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On Musielak-Orlicz spaces isometric to L_2 or L_∞

Anna Kamińska

The University of Memphis, Department of Mathematical Sciences,

Memphis, TN 38152, U.S.A.

Abstract

It is proved that a Musielak-Orlicz space L_Φ of real valued functions which is isometric to a Hilbert space coincides with L_2 up to a weight, that is $\Phi(u,t)= c(t)u^2$. Moreover it is shown that any surjective isometry between L_Φ and L_∞ is a weighted composition operator and a criterion for L_Φ to be isometric to L_∞ is presented.

Isometries in complex function spaces have been studied successfully for fairly long time (see review article by Fleming and Jamison [3]). The real case appeared more difficult and since the well known characterization of isometries in L_p spaces by S. Banach in 1932 and some partial results in other spaces, only very recently such isometries in rearrangement invariant real function spaces have been characterized by Kalton and Randrianantoanina in [7]. Isometries between r.i. real spaces and L_p have been also studied in [2]. Here we study surjective linear isometries in the class of Musielak-Orlicz spaces of real-valued functions. We show that any Musielak-Orlicz space L_Φ isometric to a Hilbert space must coincide with L_2 "up to a weight", namely there exists a positive measurable function $c(t)$, such that $\Phi(u,t) = c(t)\, u^2$. We also present a criterion for L_Φ to be isometric to L_∞, and moreover we show that any such isometry has disjoint support property and in consequence is a weighted composition operator.

In the sequel let (Ω, Σ, μ) be a σ-finite, complete and nonatomic measure space. Symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ stand, as usual, for reals, nonnegative reals, natural numbers and for integers respectively. The space of all (equivalence classes of) Σ-measurable real functions defined on Ω is denoted by L^0. L^0 is a lattice with the pointwise order, that is $f \le g$ whenever $f(t) \le g(t)$, a.e. As usual L_2 and L_∞ stand for Lebesgue spaces of 2-integrable functions and for μ-essentially bounded functions on Ω, respectively. *A Banach function space* E over the measure space (Ω, Σ, μ) is a Banach lattice, which as a vector lattice is an ideal of vector lattice L^0 ([9]).

If $U : E \to F$ is a linear isometry between Banach function spaces E and F, then we say that U has *disjoint support property* if for any f, g with $fg = 0$ a.e. it holds $Uf \cdot Ug = 0$ a.e. A linear operator $U : E \to F$ is called a *weighted composition operator* if $Uf(t) = w(t)f \circ \tau(t)$ a.e. for all $f \in E$, where $w : \Omega \to \mathbb{R}$ is Σ-measurable and $\tau : \Sigma \to \Sigma$ is a set automorphism ([3]). If a weighted composition operator is a surjective isometry, then τ is a regular set isomorphism defined modulo null sets on Σ ([4, 5, 6]). Observe that any surjective linear isometry between Banach function spaces with disjoint support property, is a weighted composition operator. Indeed, there exists a partition $\{\Omega_k\}$ of Ω such that $\chi_{\Omega_k} \in E$ (Cor. 2, p. 95 in [9]). Then setting $w(t) = U(\chi_{\Omega_k})(t)$ for $t \in \text{supp } U(\chi_{\Omega_k})$ and $\tau(A) = \bigcup_{k=1}^{\infty} \text{supp } (\chi_{A \cap \Omega_k})$ we easily show that U is a weighted composition operator (compare e.g. [6]).

Let $\Phi(u, t) : \mathbb{R}_+ \times \Omega \to [0, +\infty]$ be a *Young function with parameter* i.e. for $t \in \Omega$, $\Phi(0, t) = 0$, $u \mapsto \Phi(u, t)$ is a left continuous convex function, and it is not identically zero or infinity for any $t \in \Omega$. For a Young function Φ with parameter we associate the *Musielak-Orlicz space* L_Φ defined as the subset of L^0 of all functions f for which

$$I_\Phi(\lambda f) = \int_\Omega \Phi(\lambda |f(t)|, t)\, d\mu(t) < \infty$$

for some $\lambda > 0$. L_Φ is a Banach space under the Luxemburg norm

$$||f|| = \inf\{\epsilon > 0 : I_\Phi(f/\epsilon) \le 1\}.$$

Define

$$a(t) = \sup\{u > 0 : \Phi(u, t) < \infty\}.$$

By the assumptions on Φ, $a(t) > 0$ a.e. on Ω. More information on Musielak-Orlicz spaces may be found in [12].

