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Some basic properties of generalized Calderon-Lozanovskiĭ spaces

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Abstract

Generalized Calderon-Lozanovskiĭ spaces $\Psi_\varphi(E_1, E_2)$ introduced in [3] are investigated. These spaces are generated by a function $\Psi : T \times R^2 \to R_+$ such that $\Psi(\cdot, u)$ is a Σ-measurable function for any $u \in R^2$ and $\Psi(t, \cdot)$ is a homogeneous, concave function vanishing at zero and by a couple of Banach function lattices E_1 and E_2 over a nonatomic measure space (T, Σ, μ). We investigate the special class of these spaces, namely the spaces E_φ corresponding to $\Psi(E, L^\infty)$, where E is an arbitrary Banach function lattice. We investigate the problem of order continuity, Fatou property, property $\mathbf{H}_\mu$ and order isomorphically isometric copies of l^∞ in E_φ. We also consider some relationships between the norm and the modular as well as between the modular convergence and the norm convergence in E_φ. In order to do so, we define a regularity condition Δ_2^E.

0. Introduction

Throughout the paper R, R_+ and N denote the sets of reals, nonnegative reals and natural numbers, respectively. A triple (T, Σ, μ) stands for a nonatomic, positive, complete and σ-finite measure space, while $L^0 = L^0(\mu)$ denotes the space of all (equivalence classes of) Σ-measurable functions $x : T \to R$. For any $x \in L^0$, we denote by $\mid x \mid$ the absolute value of x, i.e. $\mid x \mid(t) = \mid x(t) \mid$ for μ-a.e. $t \in T$. Moreover, φ stands for a Musielak-Orlicz function, i.e. a function defined on $T \times R$ with values in R_+ such that $\varphi(t, \cdot)$ is a nonzero function, it vanishes at zero, it is

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convex and even for μ-a.e. $t \in T$ and $\varphi(\cdot, u)$ as well as $\varphi^{-1}(\cdot, u)$ are Σ-measurable functions for any $u \in R_+$.

The letter E stands for a Banach function lattice over the measure space (T, Σ, μ), i.e. E is a Banach subspace of L^0 satisfying the following conditions:

(i) if $x \in E$, $y \in L^0$ and $|\, y \,| \leq |\, x \,|$ μ-a.e., then $y \in E$ and $\| y \|_E \leq \| x \|_E$.
(ii) there exists a function x in E which is strictly positive on the whole T.

The positive cone in E is denoted by E_+.

We say a Musielak-Orlicz function φ satisfies the condition Δ_2^E if there exist a set $A \in \Sigma$ with $\mu(A) = 0$, a constant $K > 0$ and a nonnegative function $h \in E$ such that the inequality

$$\varphi(t, 2u) \leq K\varphi(t, u) + h(t)$$

holds for all $t \in T \backslash A$ and $u \in R$ (cf. [22]).

We say that φ satisfies the condition $\Delta_2^E(\varepsilon)$ if it satisfies the condition Δ_2^E with $\| h \|_E < \varepsilon$. The condition Δ_2^E can be equivalently formulated in the form

$$\varphi(t, 2u) \leq K_1 \varphi(t, u)$$

for all $t \in T \backslash A$ and $u \geq f(t)$, where f is a nonnegative function on T such that $\varphi \circ 2f \in E$ and $K_1 > 0$ is independent of $t \in T \backslash A$ and $u \geq f(t)$ (cf. [7]). We understand $(\varphi \circ 2f)(t) = \varphi(t, 2f(t))$ μ-a.e..

Given any Musielak-Orlicz function φ such that $\varphi(t, \cdot)$ vanishes only at zero for μ-a.e. $t \in T$, we define a concave function $\Psi_\varphi : T \times R_+^2 \to R_+$ by the formula

$$\Psi_\varphi(t, u, v) = v\varphi^{-1}(t, u/v)$$

for $t \in T$, $u, v \in R_+$, $v \neq 0$ and $\Psi_\varphi(t, u, 0) = 0$ for all $t \in T$ and $u \in R_+$, where $\varphi^{-1}(t, \cdot)$ is the inverse function of $\varphi(t, \cdot)$. It is easy to see that $\Psi_\varphi(t, \cdot, \cdot)$ is a continuous concave function of two variables and that it is homogeneous, i.e.

$$\Psi_\varphi(t, \lambda u, \lambda v) = \lambda \Psi_\varphi(t, u, v)$$

for all $\lambda > 0$ and $u, v \in R_+$. Denote the class of such functions by $\mathcal{C}$. Then, conversely, if $\Psi \in \mathcal{C}$ is such a function that for μ-a.e. $t \in T$, $\Psi(t, u, v) = 0$ iff $u = 0$ whenever $v \neq 0$ and $\Psi(t, u, 0) = 0$ for any $u \in E$, defining $\varphi(t, \cdot)$ as the inverse function of $\Psi(t, \cdot, 1)$, we get $\Psi_\varphi = \Psi$. Therefore, there exist *one−to−one* correspondence between the class of Musielak-Orlicz functions and the class $\mathcal{C}$.

Given a Musielak-Orlicz function φ and a couple of Banach function lattices E_1 and E_2 we define following Calderon (cf. [3], pp. 162–163 and 165–166) the generalized Calderon-Lozanowskiĭ space

$$\Psi_\varphi(E_1,E_2) = \{x \in L^0 : |\,x(\cdot)\,| \le \lambda \Psi_\varphi(\cdot, |\,x_1(\cdot)\,|, |\,x_2(\cdot)\,|) \\ \mu\text{-a.e. for some } \lambda > 0 \text{ and } x_i \in B(E_i)\ (i=1,2)\},$$

where $B(E_i)$ denote the unit ball of E_i $(i = 1, 2)$. A norm in $\Psi_\varphi(E_1, E_2)$ is defined by

$$\|\,x\,\|_{\Psi_\varphi(E_1,E_2)} = \inf\{\lambda > 0 : |\,x(\cdot)\,| \le \lambda \Psi_\varphi(\cdot, |\,x_1(\cdot)\,|, |x_2(\cdot)\,|) \\ \mu\text{-a.e. for some } x_i \in B(E_i)\ (i=1,2)\}.$$

