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A Liouville-type theorem for very weak solutions of nonlinear partial differential equations

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Abstract

Let us consider the variational equation in $\mathbb{R}^n$

$$div\left(a(x)F'(\mid \nabla u\mid)\frac{\nabla u}{\mid \nabla u\mid}\right)=0$$

where $0<\lambda_0\le a(x)\le \Lambda_0<\infty$ and F is a convex increasing function verifying suitable conditions. We prove that the *very weak solutions* of such equation, whose gradient belongs to a suitable Orlicz space, must be constant almost everywhere. The result applies, in particular, to the case in which F is the power $F(t)=t^p$ $(p>1)$, i.e. to the variational equation in $\mathbb{R}^n$

$$div\left(a(x)\mid \nabla u\mid^{p-2}\nabla u\right)=0.$$

1. Introduction

Throughout the paper we will denote by $F=F(t)$ a convex differentiable increasing function on $[0,\infty[$ such that $pF(t)\le tF'(t)\le qF(t)$ $\forall t\ge 0$ where $1<p\le q<\infty$, and such that $\liminf_{t\to 0}\frac{tF'(t)}{F(t)}>n$ or $\limsup_{t\to\infty}\frac{tF'(t)}{F(t)}\le n$. Let us consider the *very weak* solutions of the variational equation in $\mathbb{R}^n$

$$div\left(a(x)F'(\mid \nabla u\mid)\frac{\nabla u}{\mid \nabla u\mid}\right)=0, \tag{1.1}$$

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where $a(x)$ is a measurable function such that $0 < \lambda_0 \leq a(x) \leq \Lambda_0 < \infty$, i.e. (see Iwaniec-Sbordone [8]) the functions $u \in W^{1,1}_{loc}(\mathbb{R}^n)$, $|\nabla u| \in L_{F_r}(\mathbb{R}^n)$, $F_r(t) = F(t)t^{r-p}$, $\max\{1, p-1\} \leq r < p$, such that

$$\int_{\mathbb{R}^n} a(x)F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\phi = 0, \qquad \forall\phi \in W^{1,\infty}(\mathbb{R}^n) \quad \text{with compact support.}$$

The definition of *very weak* solution is best visualized when F is the power $F(t) = t^p$ $(p > 1)$. In this case the equation (1.1) reduces to the variational equation in $\mathbb{R}^n$

$$div\left(a(x)|\nabla u|^{p-2}\nabla u\right) = 0, \tag{1.2}$$

and any *weak* solution $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ of (1.2) must satisfy the identity

$$\int_{\mathbb{R}^n} a(x)|\nabla u|^{p-2}\nabla u\nabla\phi = 0, \qquad \forall\phi \in W^{1,\infty}(\mathbb{R}^n) \quad \text{with compact support.} \tag{1.3}$$

In order to give meaning to the integral in (1.3), the assumption $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ is not necessary. Actually, it will be sufficient to assume

$$u \in W^{1,r}_{loc}(\mathbb{R}^n), \qquad \max\{1, p-1\} \leq r < p. \tag{1.4}$$

Any function u verifying (1.4) is called a *very weak solution* (see [10]) of equation (1.2) if (1.3) holds for any $\phi \in W^{1,\infty}(\mathbb{R}^n)$ with compact support.

The aim of this paper is to prove the following Liouville-type theorem.

Theorem 1.1

There exists $r_0 < p$ such that, if u is a very weak solution of (1.1) such that $|\nabla u| \in L_{F_r}(\mathbb{R}^n)$, with $r_0 < r < p$, then u is constant.

If F is such that $\liminf_{t\to 0} \frac{tF'(t)}{F(t)} > n$ or $\limsup_{t\to\infty} \frac{tF'(t)}{F(t)} \leq n$, then from Theorem 1.1 we can deduce, in particular, the main results of [4], [2] in which, under a further assumption of integrability on u, it is proved that u must be zero a.e. The proof of Theorem 1.1 will be deduced by using the same technique (introduced by Lewis in [9]) as in [4], without any integrability assumption on u.