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be convex, $\varphi(0) = 0$ and $\varphi(1) = 1$. A number $a > 0$ is called *multiplier* of φ if $\varphi(au) = \varphi(a)\varphi(u)$ for every $u \in \mathbb{R}_+$. Let's denote by M_φ the group of multipliers of φ. It was proved by Lamperti in [10] that either $M_\varphi = \{1\}$ or $M_\varphi = \{a^k : k \in \mathbb{Z}\}$ for some $a > 0$ or $M_\varphi = \mathbb{R}_+$. In the latter case φ is a power function i.e. $\varphi(u) = u^p$ for some $1 \le p < \infty$.

Further we will need the following measure theoretic lemma from [4].

Lemma 1

Let ν be a measure on Σ absolutely continuous with respect to the measure μ. Let $0 < c < \nu(\Omega)$ and f be a measurable real function such that $\int_A f(t)\, d\mu(t) = 0$ for any A with $\nu(A) = c$. Then $f = 0$ a.e. on Ω.

Theorem 2

If $\Phi(1,t) \equiv 1$ and L_Φ is isometric to a Hilbert space, then $\Phi(u,t) \equiv u^2$.

Proof. Let's first observe that Φ must assume only finite values, since otherwise L_Φ contains an isomorphic copy of l_∞. Moreover, there exists a disjoint sequence $\{\Omega_i\}$, a partition of Ω, such that

$$\sup_{t\in\Omega_i} \Phi(u,t) < \infty \tag{1}$$

for every $u \geq 0$ and all $i \in \mathbb{N}$ (see (0.1) in [8]). Then without loss of generality we suppose, that $\int_\Omega \Phi(u,t)\,d\mu(t) < \infty$ for every $u > 0$. This, among others, implies that for any function f with norm 1, $I_\Phi(f) = 1$. Let $\lambda > 0$ be such that $\int_\Omega \Phi(\lambda,t)\,d\mu(t) > 2$. For any two disjoint sets A and B with $||\lambda\chi_A|| = ||\lambda\chi_B|| = 1$, applying the parallelogram law we get that $||\lambda\chi_A + \lambda\chi_B||^2 = 2$. Hence

$$1 = \Big\|\frac{\lambda}{\sqrt{2}}\chi_{A\cup B}\Big\| = \int_{A\cup B} \Phi\Big(\frac{\lambda}{\sqrt{2}}, t\Big)\,d\mu(t) = \frac{1}{2}\int_{A\cup B} \Phi(\lambda,t)\,d\mu(t).$$

Thus

$$\int_{A\cup B} \Big[\Phi\Big(\frac{\lambda}{\sqrt{2}}, t\Big) - \frac{1}{2}\Phi(\lambda,t)\Big]\,d\mu(t) = 0,$$

for any disjoint sets A and B such that $\int_A \Phi(\lambda,t)\,d\mu(t) = \int_B \Phi(\lambda,t)\,d\mu(t) = 1$. Applying now Lemma 1 for any C with $\nu(C) = \int_C \Phi(\lambda,t)\,d\mu(t) = 2 < \nu(\Omega)$, we obtain that $\Phi(\frac{\lambda}{\sqrt{2}}, t) = \frac{1}{2}\Phi(\lambda,t)$ for a.a. $t \in \Omega$. The same equation holds for any number bigger than λ. Thus there exists a nonnegative β such that for every $\lambda \geq \beta$

$$\Phi\Big(\frac{\lambda}{\sqrt{2}}, t\Big) = \frac{1}{2}\Phi(\lambda,t)$$

a.e. on Ω. Now let $\lambda < \beta$. Choose any $0 < c < \min\{1, \frac{1}{2}\int_\Omega \Phi(\lambda,t)\,d\mu(t)\}$. For any two disjoint sets A and B with $I_\Phi(\lambda\chi_A) = I_\Phi(\lambda\chi_B) = c$ we choose $\gamma \geq \beta$ and two other sets A_1 and B_1 such that all sets A, B, A_1, B_1 are disjoint and $I_\Phi(\gamma\chi_{A_1}) = I_\Phi(\gamma\chi_{B_1}) = 1 - c$. Setting

$$f = \lambda\chi_A + \gamma\chi_{A_1} \quad \text{and} \quad g = \lambda\chi_B + \gamma\chi_{B_1},$$