The couple $(\Psi_\varphi(E_1, E_2), \|\cdot\|_{\Psi_\varphi(E_1,E_2)})$ is a Banach space. In the case when $E_1 = E$ and $E_2 = L^\infty$ we write E_φ in place of $\Psi_\varphi(E, L^\infty)$. We have

$$E_\varphi = \{x \in L^0 : \varphi \circ \lambda x \in E \text{ for some } \lambda > 0\}.$$

Moreover, the norm $\|\cdot\|_{\Psi_\varphi(E,L^\infty)}$ coincides with the norm $\|\cdot\|_\varphi$ defined by

$$\|\,x\,\|_\varphi = \inf\{\lambda > 0 : \rho(x/\lambda) \le 1\},$$

where

$$\rho(x) = \begin{cases} \|\ \varphi \circ x\ \|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise}. \end{cases}$$

The subspace E_φ^o of E_φ defined by $E_\varphi^o = \{x \in L^0 : \rho(\lambda x) < \infty \text{ for any } \lambda > 0\}$ is considered with the norm $\|\cdot\|_\varphi$ induced from E_φ.

In the case when φ is an Orlicz function, i.e. there is a set $A \in \Sigma$ with $\mu(A) = 0$ such that $\varphi(t_1,\cdot) = \varphi(t_2,\cdot)$ for all $t_1, t_2 \in T\backslash A$, these Calderon-Lozanovskiĭ spaces were investigated by Lozanovskiĭ in [15], [16] and [17] and the investigations were continued in the papers [4], [8], [19], [20], [21], [24] and [25]. For the concave functions

$$\Psi(u,v) = v\left(\frac{u}{v}\right)^p = u^p v^{1-p} \quad (0 < p < 1)$$

the spaces $\Psi_\varphi(E_1, E_2)$ were mainly considered by Calderon [3].

It is easy to see that for each Musielak-Orlicz function φ the space $\Psi_\varphi(L^1, L^\infty)$ become the Musielak-Orlicz space L^φ and the norm $\|\cdot\|_\varphi$ coincides in this case with the Luxemburg norm. For the theory of Musielak-Orlicz spaces (Orlicz spaces) we refer to [22] (resp. [13], [18], [20] and [23]).

We assume in the whole paper that E has the Fatou property ($E \in \mathbf{FP}$ for short), i.e. for any $x \in L^0$ and $(x_n)_{n=1}^{\infty}$ in E such that $0 \le x_n \nearrow x$ μ-a.e. and $\sup_n \| x_n \|_E < \infty$ we have $\| x \|_E = \lim_n \| x_n \|_E$ (cf. [1], [12], [14]). This assumption guarantees that for any Musielak-Orlicz function φ the modular ρ is left continuous, i.e. $\sup \{\rho(\lambda x) : |\lambda| \le \lambda_0\} = \rho(\lambda_0 x)$ for any $\lambda_0 > 0$. We also have $\rho(x) \le \| x \|_\varphi$ whenever $\| x \|_\varphi \le 1$ and $\| x \|_\varphi \le \rho(x)$ whenever $\| x \|_\varphi \ge 1$. Moreover, $\| x_n \|_\varphi \to 0$ if and only if $\rho(\lambda x_n) \to 0$ for any $\lambda > 0$.

An element $x \in E$ is said to be order continuous if for any sequence (x_n) in E such that $0 \swarrow x_n \le |x|$, we have $\| x_n \|_E \to 0$. The subset E_a of all order continuous elements in E is a sublattice of E. The lattice E is said to be order continuous ($E \in \mathbf{OC}$ for short) if $E_a = E$ (cf. [1], [12], [14]). In the following we assume that E_a satisfies condition (ii) from the definition of a Banach function lattice, i.e. $\operatorname{supp} E_a = T$.

We say that E has the property $\mathbf{H}_\mu$ if for any $x \in E$ and any sequence (x_n) in E such that $\| x_n \|_E \to \| x \|_E$ and $x_n \chi_A \to x \chi_A$ in measure for any $A \in \Sigma$ with $\mu(A) < \infty$, we have $\| x - x_n \|_E \to 0$ (cf. [9]).

By σ-finiteness of the measure space (T, Σ, μ) in the definition of the property $\mathbf{H}_\mu$ we can replace the local convergence in measure by the μ-a.e. convergence of (x_n) to x.

1. Results

We start with the following fundamental lemma.

Lemma 1

For any Musielak-Orlicz function φ we have $\operatorname{supp} E_\varphi = \operatorname{supp} E_\varphi^o = \operatorname{supp} (E_a)_\varphi = T$. Namely, there exists a sequence $(T_n)_{n=1}^{\infty}$ in Σ such that $\bigcup_n T_n = T$ and $\chi_{T_n} \in E_\varphi^o \cap (E_a)_\varphi$ for each $n \in N$.

Proof. Kamińska [11] has proved that there exists an ascending sequence $(T_n')_{n=1}^{\infty}$ such that $\bigcup_n T_n' = T$ and

$$\sup_{t \in T_n'} \varphi(t, u) < \infty$$

for each $n \in N$ and $u \in R_+$. Moreover, there exist an ascending sequence $(A_n)_{n=1}^{\infty}$ such that $\chi_{A_n} \in E$ and $\bigcup_n A_n = T$ (cf. [12], Corollary 2, p. 138). Define $T_n =$

$T_n' \cap A_n$. It is obvious that $\bigcup_n T_n = T$. First, we will show that $\chi_{T_n} \in E_\varphi^o$, i.e. $\rho(\lambda\chi_{T_n}) < \infty$ for each $\lambda > 0$ and $n \in N$. We have

$$a_{n,\lambda} = \sup_{t\in T_n} \varphi(t,\lambda) < \infty\,.$$

Hence

$$\begin{aligned}\rho(\lambda\chi_{T_n}) = \|\, \varphi\circ\lambda\chi_{T_n}\,\|_E \leq\ &\|\, a_{n,\lambda}\chi_{T_n}\,\|_E = a_{n,\lambda}\|\,\chi_{T_n}\,\|_E \\ \leq\ &a_{n,\lambda}\|\,\chi_{A_n}\,\|_E < +\infty\,.\end{aligned}$$