We remark that the Liouville theorem for *weak* solutions of the p-harmonic equation is well-known (see [7], for instance, in which also nonhomogeneous equations are considered).

2. Notations and preliminary results

We begin with the following

Remark 2.1. If

$$\liminf_{t\to 0} \frac{tF'(t)}{F(t)} > n, \quad \text{then} \quad \lim_{t\to 0} \frac{F(t)}{t^n} = 0.$$

The statement follows by noticing that for small $\epsilon > 0$ the function $\frac{F(t)}{t^{n+\epsilon}}$ has first derivative positive near zero, and therefore has a finite limit when $t \to 0$.

Next theorem is well known in the theory of Sobolev spaces. We will use the following version, which is a generalization in the context of the Orlicz-Sobolev spaces theory.

Theorem 2.2 ([11], [3])

If $pF(t) \leq tF'(t) \leq qF(t)$, $\forall t \geq 0$ with $1 < p \leq q < n$, and if $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ is such that $\mid Du \mid \in L_F(\mathbb{R}^n)$, then there exists a constant $c \in \mathbb{R}$ such that $u - c \in L_{F_*}(\mathbb{R}^n)$, where F_* is the Sobolev conjugate function of F defined by

$$F_*^{-1}(t) = \int_0^t \frac{F^{-1}(\tau)}{\tau^{1+1/n}} d\tau \qquad \forall t \geq 0.$$

Let us remark that, more generally, Theorem 2.2 is true under the assumption $1 < i(F) \leq I(F) < n$, where $i(F)$, $I(F)$ are the reciprocal of the Boyd indices of F: this fact can be deduced by using some relations between the Simonenko indices and the Boyd indices (see [5]).

Let us note also that functions u verifying the assumptions of Theorem 2.2 are such that Mu is almost everywhere finite, where M is the Hardy-Littlewood maximal operator defined by

$$Mu(y) = \sup_{Q \ni y} \fint_Q u(x)dx$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing y. The proof of Theorem 2.2 may be carried out by using the Riesz potential in a standard way. It is easy to realize also that

$$c = \lim_{\rho\to\infty} \fint_{B_{\rho(y)}} u(x)dx \qquad \forall y \in \mathbb{R}^n$$

where $B_\rho(y) = \{x \in \mathbb{R}^n :| y - x |< \rho\}$. We observe also that by using results proved in [1] about Riesz potentials, if

$$\int_0^\infty \frac{\widetilde{F}(t)}{t^{1+n/n-1}} dt < \infty$$

where $\widetilde{F}$ denotes the conjugate function of F, and if $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ is such that $| Du |\in L_F(\mathbb{R}^n)$, then u is bounded. This result is proved in [1] for functions belonging to the Orlicz-Sobolev space $W^1 L_F(\mathbb{R}^n)$.

Next theorem, due to Gustavsson-Peetre ([6]), is from Interpolation theory, and is a particular case of the original statement.

Theorem 2.3

Let p^*, p, q^*, $q \in]1, \infty[$ and let T be a continuous linear operator

$$T : L_{p^*}(\Omega) \to L_p(\Omega)$$

$$T : L_{q^*}(\Omega) \to L_q(\Omega)$$

respectively with norm $\|T\|_{p^*,p}$, $\|T\|_{q^*,q}$, where Ω is a bounded open set in $\mathbb{R}^n$.

Let $\eta :]0, \infty[\to]0, \infty[$ be such that

$$\bar{\alpha}\eta(t) \le t\eta'(t) \le \bar{\beta}\eta(t) \qquad \forall t > 0$$

for some $\bar{\alpha}$, $\bar{\beta} \in]0, 1[$ and let

$$A^{-1}(t) = t^{1/q^*} \eta \left(t^{1/p^* - 1/q^*}\right) \quad , \quad B^{-1}(t) = t^{1/q} \eta \left(t^{1/p-1/q}\right).$$

Then the operator

$$T : L_A(\Omega) \to L_B(\Omega)$$

is a continuous linear operator with norm $\|T\|_{A,B} \le max(\|T\|_{p^*,p}, \|T\|_{q^*,q})$.