$||f|| = ||g|| = 1$. Therefore, by the parallelogram law applied to f and g we obtain

$$\int_{A\cup B} \Phi\Big(\frac{\lambda}{\sqrt{2}}, t\Big)\,d\mu(t) + \int_{A_1\cup B_1} \Phi\Big(\frac{\gamma}{\sqrt{2}}, t\Big)\,d\mu(t) = 1.$$

By the choice of γ and the sets A_1 and B_1, and by the equality $\Phi(\frac{\gamma}{\sqrt{2}},t) = \frac{1}{2}\Phi(\gamma,t)$,

$$\int_{A_1\cup B_1} \Phi\Big(\frac{\gamma}{\sqrt{2}},t\Big)\,d\mu(t) = 1 - c.$$

Then $\int_{A\cup B}[\Phi(\frac{\lambda}{\sqrt{2}},t) - \frac{1}{2}\Phi(\lambda,t)]\,d\mu(t) = 0$ for any disjoint sets A, B with $\int_A \Phi(\lambda,t)\,d\mu(t) = \int_B \Phi(\lambda,t)\,d\mu(t) = c$. Applying Lemma 1 again we get that $\Phi(\frac{\lambda}{\sqrt{2}},t) = \frac{1}{2}\Phi(\lambda,t)$ a.e. on Ω.

The above two steps show that $\frac{1}{\sqrt{2}}$ is a multiplier of the function $u \mapsto \Phi(u,t)$ a.e. on Ω that is $\Phi(\frac{u}{\sqrt{2}},t) = \Phi(\frac{1}{\sqrt{2}},t)\cdot\Phi(u,t)$ for every $u \geq 0$.

Now, let's sketch only the proof that this function has another multiplier, e.g. $\frac{1}{\sqrt{3}}$. Consider three disjoint sets A, B and C such that $\int_A \Phi(\lambda,t)\,d\mu(t) = \int_B \Phi(\lambda,t)\,d\mu(t) = \int_B \Phi(\lambda,t)\,d\mu(t) = 1$. Then by the parallelogram law applied to the functions $\lambda\chi_A$, $\lambda\chi_B$ and $\lambda\chi_C$, we get that $||\lambda\chi_{A\cup B\cup C}|| = \sqrt{3}$, whence

$$\int_{A\cup B\cup C} \Big[\Phi\Big(\frac{\lambda}{\sqrt{3}},t\Big) - \frac{1}{3}\Phi(\lambda,t)\Big]\,d\mu(t) = 0.$$

Further we apply Lemma 1 and proceed analogously as before.

We showed that $u \mapsto \Phi(u,t)$ has two different multipliers and so by the Lamperti's result ([11]), $\Phi(u,t)$ must be a power function. Finally, since L_Φ is isometric to a Hilbert space, $\Phi(u,t) \equiv u^2$. □

Corollary 3

If a Musielak-Orlicz space L_Φ is isometric to a Hilbert space, then there exists a measurable positive function $c(t)$ such that $\Phi(u,t) = c(t)\,u^2$.

Proof. Setting $\overline{\Phi}(u,t) = \Phi(b(t)u,t)$ with $\Phi(b(t),t) = 1$, L_Φ and $L_{\overline{\Phi}}$ are isometric. Applying now the previous theorem to $L_{\overline{\Phi}}$, we get that $\overline{\Phi}(u,t) \equiv u^2$. Hence $\Phi(u,t) = c(t)\,u^2$, with $c(t) = \Phi(1,t)$. □

Before we present our last result, let us recall the well known fact, due to Abramovich and Wojtaszczyk, that a Banach lattice E is isomorphic to an M-space if and only if there exists $c > 0$ such that $||x_1 + \ldots + x_n|| \leq c\,\max\{||x_1||, \ldots, ||x_n||\}$ for all disjoint elements $x_1, \ldots, x_n$ in E (cf. [1], p. 324).