To prove that $\chi_{T_n} \in (E_a)_\varphi$ for any $n \in N$, take any sequence $\{x_k\}$ in $(E_a)_\varphi$ such that $0 \swarrow x_k \leq \chi_{T_n}$. Then we have for any $\lambda > 0$:

$$0 \swarrow \varphi\circ\lambda x_k \leq \varphi\circ\lambda\chi_{T_n} \leq a_n\chi_{T_n} \leq a_n\chi_{A_n},$$

where $a_n = \sup_{t\in T_n'}\varphi(t,\lambda)$. Since $a_n\chi_{A_n} \in E_a$, we get $\rho(\lambda x_k) = \|\,\varphi\circ\lambda x_k\,\|_E \to 0$. By the arbitrariness of $\lambda > 0$, $\|\, x_k\,\|_\varphi \to 0$. This finishes the proof. □

Lemma 2

*If $\varphi(t,\cdot)$ vanishes only at zero for μ-a.e. $t \in T$ and $E \in$ **OC**, then $\varphi \in \Delta_2^E$ if and only if for every $\varepsilon > 0$ there exist $K \geq 2$, a set $A \in \Sigma$ with $\mu(A) = 0$ and a function $h_\varepsilon : T \to R_+$ such that $\varphi\circ 2h_\varepsilon \in E$, $\|\,\varphi\circ 2h_\varepsilon\,\|_E < \varepsilon$ and $\varphi(t,2u) \leq K\varphi(t,u)$ for all $t \in T\backslash A$ and $u \geq h_\varepsilon(t)$.*

Proof. It is enough to show that $\varphi \in \Delta_2^E$ implies the condition from the lemma. Take an arbitrary $\varepsilon > 0$ and let h be the function determined by $\varphi \in \Delta_2^E$, i.e. $\varphi\circ 2h \in E$ and $\varphi(t,2u) \leq K\varphi(t,u)$ for all $u \geq h(t)$ μ-a.e. in T. Then

$$0 \leq \left\|\frac{1}{n}\,\varphi\circ 2h\right\|_E \leq \frac{1}{n}\,\|\,\varphi\circ 2h\,\|_E \to 0$$

as $n \to \infty$. So, there exists $n_0 \in N$ such that $\|\, 1/n_0\varphi\circ 2h\,\|_E < \varepsilon/2$. Define

$$A_m = \left\{t \in T : \varphi(t,2u) \leq 2^m\varphi(t,u) \text{ if } \frac{1}{n_0}\varphi\big(t,2h(t)\big) \leq \varphi\big(t,u\big) \leq \varphi\big(t,2h(t)\big)\right\}.$$

Clearly, $A_m \uparrow$ and $\mu\big(T\backslash\bigcup_m A_m\big) = 0$. Denote

$$x_m = \varphi\circ 2h\chi_{T\backslash A_m}\,.$$

We have $x_m \searrow 0$, whence by $E \in \mathbf{OC}$, we get $\| x_m \|_E \to 0$ as $m \to \infty$. Therefore, there exists $m_0 \in N$ such that $\| x_{m_0} \|_E < \varepsilon/2$. Let g be the function satisfying $1/n_0\varphi(t, 2h(t)) = \varphi(t, g(t))$ and define

$$h_\varepsilon(t) = \frac{1}{2}\, g(t)\chi_{A_{m_0}}(t) + h(t)\chi_{T\setminus A_{m_0}}(t)\,.$$

Clearly $\varphi \circ 2h_\varepsilon \in E$ and $\varphi(t, 2u) \le \max(K, 2^{m_0})\varphi(t, u)$ for $u \ge h_\varepsilon(t)$ μ-a.e. in T. Moreover,

$$\| \varphi \circ 2h_\varepsilon \|_E \le \| \varphi \circ g\chi_{A_{m_0}} \|_E + \| \varphi \circ 2h\chi_{T\setminus A_{m_0}} \|_E < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

So, h_ε has the desired properties. □

Note that Lemma 2 generalizes Lemma 1.6 in [6] from Orlicz spaces L^φ to Calderon-Lozanovskiĭ spaces E_φ.

Lemma 3

Assume that $\varphi(t, \cdot)$ vanishes only at zero for μ-a.e. $t \in T$, $\varphi \in \Delta_2^E$ and $E \in \mathbf{OC}$. Then for any sequence (x_n) in E_φ we have $\| x_n \|_\varphi \to 1$ if and only if $\rho(x_n) \to 1$.

Proof. We have $\rho(x) \le \| x \|_\varphi$ if $\| x \|_\varphi \le 1$ and $\rho(x) \ge \| x \|_\varphi$ if $\| x \|_\varphi \ge 1$, whence it follows that $\rho(x_n) \to 1$ implies $\| x_n \|_\varphi \to 1$. We will prove the opposite implication. It is enough to consider only the cases when $\| x_n \|_\varphi \le 1$ for all $n \in N$ or $\| x_n \|_\varphi \ge 1$ for all $n \in N$.

1°. $\| x_n \|_\varphi \le 1$ for all $n \in N$ and $\| x_n \|_\varphi \to 1$. Assume for the contrary that $\rho(x_n) \not\to 1$. Then we may assume without loss of generality that there exists $\delta > 0$ such that $\rho(x_n) \le 1 - \delta$ for any $n \in N$. By Lemma 2, there exist $K \ge 2$, a set $A \in \Sigma$ with $\mu(A) = 0$ and a function $h : T \to R_+$ such that $\varphi \circ 2h \in E$, $\| \varphi \circ 2h \|_E < \delta/2$ and the inequality $\varphi(t, 2u) \le K\varphi(t, u)$ holds for all $t \in T\setminus A$ and $u \ge h(t)$. Denoting the right derivative of $\varphi(t, \cdot)$ by $p(t, \cdot)$, we get

$$up(t, u) \le \varphi(t, 2u) \le K\varphi(t, u)$$

for all $t \in T\setminus A$ and $u \ge h(t)$. Hence for all $\alpha \ge 1$, $t \in T\setminus A$ and $u \ge h(t)$, we get

$$\int_u^{\alpha u} \frac{p(t, s)}{\varphi(t, s)} ds \le K \int_u^{\alpha u} \frac{ds}{s},$$

i.e. $\varphi(t,\alpha u) \leq \alpha^K \varphi(t,u)$. Let $\alpha \in (1,2]$ be such that $1 < \alpha^K \leq \frac{1-\delta/2}{1-\delta}$ and define

$$B_n = \{t \in T : |\, x_n(t)\,| \geq h(t)\}, \quad n = 1,2,\dots .$$

Then we have for all $n \in N$:

$$\begin{aligned}\rho(\alpha x_n) \leq \rho(\alpha x_n \chi_{B_n}) + \rho(\alpha x_n \chi_{T\setminus B_n}) &\leq \alpha^K \rho(x_n \chi_{B_n}) + \rho(2x_n \chi_{T\setminus B_n}) \\ &\leq \frac{1-\delta/2}{1-\delta}(1-\delta) + \frac{\delta}{2} = 1\,.\end{aligned}$$