Next lemmas are parts of the proof of the main theorem of [4].

Lemma 2.4 ([4])

If $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ is a very weak solution of (1.1) such that $| \nabla u |\in L_{F_r}(\mathbb{R}^n)$ with $r_0 < r < p$, and if u^k is the truncation of u at levels k and $-k$, $u^k_\rho = u^k \phi_\rho$ where ϕ_ρ are cut-off, $\lambda_\rho = c\rho^{-n} \int_{B_{2\rho}} | \nabla u^k_\rho(y) | \, dy$, and

$$E(\lambda_\rho) = \{x \in \mathbb{R}^n : M(| \nabla u^k_\rho(x) |) \le \lambda_\rho\},$$

then we have $\lim_{\rho\to\infty} \lambda_\rho = 0$ and $\chi_{E(\lambda_\rho)\longrightarrow 0}$ a.e. in $\mathbb{R}^n$.

Lemma 2.5 ([4])

If $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ *is a very weak solution of* (1.1) *such that* $|\nabla u| \in L_{F_r}(\mathbb{R}^n)$ *with* $r_0 < r < p$, *then if* $\delta = p - r > 0$ *the following inequality holds*

$$\frac{1}{\delta}\int_{B_{4\rho}\setminus E(\lambda_{\rho_h})} aF'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla u^k_\rho (M(|\nabla u^k_\rho|))^{-\delta}dx \tag{2.1}$$

$$+\frac{\lambda_\rho^{-\delta}}{\delta}\int_{E(\lambda_{\rho_h})} aF'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla u^k_\rho dx$$

$$\leq \frac{c}{1-\delta}\int_{B_{4\rho}} aF'(|\nabla u|)(M(|\nabla u^k_\rho|))^{1-\delta}dx.$$

3. Proof of Theorem 1.1

We will proceed as follows: first we prove it suffices to show that there exists $r_0 < p$ such that for every $k > 0$, $r_0 < r < p$ there exists a sequence $(\rho_h)_{h\in\mathbb{N}}$ such that

$$\lim_{h\to\infty}\frac{\|u^k\|_{L_{F_r(\Omega_{\rho_h})}}}{\rho_h} = 0 \tag{3.1}$$

where u^k for $k > 0$ is defined by

$$u^k(x) = \begin{cases} u & \text{if} \quad |u(x)| \leq k \\ k & \text{if} \quad u(x) \geq k \\ -k & \text{if} \quad u(x) \leq -k \end{cases}$$

and $\Omega_\rho = B_{2\rho} - B_\rho \quad \forall \rho > 0$.

Then we will prove (3.1) by considering the following three cases:

Case 1: $\displaystyle\liminf_{t\to 0}\frac{tF'(t)}{F(t)} > n.$

Case 2: $\displaystyle\limsup_{t\to 0}\frac{tF'(t)}{F(t)} \leq n$ and $\displaystyle\limsup_{t\to\infty}\frac{tF'(t)}{F(t)} \leq n.$

Case 3: $\displaystyle\limsup_{t\to 0}\frac{tF'(t)}{F(t)} \leq n$ and $\displaystyle\limsup_{t\to\infty}\frac{tF'(t)}{F(t)} > n.$

To show that (3.1) is sufficient to prove Theorem 1.1, we begin by noticing that

$$\begin{aligned}\|\nabla u^k_\rho - \nabla u^k\|_{L_{F_r}(\mathbb{R}^n)} &= \|(\nabla u^k)(\phi_\rho - 1) + u^k \nabla\phi_\rho\|_{L_{F_r}(\mathbb{R}^n)}\\ &\le \|(\nabla u^k)(\phi_\rho - 1)\|_{L_{F_r}(\mathbb{R}^n)} + \|u^k\nabla\phi_\rho\|_{L_{F_r}(\mathbb{R}^n)}.\end{aligned}$$