Theorem 4

A Musielak-Orlicz space L_Φ is isometric to L_∞ if and only if $I_\Phi(a) \leq 1$. Moreover, if $U : L_\Phi \to L_\infty$ is a surjective isometry, then U has disjoint support property and in consequence U is a weighted composition operator.

Proof. Let L_Φ be isometric to L_∞. At first let's show that $a(t) < \infty$ a.e. For a contrary, without loss of generality, assume that $a(t) = \infty$ on the whole set Ω and that $\Phi(1,t) \equiv 1$. Let $\{\Omega_i\}$ be a disjoint partition of Ω satisfying condition (1). There exist $\beta > 0$ and a set $A \in \Sigma$ such that $||\beta\chi_A|| = 1$. By the Fatou property of L_Φ, $||\beta\chi_{A\cap\cup_{k=1}^n\Omega_k}|| \uparrow ||\beta\chi_A|| = 1$. Hence there exists $n \in \mathbb{N}$ such that $||\beta\chi_{A\cap\cup_{k=1}^n\Omega_k}|| \geq \frac{1}{2}$. We have that $I_\Phi(\lambda\chi_{\cup_{k=1}^n\Omega_k}) < \infty$ for every $\lambda > 0$. Therefore, for every $K > 0$ there exist $M \geq K$ and a finite disjoint partition $\{A_1, A_2, \ldots, A_m\}$ of $\cup_{k=1}^n\Omega_k$ such that $||M\chi_{A_i}|| = 1$ for $i = 1, 2, \ldots, m$. Thus for an arbitrary large number K we construct a finite sequence of disjoint functions $f_i = M\chi_{A_i}$, with $\|f_i\| = 1$ and $||f_1 + f_2 + \ldots + f_m|| \geq \frac{K}{2}$. Then in view of [1], L_Φ cannot be M-space, and so we obtain a contradiction.

Now let $I_\Phi(a) > 1$. Then there exist two sets C_1 and C_2 with $\mu(C_1 \triangle C_2) > 0$, $I_\Phi(a\chi_{C_1}) = I_\Phi(a\chi_{C_2}) = 1$ and $I_\Phi(a\chi_{C_1} - a\chi_{C_2}) \leq \frac{3}{2}$. The functions $f = a\chi_{C_1}$ and $g = a\chi_{C_2}$ are extreme points of the unit ball in L_Φ and $||f - g|| \leq \frac{3}{2}$. Hence Uf and Ug are extreme points of the unit ball in L_∞. Therefore $|Uf| \equiv |Ug| \equiv 1$. Moreover $Uf \neq Ug$, since $f \neq g$. Thus there is a set C with positive measure such that $Uf(t) \neq Ug(t)$ for all $t \in C$, whence $|Uf(t) - Ug(t)| = 2$ on C. So we obtain a contradiction, since

$$2 = ||Uf - Ug||_\infty = ||f - g|| \leq \frac{3}{2}.$$

If $I_\Phi(a) \leq 1$, then clearly $U : f \mapsto \frac{f}{a}$ is an isometry of L_Φ onto L_∞.

Finally we shall show that every onto isometry U between L_Φ and L_∞ has disjoint support property. Since $I_\Phi(a) \leq 1$, any function of the form $|u(t)|a(t)$, where u is an unimodular function, is an extreme point of the unit ball in L_Φ. If f and g have disjoint supports and $f + g$ is an extreme point, then $f - g$ is also extreme, and so

$$|Uf + Ug| \equiv 1 \equiv |Uf - Ug|.$$

Thus Uf and Ug have disjoint supports. This implies that if u is an unimodular function and A any measurable subset of $[0, 1]$, then there exists F such that

$$|U(ua\chi_A)| = \chi_F.$$

Let now A and B be two disjoint sets. Then for any unimodular function u, all functions $U(ua\chi_A)$, $U(ua\chi_B)$, $U(ua\chi_A + ua\chi_B)$ and $U(ua\chi_A - ua\chi_B)$ take as values either 0 or 1 or -1. Hence $U(ua\chi_A)$ and $U(ua\chi_B)$ have disjoint supports.