Hence it follows that $\|\, x_n \,\|_\varphi \leq \alpha^{-1} < 1$ for all $n \in N$, which contradicts the condition $\|\, x_n \,\|_\varphi \to 1$.

2°. $\|\, x_n \,\|_\varphi \geq 1$ for all $n \in N$ and $\|\, x_n \,\|_\varphi \to 1$. Assume for the contrary that $\rho(x_n) \not\to 1$. We may assume without loss of generality that there exists $\delta > 0$ such that $\rho(x_n) \geq 1 + \delta$ for any $n \in N$. By Lemma 2, we can find a function $h : T \to R_+$ such that $\varphi \circ h \in E$, $\|\, \varphi \circ h \,\|_E < \delta/2$ and $\varphi(t,2u) \leq K\varphi(t,u)$ for all $t \in T\setminus A$ and $u \geq h(t)/2$, where $\mu(A) = 0$ and $K \geq 2$ is independent of t. Analogously, as in the first case, we can show that

$$\varphi(t,\alpha u) \geq \alpha^K \varphi(t,u)$$

for all $t \in T\setminus A$, $u \geq h(t)$ and $1/2 \leq \alpha \leq 1$. Defining

$$C_n = \{t \in T : |\, x_n(t)\,| \geq h(t)\}, \quad n = 1,2,\dots ,$$

we have $\rho(x_n \chi_{T\setminus C_n}) \leq \rho(h) < \delta/2$, whence $\rho(x_n \chi_{C_n}) > 1 + \delta/2$ for each $n \in N$. Therefore, for n large enough such that $\|\, x_n \,\|_\varphi \leq 2$, we get

$$1 = \rho\left(\frac{x_n}{\|\, x_n \,\|_\varphi}\right) \geq \rho\left(\frac{x_n}{\|\, x_n \,\|_\varphi}\chi_{C_n}\right) \geq \frac{1}{(\|\, x_n \,\|_\varphi)^K}\rho(x_n \chi_{C_n}) \geq \frac{1+\delta/2}{(\|\, x_n \,\|_\varphi)^K},$$

which yields a contradiction for large $n \in N$. □

Note that in the case when $E = L^1$ and $\rho(x_n) \leq 1$, Lemma 3 coincides with Lemma 1.5 from [6].

Lemma 4

If $\varphi \notin \Delta_2^E$, then E_φ contains an order isomorphically isometric copy of l^∞.

Proof. Define for μ-a.e. $t \in T$

$$g_n(t) = \sup \left\{ u \in R_+ : \varphi\left(t, \left(1 + \frac{1}{n}\right) u\right) \geq 2^{n+1}\varphi(t, u) \right\}.$$

Functions g_n are Σ-measurable for all $n \in N$ (see S. Chen, *Geometry of Orlicz spaces*, Dissertationes Math. **356** (1996), Lemma 5.4, p. 176). Assume by definition $\varphi(t, \infty) = \infty$ for μ-a.e. $t \in T$. It is easy to see that $\varphi \notin \Delta_2^E$ is equivalent to $\varphi \notin \Delta_{1+\frac{1}{n}}^E$ for $n = 1, 2, ...$ and that the last fact is equivalent to $\| \varphi \circ g_n \|_E = \infty$ for $n = 1, 2, ...$

Define $T_n = \{t \in T : g_n(t) = \infty\}$. Then $T_{n+1} \subset T_n$ for any $n \in N$. We will show that there exists a sequence (x_n) of elements with pairwise disjoint support and such that $\rho(x_n) < 2^{-n}$ and $\| x_n \|_E = 1$ for all $n \in N$, $\rho(x) \leq 1/2$ and $\| x \|_E = 1$, where $x = \sum_{n=1}^{\infty} x_n$. To build such a sequence of elements consider three cases.

1°. Assume first that $\mu(T_n) = 0$ for n large enough. We can assume without loss of generality that $\mu(T_1) = 0$. Then, by the continuity of φ, we have

$$\varphi(t, (1 + \frac{1}{n})g_n(t)) \geq 2^{n+1}\varphi(t, g_n(t)),$$

for μ-a.e. $t \in T$. Let $(A_n)_{n=1}^{\infty}$ be an ascending sequence of measurable sets of finite measure such that $\chi_{A_n} \in E$ for $n = 1, 2, ...$ and $\bigcup_n A_n = T$ (see [12], p. 132, Corollary 2). Defining

$$B_n^1 = \{t \in T : \varphi(t, g_1(t)) \leq n\}, \quad n = 1, 2, ... ,$$

we have $B_1^1 \subset B_2^1 \subset ...$ and $\mu(T \backslash \bigcup_n B_n^1) = 0$. We get by $E \in \mathbf{FP}$ that

$$\| \varphi \circ g_1 \chi_{B_n^1 \cap A_n} \|_E \to \| \varphi \circ g_1 \|_E = \infty.$$

So, there exist $n_1 \in N$ and $B_1 \subset B_{n_1}^1 \cap A_{n_1}$, $B_1 \in \Sigma$, such that

$$\| \varphi \circ g_1 \chi_{B_1} \|_E = \frac{1}{4}.$$

We applied here the Dobrakov result from [5] which says that submeasures which are absolutely continuous with respect to μ being nonatomic have the Darboux property.