The first term on the right hand side goes to 0 as $\rho \to \infty$ because of the Lebesgue Dominated Convergence Theorem and

$$\|u^k\nabla\phi_\rho\|_{L_{F_r}(\mathbb{R}^n)} \le \frac{\|u^k\|_{L_{F_r(\Omega_\rho)}}}{\rho}.$$

By using (3.1) we get that there exists a sequence $(\rho_h)_{h\in\mathbb{N}}$ such that $\nabla u^k_{\rho_h} \to \nabla u^k$ in $L_{F_r}(\mathbb{R}^n)$. Now let us pass to the limit in the second term on the left hand side of (2.1). We will call $H(x)$ a function in $L_{F_r}(\mathbb{R}^n)$ that majorizes a subsequence of $\nabla u^k_{\rho_h}$ and we will denote, for simplicity of notations, by ρ_h again, the relative subsequence of indices. We have

$$\begin{aligned}&\Big|\int_{E(\lambda_{\rho_h})} aF'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla u^k_{\rho_h}\big(M(|\nabla u^k_{\rho_h}|)\big)^{-\delta}dx\Big|\\ &\quad= \Big|\int_{E(\lambda_{\rho_h})\cap\Omega_{\rho_h}} aF'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla u^k_{\rho_h}\big(M(|\nabla u^k_{\rho_h}|)\big)^{-\delta}dx\Big|\\ &\quad\le \int_{\Omega_{\rho_h}} aF'(|\nabla u|)\big(M(|\nabla u^k_{\rho_h}|)\big)^{1-\delta}dx\\ &\quad\le \int_{\Omega_{\rho_h}} aF'(|\nabla u|)\big(M(H)\big)^{1-\delta}dx \to 0\end{aligned}$$

because the last integrand is in $L_1(\mathbb{R}^n)$ by virtue of Lemma 2.3 of [4].

At this point we can conclude the proof by using Lemma 2.4 exactly as in [4], one has only to remember that the δ in Step 4 has to be chosen also such that $0 < \delta < p - r_0$.

Now we prove (3.1).

Case 1: $\displaystyle\liminf_{t\to 0}\frac{tF'(t)}{F(t)} > n.$

Since

$$\frac{tF_r'(t)}{F_r(t)} = \frac{tF'(t)}{F(t)} - (p-r),$$

let $1 < r_0 < p$ be such that for every $r_0 < r < p$

$$\liminf_{t\to 0} \frac{tF_r'(t)}{F_r(t)} > n$$

and let $(t_h)_{h\in\mathbb{N}}$ any decreasing sequence such that $t_h \to 0$, and therefore, by Remark 2.1, such that

$$\lim_{h\to\infty} \frac{F_r(t_h)}{t_h^n} = 0\,.$$

Set $c_n = [(2^n - 1)\omega_n]^{-1}$ where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$, and

$$\rho_h = \left(\frac{c_n}{F_r(t_h)}\right)^{1/n}.$$

We have

$$\frac{\|u^k\|_{L_{F_r(\Omega_{\rho_h})}}}{\rho_h} \le \frac{k\|1\|_{L_{F_r(\Omega_{\rho_h})}}}{\rho_h} = \frac{k}{\rho_h F_r^{-1}(c_n\rho_h^{-n})} = k\left(\frac{F_r(t_h)}{c_n}\right)^{1/n}\frac{1}{t_h} \to 0$$

as $h \to \infty$ and therefore in this case the proof is complete.

