riesz potentials in a standard way. The very general Corollary 4.3 contains the assumption about existence of a suitable function $\varphi$, hence to derive in this way the desired estimates consists in finding a "good" $\varphi$ as the first step. This can be done, for instance, when the weights are those considered in Theorems 4.4 and 4.6. The explicit dependence of the norms of the imbeddings is given by

**Theorem 4.9**

*Let*

$$If(x) = \int_B |x-y|^{1-N} f(y)\, dy, \qquad f \geq 0, \quad f \in L^1_{\mathrm{loc}}.$$

*(i) If $-N < \delta \leq 0$, then*

$$\left(\int_B |If(x)|^q |x|^{\delta}\, dx\right)^{1/q} \leq c q^{1/N'} \left(\int_B |f(x)|^N\, dx\right)^{1/N}$$

*for large $q$'s, with $c$ independent of $f$ and $q$. (ii) If $0 < \delta < N(N-1)$, then*

$$\left(\int_B \left| |x|^{\delta/N} If(x)\right|^q |x|^{-N}\, dx\right)^{1/q} \leq c q^{1/N'} \left(\int_B |f(x)|^N |x|^{\delta}\, dx\right)^{1/N}$$

*for large $q$'s, with $c$ independent of $f$ and $q$. (iii) If $\delta < N-1$, then*

$$\left(\int_B \left| \left(\log \frac{e}{|x|}\right)^{\delta/N - 1/N'} If(x)\right|^q |x|^{-N} \left(\log \frac{e}{|x|}\right)^{-1} dx\right)^{1/q}$$
$$\leq c q^{1/N'} \left(\int_B |f(x)|^N \left(\log \frac{e}{|x|}\right)^{\delta} dx\right)^{1/N}$$

*for $q > N$, with $c$ independent of $f$ and $q$.*

*(iv) Let $\eta < 0$. Then*

$$\left(\int_B |If(x)|^q \left(\log\frac{e}{|x|}\right)^{\eta-1} |x|^{-N}\,dx\right)^{1/q} \leq cq^{1/N'}\left(\int_B |f(x)|^N \left(\log\frac{e}{|x|}\right)^{N-1} dx\right)^{1/N},$$

*for $q > N$, with $c$ independent of $f$ and $q$.*

The function $\varphi$ in all the above spaces can be found as a suitable combinations of powers of $|x|$ and powers of $|\log|x||$. This reduces the problem, after some effort, to estimates for integrals of type

$$F(x) = \int_B |x-y|^\alpha |y|^\beta \left(\log\frac{e}{|y|}\right)^\gamma dy, \qquad x \in B.$$

Once this is done the next step consists in establishing estimates for norms of $If$ in corresponding weighted spaces. By the choice of $\varphi$ this happens to be a repeated application of estimates for $F(x)$.

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