Since $g_2 \leq g_1$, we get $\| \varphi \circ g_2 \chi_{B_1} \|_E \leq \frac{1}{4}$, whence $\| \varphi \circ g_2 \chi_{T \backslash B_1} \|_E = \infty$. Define the sets

$$B_n^2 = \{t \in T \backslash B_1 : \varphi(t, g_2(t)) \leq n\}, \quad n = 1, 2, ...$$

Then $\| \varphi \circ g_2\chi_{B^2_n \cap A_n} \|_E \to \| \varphi \circ g_2\chi_{T\setminus B_1} \|_E$. So, there exist $n_2 \in N$ and $B_2 \subset B^2_{n_2} \cap A_{n_2}$ such that $B_2 \in \Sigma$ and

$$\| \varphi \circ g_2\chi_{B_2} \|_E = \frac{1}{8}, \ \| \varphi \circ g_3\chi_{T\setminus B_2} \|_E = \infty.$$

Continuing this procedure by induction, we can find a sequence $(B_n)_{n=1}^{\infty}$ of pairwise disjoint measurable sets such that

$$\| \varphi \circ g_n\chi_{B_n} \|_E = \frac{1}{2^{n+1}}, \ (n = 1, 2, ...).$$

Define

$$x = \sum_{n=1}^{\infty} g_n\chi_{B_n}.$$

Then we have $\rho(x) \le \sum_{n=1}^{\infty} \| \varphi \circ g_n\chi_{B_n} \|_E = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$, whence $\| x \|_\varphi \le 1$. Let $\lambda > 1$. There is $m \in N$ such that $\lambda \ge 1 + \frac{1}{m}$. Therefore,

$$\rho(\lambda x) \ge \| \varphi \circ (1 + \tfrac{1}{m})g_m\chi_{B_m} \|_E \ge \| 2^{m+1}\varphi \circ g_m\chi_{B_m} \|_E = 2^{m+1}\frac{1}{2^{m+1}} = 1,$$

whence $\| \lambda x \|_\varphi \ge 1$ and, by the arbitrariness of $\lambda > 1$, $\| x \|_\varphi \ge 1$. So, we have $\| x \|_\varphi = 1$. Define

$$x_1 = g_1\chi_{B_1} + g_3\chi_{B_3} + g_5\chi_{B_5} + ...,$$
$$x_2 = g_2\chi_{B_2} + g_6\chi_{B_6} + g_{10}\chi_{B_{10}} + ...$$

and by induction we define x_n to be the sum of every second term $g_l\chi_{B_l}$ of $\sum_{i=1}^{\infty} g_i\chi_{B_i} - \sum_{k=1}^{n-1} x_k$ beginning from the first term of the rest. Then we can prove in the same way as for x that $\rho(x_n) < 2^{-n}$ and $\| x_n \|_\varphi = 1$ for each $n \in N$.

2°. Assume that $\mu(\bigcap_{n=1}^{\infty} T_n) = a > 0$. By σ-finiteness of μ, we know that there exists a sequence of Σ-measurable functions and a sequence $(B'_n)_{n=1}^{\infty}$ of pairwise disjoint sets such that $B'_n \subset \bigcap_{n=1}^{\infty} T_n$ and $\mu(B'_n) = 2^{-n-1}a$ for any $n \in N$. Note that $\| \varphi \circ g_n\chi_{B'_n} \|_E = \infty$ for any $n \in N$. Now, for any $n \in N$ there exists a sequence $(f^n_k)_{k=1}^{\infty}$ of Σ-measurable finitely valued nonnegative functions such that $f^n_k(t) \nearrow g_n(t)$ and $\varphi(t, (1 + \frac{1}{n})f^n_k(t)) \ge 2^{n+1}\varphi(t, f^n_k(t))$ for μ-a.e. $t \in T$(see S. Chen, *Geometry of Orlicz spaces*, Dissertationes Math. **356** (1996), Lemma 5.4, p. 176). Since E has the Fatou property, we get

$$\| \varphi \circ f^n_k\chi_{B'_n} \|_E \to \| \varphi \circ g_n\chi_{B'_n} \|_E = \infty$$

for any $n \in N$. Therefore, for any $n \in N$ there are $k_n \in N$ and a measurable set $B_n \subset B'_n$ such that $\| \varphi \circ f^n_k\chi_{B_n} \|_E = 2^{-n-1}$.

In an analogous way as in case 1°, we can construct $x \in E_\varphi$ and a sequence (x_m) in E_φ, which have the desired properties.

3°. Finally, consider the case when $\mu(T_n) > 0$ for any $n \in N$ and $\mu(\bigcap_{n=1}^{\infty} T_n) = 0$. Then there is a sequence (m_n) of natural numbers such that $\mu(T_{m_n} \backslash T_{m_{n+1}}) > 0$ for any $n \in N$. Let, for any $n \in N$, B_n be a Σ-measurable set in $T_{m_n} \backslash T_{m_{n+1}}$ such that $\mu(B_n) > 0$. Then $\| \varphi \circ g_n \chi_{B_n} \|_E = \infty$. Now, we can proceed as in case 2°.

Define an operator $P : l^\infty \to E_\varphi$ by

$$(Pz)(t) = \sum_{n=1}^{\infty} z_n x_n(t) \ \ (\forall z = (z_n) \in l^\infty).$$

It is obvious that P is a linear operator. We have for all $z \in l^\infty \backslash \{0\}$:

$$\rho\left(\frac{Pz}{\|z\|_\infty}\right) = \left\| \sum_{n=1}^{\infty} \varphi \circ \frac{| \ x \ | z_n}{\|z\|_\infty} | \ x \ | x_n \right\|_E \leq \sum_{n=1}^{\infty} \| \varphi \circ x_n \|_E \leq \sum_{n=1}^{\infty} 2^{-n} = 1,$$

whence $\| Pz \|_\varphi \leq \|z\|_\infty$. Taking any $\lambda < 1$, we can find $m \in N$ such that $\frac{|z_m|}{\lambda \| z \|_\infty} = \lambda_o > 1$. Therefore,

$$\left\| \frac{Pz}{\lambda \| z \|_\infty} \right\|_\varphi \geq \left\| \frac{|z_m|}{\lambda \| z \|_\infty} |x_m| \right\|_\varphi = \| \lambda_o x_m \|_\varphi = \lambda_o > 1,$$

whence $\| Pz \|_\varphi > \lambda \|z\|_\infty$ and, by the arbitrariness of $\lambda < 1$, $\| Pz \|_\varphi \geq \| z \|_\infty$. Consequently, $\| Pz \|_\varphi = \|z\|_\infty$ for all $z \in l^\infty$. Since the functions x_n are nonnegative, we have $Pz \geq 0$ for $z \geq 0$, so P is an order isomorphic isometry. The proof is complete. □

In the case when φ does not depend on the parameter, Lemma 4 has been proved in [8].