Case 2: $\displaystyle\limsup_{t\to 0} \frac{tF'(t)}{F(t)} \le n$ and $\displaystyle\limsup_{t\to\infty} \frac{tF'(t)}{F(t)} \le n.$

Since the inverse $I(F)$ of the upper Boyd index of F is such that (see [5])

$$I(F) \le \max\left\{\limsup_{t\to 0}\frac{tF'(t)}{F(t)}, \limsup_{t\to\infty}\frac{tF'(t)}{F(t)}\right\},$$

we have $I(F) \le n$ and therefore (see [5]) $I(F_r) \le n-(p-r) < n$, so we may assume, eventually considering a function equivalent to F_r, that

$$p_rF_r(t) \le tF_r'(t) \le q_rF_r(t) \qquad \forall t \ge 0 \tag{3.2}$$

for some $1 < p_r, q_r < n$. Just to simplify the notations, let us drop the index r in all the symbols of (3.2), until the end of the proof of this case: no confusion arise, because we never need to consider the function F.

The proof of (3.1) easily follows from the following inequality

$$\|v\|_{L_F(\Omega_\rho)} \le c(n)\rho\|v\|_{L_{F_*}(\Omega_\rho)} \qquad \forall v \in L_{F_*(\mathbb{R}^n)}, \quad \forall \rho > 0 \tag{3.3}$$

where F^* is the Sobolev-conjugate function of F, in fact, after (3.3), we have

$$0 \leq \lim_{\rho\to\infty} \frac{\|u^k\|_{L_{F_r(\Omega_\rho)}}}{\rho_h} \leq c(n) \lim_{\rho\to\infty} \|u^k\|_{L_{F_*}(\Omega_\rho)} = 0$$

because we may assume, without loss of generality, that $u^k \in L_{F_*(\mathbb{R}^n)}$ by virtue of Theorem 2.2.

In the case of powers, i.e. $F(t) = t^r$ with $r \in]1, n[$, inequality (3.3) follows from Hölder inequality:

$$\text{(3.4)} \qquad \|v\|_{L_r(\Omega_\rho)} \leq \|v\|_{L_{r^*}(\Omega_\rho)} \mid \Omega_\rho \mid^{(1-r/r^*)1/r} = c(n)\rho\|v\|_{L_{r^*}(\Omega_\rho)}$$

where $c(n) = [(2^n-1)\omega_n]^{1/n}$ and $r^* = \dfrac{nr}{n-r}$ is the Sobolev-conjugate exponent of r.

If F is not a power, we will proceed by the following interpolation argument. By (3.2) we have

$$\frac{1}{q}F^{-1}(t) \leq t\big(F^{-1}(t)\big)' \leq \frac{1}{p}F^{-1}(t) \qquad \forall t > 0,$$

with $\frac{1}{n} < \frac{1}{q} \leq \frac{1}{p} < 1$. Therefore F^{-1} may be written as follows

$$F^{-1}(t) = t^{1/q_1}\eta\left(t^{1/p_1 - 1/q_1}\right) \qquad \forall t \geq 0$$

with η as in Theorem 2.3 and $\frac{1}{n} < \frac{1}{q_1} < \frac{1}{q} \leq \frac{1}{p} < \frac{1}{p_1} < 1$. Let p_1^*, q_1^* be the Sobolev-conjugate exponents of p_1, q_1 and let T be the identity operator. We can apply Theorem 2.3 with $\Omega = \Omega_\rho$ because (3.4) shows that

$$\|T\|_{p_1^*, p_1} \leq c(n)\rho, \qquad \|T\|_{q_1^*, q_1} \leq c(n)\rho.$$

Then we have

$$\|v\|_{L_B(\Omega_\rho)} \leq c(n)\rho\|v\|_{L_A(\Omega_\rho)}$$

with $B(t) = F(t)$ and $A(t)$ given by

$$A^{-1}(t) = t^{1/q_1^*}\eta\left(t^{1/p_1^* - 1/q_1^*}\right) \qquad \forall t \geq 0.$$