For $f = ua\chi_A$, with an unimodular function u and $\mu A > 0$, and g with $|g| \leq a$ and $\operatorname{supp} g \cap A = \emptyset$, we shall show that Uf and Ug have disjoint supports. Clearly, $||f \mp g|| = 1$ and $||g|| \leq 1$. Hence $||Uf \mp Ug|| = 1$ and $||Ug|| \leq 1$. Moreover, there exists F such that $|Uf| = \chi_F$. Assuming for some t that $Uf(t) = 1$ or -1 and $Ug(t) = b$ where $|b| \leq 1$, we get $|1 \mp b| \leq 1$, whence $b = 0$. Thus $Uf \cdot Ug = 0$.

Now let f and g be such that $|f| \leq a$, $|g| \leq a$, supp $f = A$, supp $g = B$ and $A \cap B = \emptyset$.

Assume first that $A \cup B = \Omega$. By the preceding step, $U(a\chi_A)$ and $U(g)$ have disjoint supports as well as $U(a\chi_B)$ and $U(f)$. Moreover, supp $U(a\chi_A) = C$ and supp $U(a\chi_B) = D$, where C and D are disjoint and their union is the whole set Ω. Hence supp $Ug \subset D$ and supp $Uf \subset C$.

Now let the set $(A \cup B)^c$ have positive measure. Then setting $h = a\chi_{(A\cup B)^c}$ and applying the above paragraph to $f + h$ and g, we have that $U(f + h) \cdot Ug = 0$. Since $Ug \cdot Uh = 0$, we have $Uf \cdot Ug = 0$.

Observe, that for any $0 \neq f \in L_\Phi$, $|\frac{f}{||f||}| \leq a$. This completes the proof. $\square$

Let's mention that for complex Musielak-Orlicz spaces Theorem 4 may be quickly derived from the results in [6].

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References

1. Yu.A. Abramovich and P. Wojtaszczyk, On the uniqueness of order in spaces $L_p[0,1]$ and l_p, *Mat. Zametki* **18**(3) (1975), 313–325 (in Russian).
2. Yu.A. Abramovich and M. Zaidenberg, A rearrangement invariant space isometric to L_p coincides with L_p, *Interaction Between Functional Analysis, Harmonic Analysis, and Probability*, Lecture Notes in Pure and Applied Mathematics Series 175, Marcel Dekker, 1995, 13–18.
3. R. Fleming and J.E. Jamison, Isometries on Banach Spaces: A Survey, *Analysis, Geometry and Groups: A Riemann Legacy Volume*, Hadronic Press, Palm Harbor, 1993, 52–123.
4. R. Fleming, J.E. Jamison and A. Kamińska, Isometries of Musielak-Orlicz spaces, *Proceedings of the Conference on Functional Analysis*, Marcel Dekker, 1992, 139–154.
5. P. Greim, J.E. Jamison and A. Kamińska, Almost transitivity of some function spaces, *Math. Proc. Cambridge Philos. Soc.* **116** (1994), 475–488.
6. J.E. Jamison, A. Kamińska and Pei-Kee Lin, Isometries of Musielak-Orlicz spaces II, *Studia Math.* **104**(1) (1993), 75–89.
7. N.J. Kalton and B. Randrianantoanina, Surjective isometries of rearrangement-invariant spaces, *Quart. J. Math. Oxford* **45** (1994), 301–327.
8. A. Kamińska, Some convexity properties of Musielak-Orlicz spaces of Bochner type, *Rend. Circ. Mat. Palermo (2) Suppl.* **10** (1985), 63–73.
9. L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Pergamon Press, 1982.
10. J. Lamperti, On the isometries of certain function spaces, *Pacific. J. Math.* **8** (1958), 459–466.

11. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991.
12. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer–Verlag, 1983.