Lemma 5

The property that $\| x \|_\varphi = 1$ *if and only if* $\rho(x) = 1$ *holds true for any* $x \in E_\varphi$ *if and only if* $\varphi \in \Delta_2^E$.

Proof. Sufficiency. Assume that $\varphi \in \Delta_2^E$, $\| x \|_\varphi = 1$ and $\rho(x) < 1$. Define the function $f(\lambda) = \rho(\lambda x)$ for $\lambda > 0$. It is clear that f is convex and (by $\varphi \in \Delta_2^E$) finitely valued. So, f is continuous and consequently it has the Darboux property. Since $f(1) = \rho(x) < 1$, there exist $\lambda_0 > 1$ such that $f(\lambda_0) = \rho(\lambda_0 x) < 1$. This yields $\| x \|_\varphi \leq 1/\lambda_0 < 1$, a contradiction.

Necessity. If $\varphi \notin \Delta_2^E$ then, by the proof of Lemma 4, there exists $x \in E_\varphi$ such that $\rho(x) \leq 1/2$ and $\| x \|_\varphi = 1$. □

Lemma 6

Let $E \in \mathbf{OC}$. We have $\| x_n \|_\varphi \to 0$ if and only if $\rho(x_n) \to 0$ for any sequence $(x_n)_{n=1}^\infty$ in E_φ if and only if $\varphi \in \Delta_2^E$ and $\varphi(t, \cdot)$ vanishes only at zero for μ-a.e. $t \in T$.

Proof. Sufficiency. By Lemma 2, the assumptions yields that $\varphi \in \Delta_2^E(\varepsilon)$ for any $\varepsilon > 0$. Let (x_n) be a sequence in E_φ such that $\rho(x_n) \to 0$. We need to show that $\rho(2x_n) \to 0$. Take an arbitrary $\varepsilon > 0$. Then, by $\varphi \in \Delta_2^E(\varepsilon/2)$, there exist a set $A \in \Sigma$ with $\mu(A) = 0$, a constant $K \geq 2$ and a nonnegative function $h \in L^0$ such that $\| h \|_E < \varepsilon/2$ and

$$\varphi(t, 2u) \leq K\varphi(t, u) + h(t)$$

for all $t \in T \backslash A$ and $u \in R$. If $m \in N$ is such that $\rho(x_n) < \varepsilon/2K$ for all $n \geq m$, we get

$$\rho(2x_n) \leq K\rho(x_n) + \| h \|_E < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq m$, which shows that $\rho(2x_n) \to 0$, finishing the proof of the sufficiency.

Necessity. If $\varphi \notin \Delta_2^E$, then taking the sequence (x_n) defined in the proof of Lemma 4, we have $\rho(x_n) = 2^{-n}$ and $\| x_n \|_\varphi = 1$ for all $n \in N$.

Assume now that φ vanishes outside zero, i.e. there is $k \in N$ such that the set $A = \{t \in T : \varphi(t, 1/k) = 0\}$ has positive measure. Defining $x_n = 1/k\chi_A$ for all $n \in N$, we have $\rho(x_n) = 0$ and $\| x_n \|_\varphi = \| k^{-1}\chi_A \|_\varphi > 0$ for $n = 1, 2, ...$ □

Corollary 7

If $\varphi(t, \cdot)$ vanishes outside zero for $t \in A \in \Sigma$ with $\mu(A) > 0$, then $\varphi \notin \Delta_2^E(\varepsilon)$ for some $\varepsilon > 0$.

Proof. Otherwise, $\varphi \in \Delta_2^E(\varepsilon)$ for any $\varepsilon > 0$, and by the proof of the sufficiency in Lemma 6, we have $\| x \|_\varphi \to 0$ whenever $\rho(x_n) \to 0$. However, this contradicts the necessity part of the proof of Lemma 6. □

Lemma 8

If (x_n) is in E_φ, $x_n \to 0$ μ-a.e. or locally in measure, $\rho(x_n) \to 0$, $E \in \mathbf{OC}$ and $\varphi \in \Delta_2^E$, then $\| x \|_\varphi \to 0$.

Proof. By the σ-finiteness of the measure space, it is enough to consider the case when $x_n \to 0$ μ-a.e. Then we have $\varphi \circ x_n \to 0$ μ-a.e. and $\| \varphi \circ x_n \|_E \to 0$. Therefore, there exist $y \in E_+$, a subsequence (x_{n_k}) of (x_n) and a sequence (ε_{n_k}) in $R_+ \backslash \{0\}$ with $\varepsilon_{n_k} \searrow 0$ such that

$$\varphi \circ x_{n_k} \le \varepsilon_{n_k} y$$

for all $k \in N$ and μ-a.e. $t \in T$ (cf. [12], Lemma 2, p. 141). We may assume without loss of generality that $\varepsilon_{n_k} \le 1$ for all $k \in N$, whence

$$\varphi \circ x_{n_k} \le y$$

for all $k \in N$. Let $\lambda > 0$ be arbitrary. Then $\lambda \le 2^l$ for some $l \in N$, so applying $\varphi \in \Delta_2^E$, we get

$$\varphi \circ \lambda x_{n_k} \le \varphi \circ 2^l x_{n_k} \le K^l \varphi \circ x_{n_k} + \left(\sum_{i=1}^{l-1} K^i \right) h \le K^l y + \left(\sum_{i=1}^{l-1} K^i \right) h = z \in E_+.$$

Since $\varphi \circ \lambda x_{n_k} \to 0$ μ-a.e., by $E \in \mathbf{OC}$, we get $\| \varphi \circ \lambda x_{n_k} \|_E = \rho(\lambda x_{n_k}) \to 0$. By the arbitrariness of $\lambda > 0$ this means that $\| x_{n_k} \|_\varphi \to 0$. In virtue of the double extract subsequence theorem this yields $\| x_n \|_\varphi \to 0$, finishing the proof. □

Theorem 9

The following hold true:

(i) The inclusions $(E_a)_\varphi^o \subset (E_\varphi)_a \subset E_\varphi^o$ *hold always true.*

(ii) If $E \in \mathbf{OC}$, *then the inclusions in (i) are equalities.*

(iii) If $\varphi \in \Delta_2^E$ *and* $\varphi(t, \cdot)$ *vanishes only at zero for* μ*-a.e.* $t \in T$, *then* $(E_\varphi)_a = E_\varphi^o$ *if and only if* $E \in \mathbf{OC}$.