But $L_A(\Omega_\rho) = L_{F_*}(\Omega_\rho)$ because the function A^{-1} is equivalent to F_*^{-1}, in fact

$$\begin{aligned} F_*^{-1}(t) &= \int_0^t \frac{F^{-1}(\tau)}{\tau^{1+1/n}}d\tau = \int_0^t \frac{\tau^{1/q_1}\eta\left(\tau^{1/p_1-1/q_1}\right)}{\tau^{1+1/n}}d\tau \\ &= \int_0^t \frac{\tau^{1/q_1^*}\eta\left(\tau^{1/p_1^*-1/q_1^*}\right)}{\tau}d\tau = \int_0^t \frac{A^{-1}(\tau)}{\tau}d\tau. \end{aligned}$$

Therefore we obtain (3.3) and Theorem 1.1 is proved also in this case.

Case 3: $\limsup_{t\to 0} \frac{tF'(t)}{F(t)} \leq n$ and $\limsup_{t\to\infty} \frac{tF'(t)}{F(t)} > n.$

Roughly speaking, this case will be treated by noticing that the behavior of F_r on big values of t does not influence substantially the norm of u^k in $L_{F_r}(\Omega_\rho)$, and therefore this case can be reduced to the previous one. Now let us see the proof in details.

Since

$$\frac{tF_r'(t)}{F_r(t)} = \frac{tF'(t)}{F(t)} - (p-r),$$

let $1 < r_0 < p$ be such that for every $r_0 < r < p$

$$\limsup_{t\to 0} \frac{tF_r'(t)}{F_r(t)} < n \qquad \text{and} \qquad \limsup_{t\to\infty} \frac{tF_r'(t)}{F_r(t)} > n$$

and let $\bar{t} > 0$ be such that

$$rF_r(t) \leq tF_r'(t) \leq \bar{q}F_r(t) \qquad \forall t \in [0, \bar{t}\,]$$

for some $r < \bar{q} < n$. Define

$$G(t) = \begin{cases} F_r(t) & \text{if } \ t \in [0, \bar{t}\,] \\ \dfrac{F_r(\bar{t})}{\bar{t}^{\,r}} t^r & \text{if } \ t \in]\bar{t}, \infty[\end{cases}$$

so that G is convex, increasing and such that

$$G(t) \leq F_r(t) \qquad \forall t \geq 0 \tag{3.5}$$

and

$$1 < r \leq \frac{tG'(t)}{G(t)} \leq \bar{q} < n \qquad \forall t > 0.$$

We may assume that

$$\limsup_{\rho\to\infty} \|u^k\|_{L_{F_r}(\Omega_\rho)} > 0,$$

otherwise the proof is trivial, and therefore there exists $(\rho_h)_{h\in\mathbb{N}}$ such that

$$\|u^k\|_{L_{F_r}(\Omega_{\rho_h})} > \bar{\epsilon} \qquad \forall h \in \mathbb{N}$$

for some $0 < \bar{\epsilon} < \frac{k}{\bar{t}}$. By (3.5) we have

$$G(t) \leq F_r(t) \leq G(t)\frac{F_r\left(\frac{k}{\bar{\epsilon}}\right)}{G(\bar{t})} \qquad \forall t \in \left[0, \frac{k}{\bar{\epsilon}}\right]$$

and therefore

$$\begin{aligned}\|u^k\|_{L_{F_r}(\Omega_{\rho_h})} &= \inf\left\{\lambda > 0 : \int_{\Omega_{\rho_h}} F_r\left(\frac{u^k}{\lambda}\right) dx \leq 1\right\} \\ &= \inf\left\{\lambda > \bar{\epsilon} : \int_{\Omega_{\rho_h}} F_r\left(\frac{u^k}{\lambda}\right) dx \leq 1\right\} \leq c(\bar{t}, k, \bar{\epsilon})\|u^k\|_{L_G(\Omega_{\rho_h})}\end{aligned}$$

from which, by the Case 2 applied to G, we have the assertion. □

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