(iv) If $\varphi(t, \cdot)$ *vanishes only at zero for* μ*-a.e.* $t \in T$*and* $\varphi \in \Delta_2^{E_a}$, *then* $(E_a)_\varphi \neq E_\varphi$ *whenever* $E \notin \mathbf{OC}$.

Proof. Let $(T_n)_{n=1}^\infty$ be the sequence of sets from Lemma 1 and let $x \in (E_\varphi)_a$. Define

$$x_n(t) = \begin{cases} x(t) & t \in T_n \text{ and } |x(t)| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x_n \in E_\varphi^o$ for each $n \in N$ and $0 \swarrow |x - x_n| \le |x|$. Since $x \in (E_\varphi)_a$, we get $\| x - x_n \|_\varphi \to 0$, i.e. $\rho(\lambda(x - x_n)) \to 0$ for each $\lambda > 0$. By the convexity of ρ, we have for any $\lambda > 0$:

$$\rho(\lambda x) = \rho(\lambda(x - x_n) + \lambda(x_n)) \le \frac{1}{2} \big\{ \rho(2\lambda(x - x_n)) + \rho(2\lambda(x_n)) \big\}.$$

By $\rho(2\lambda(x-x_n)) \to 0$, there is $m \in N$ such that $\rho(2\lambda(x-x_n)) \leq 1$ for all $n \geq m$. Since $\rho(2\lambda x_n) < \infty$ for all $\lambda > 0$ and $n \in N$, this yields that $\rho(\lambda x) < \infty$ for any $\lambda > 0$, and the inclusion $(E_\varphi)_a \subset E_\varphi^o$ is proved.

Assume now that $x \in (E_a)_\varphi^o$, i.e. $\varphi \circ \lambda x \in E_a$ for all $\lambda > 0$. Let $(x_n)_{n=1}^\infty$ be a sequence in L^0 such that $0 \swarrow x_n \leq |x|$. Then we get $0 \swarrow \varphi \circ \lambda x_n \leq \varphi \circ \lambda x \in E_a$, whence $\rho(\lambda x_n) = \| \varphi \circ \lambda x_n \|_E \to 0$. By the arbitrariness of $\lambda > 0$, we have $\| x_n \|_\varphi \to 0$, which yields $x \in (E_\varphi)_a$. So, the proof of (i) is complete.

Let $E \in \mathbf{OC}$, then $E_a = E$. Hence, (i) implies (ii).

Now, we will prove (iii). We need only to show that under the assumptions that $\varphi \in \Delta_2^E$ and $\varphi(t, \cdot)$ vanishes only at zero for μ-a.e. $t \in T$, we have $(E_\varphi)_a \neq E_\varphi^o$ whenever $E \notin \mathbf{OC}$. By $E \notin \mathbf{OC}$ there exist $\delta > 0$, $x \in E_+$ and a sequence (x_n) in E_+ such that $0 \swarrow x_n \leq x$ and $\| x_n \|_E \geq \delta$. Defining $y_n = x_n / \| x_n \|_E$, we have $\| y_n \|_E = 1$ for each $n \in N$ and $0 \swarrow y_n \leq x/\delta \in E$. Define now $z_n = \varphi^{-1} \circ y_n$ and $z = \varphi^{-1} \circ x/\delta$. Then $\varphi \circ z_n = y_n \in E$, $\varphi \circ z = x/\delta \in E$ and

$$\rho(z_n) = \| \varphi \circ z_n \|_E = \| y_n \|_E = 1.$$

Hence, $\| z_n \|_\varphi = 1$ for all $n \in N$, which means that $z \notin (E_\varphi)_a$. However, by $\varphi \in \Delta_2^E$, we have $z \in E_\varphi^o$, which finishes the proof of (iii).

Finally, we will prove (iv). Assuming that $E \notin \mathbf{OC}$, we have $E \neq E_a$, so there exists $x \in E \backslash E_a$. Define $y = \varphi^{-1} \circ | x |$. Then $y \in L^0$ by Σ-measurability of $\varphi^{-1}(\cdot, u)$ for all $u \in R$. Since $\varphi \circ y = | x | \in E$, we have $y \in E_\varphi$. Assuming that $y \in (E_a)_\varphi$, we get $\varphi \circ \lambda y \in E_a$ for some $\lambda > 0$, whence in view of $\varphi \in \Delta_2^{E_a}$, $| x | = \varphi \circ y \in E_a$, a contradiction. □

Theorem 10

If φ is a Musielak-Orlicz function such that $\varphi(t, \cdot)$ vanishes only at zero for μ-a.e. $t \in T$, then $E_\varphi \in \mathbf{OC}$ if and only if $E \in \mathbf{OC}$ and $\varphi \in \Delta_2^E$.

Proof. Sufficiency. Take any $x \in E_\varphi$ and a sequence (x_n) in E_φ such that $0 \swarrow x_n \leq x$. Let $\lambda > 0$ be such that $\varphi \circ \lambda x \in E$. We have $0 \swarrow \varphi \circ \lambda x_n \leq \varphi \circ \lambda | x | \in E$, whence $\rho(\lambda x_n) = \| \varphi \circ \lambda x_n \|_E \to 0$. Since $\varphi \in \Delta_2^E(\varepsilon)$ for any $\varepsilon > 0$, this yields $\| x_n \|_\varphi \to 0$, which means that $x \in (E_\varphi)_a$. By the arbitrariness of $x \in E_\varphi$ this means that $E_\varphi \in \mathbf{OC}$.

Necessity. If $\varphi \notin \Delta_2^E$ then, in virtue of Lemma 4, E_φ contains an order isomorphically isometric copy of l^∞, so $E_\varphi \notin \mathbf{OC}$.

Assume now that $E \notin \mathbf{OC}$. Then there exist $x \in E$ and a sequence (x_n) in E such that $0 \swarrow x_n \leq x$ and $\| x_n \|_E = 1$ for each $n \in N$. Define $y_n = \varphi^{-1} \circ x_n$ and $y = \varphi^{-1} \circ x$. Then we get $0 \swarrow x_n = \varphi \circ y_n \leq \varphi \circ y = x$, whence $\rho(y_n) = \| \varphi \circ y_n \|_E = \| x_n \|_E = 1$, and consequently $\| y_n \|_\varphi = 1$ for each $n \in N$. Since $y \in E_\varphi$, this means that $E_\varphi \notin \mathbf{OC}$ and the proof is finished. □

Theorem 11

If $\varphi \in \Delta_2^E$ and E has the property $\mathbf{H}_\mu$, then E_φ has the property $\mathbf{H}_\mu$.

Proof. Under the assumptions of the theorem, take x and (x_n) in $(E_\varphi)_+$ such that $x_n \to x$ locally in measure and $\| x_n \|_\varphi \to \| x \|_\varphi$. We can assume without loss of generality that $\| x \|_\varphi = 1$. Then in virtue of Lemma 3, we have $\rho(x_n) \to \rho(x) = 1$. By σ-finiteness of the measure space we may assume (passing to a subsequence if necessary) that $x_n \to x$ μ-a.e. . So, $\varphi \circ x_n \to \varphi \circ x\, \mu$-a.e. and $\| \varphi \circ x_n \|_E \to \| \varphi \circ x \|_E$. By the assumption that $E \in \mathbf{H}_\mu$, this yields $\| \varphi \circ x_n - \varphi \circ x \|_E \to 0$. So, there exist $y \in E_+$ and a subsequence (x_{n_k}) of (x_n) such that

$$\left| \varphi \circ x_{n_k} - \varphi \circ x \right| \leq \varepsilon_{n_k} y$$

μ-a.e. for a sequence (ε_{n_k}) with $\varepsilon_{n_k} \searrow 0$ (cf. [12], Lemma 2, p. 141). We may assume without loss of generality that $\varepsilon_{n_k} \leq 1$ for all $k \in N$. Since $\varphi \in \Delta_2^E$ we have $\varphi \circ x \in E$ and consequently

$$\varphi \circ x_{n_k} \leq \left| \varphi \circ x_{n_k} - \varphi \circ x \right| + \varphi \circ x \leq y + \varphi \circ x \in E_+ .$$

Applying again $\varphi \in \Delta_2^E$,we get

$$\begin{aligned} \varphi \circ (x_{n_k} - x) \leq \; & \varphi \circ 2 \frac{(x_{n_k} - x)}{2} \leq \frac{K}{2} \left\{ \varphi \circ x_{n_k} + \varphi \circ x \right\} + h \\ \leq \; & \frac{K}{2} \left\{ y + \varphi \circ x + \varphi \circ x \right\} + h = K \varphi \circ x + \frac{K}{2} y + h \in E_+ . \end{aligned}$$

So, by $E \in \mathbf{OC}$, which follows by $E \in \mathbf{H}_\mu$ (cf. [10]), we get $\rho(x_{n_k} - x) = \| \varphi \circ (x_{n_k} - x) \|_E \to 0$. Applying now Lemma 8, we get $\| x_{n_k} - x \|_\varphi \to 0$. In virtue of the double extract subsequence theorem, we get $\| x_n - x \|_\varphi \to 0$. This means that $(E_\varphi)_+ \in \mathbf{H}_\mu$, and consequently (cf. [9]) $E_\varphi \in \mathbf{H}_\mu$ as well. The proof is complete. □

Note that in the case when φ does not depend on the parameter, Theorem 11 has been proved in [9].

Theorem 12

For any Musielak-Orlicz function φ the space E_φ has the Fatou property.

Proof. Assume for the contrary that $0 \le x_n \nearrow x$, $x_n \in E_\varphi$ for all $n \in N$, $x \in L^0$, $k = \sup_n \| x_n \|_\varphi < \infty$ and $\| x_n \|_\varphi \not\to \| x \|_\varphi$. Since $E \in \mathbf{FP}$, we can conclude that $\| x \|_\varphi < \infty$. In fact, by the definition of the norm $\| \cdot \|_\varphi$, we have

$$\rho\left(\frac{x_n}{k+1}\right) = \left\|\varphi \circ \frac{x_n}{k+1}\right\|_E \le 1\,,$$

for all $n \in N$ and $\varphi \circ (x_n/(k+1)) \nearrow \varphi \circ (x/(k+1))$. Therefore, by $E \in \mathbf{FP}$, we get

$$\rho\left(\frac{x}{k+1}\right) = \left\|\varphi \circ \frac{x}{k+1}\right\|_E = \lim_{n\to\infty} \left\|\varphi \circ \frac{x_n}{k+1}\right\|_E \le 1\,,$$

whence $\| x \|_\varphi \le k+1 < \infty$. We may assume without loss of generality that $\| x_n \|_\varphi \le \| x \|_\varphi - \varepsilon$, i.e. $\| x_n/(\| x \|_\varphi - \varepsilon) \|_\varphi \le 1$. Hence

$$\rho(x_n/(\| x \|_\varphi - \varepsilon)) = \| \varphi \circ (x_n/(\| x \|_\varphi - \varepsilon)) \|_E \le 1\,.$$

But $0 \le \varphi \circ (x_n/(\| x \|_\varphi - \varepsilon)) \nearrow \varphi \circ (x/(\| x \|_\varphi - \varepsilon))$. So, by $E \in \mathbf{FP}$, we get $\varphi \circ x/(\| x \|_\varphi - \varepsilon) \in E$ and $\| \varphi \circ \frac{x}{\| x \|_{\varphi - \varepsilon}} \|_E = \lim_{n\to\infty} \| \varphi \circ \frac{x_n}{\| x \|_{\varphi-\varepsilon}} \|_E \le 1$, which contradicts the definition of $\| x \|_\varphi$. So $\| x_n \|_\varphi \to \| x \|_\varphi$, i.e. $E \in \mathbf{FP}$. $\square$

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