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Locally solid topologies on vector valued function spaces

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Abstract

Locally solid topologies on vector valued function spaces are studied. The relationship between the solid and topological structures of such spaces is examined.

0. Introduction and preliminaries

The topological structure of scalar valued function spaces, in particular Banach function spaces, has been examined intensively by means of the theory of locally solid Riesz spaces (see [3], [11], [13], [25], [26]).

For a given real Banach space $(X, \|\cdot\|_X)$ and an ideal E of L^0 one can consider Banach space valued function spaces $E(X)$ defined as the subspaces of the space $L^0(X)$ of strongly measurable functions and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. When E is a Banach function space (in particular a Lebesgue space L^p or an Orlicz space L^φ) the space $E(X)$ is usually called a Köthe-Bochner space (resp. a Lebesgue-Bochner space or an Orlicz-Bochner space). The geometric and topological properties of Köthe-Bochner spaces $E(X)$ were studied by A.V. Bukhvalov [5]. The order structure of $E(X)$ when X is a Banach lattice was examined by E. de Jonge [9], [10] and A.V. Bukhvalov [6]. For X being a locally convex lattice and E being a Banach function space the order properties of $E(X)$ were studied by C.W. Mullins [18]. N.P. Cac [7] and A.L. Macdonald [13], [14], [15] examined the topological structure and the dual of function spaces consisting of measurable functions defined on the locally compact

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Hausdorff topological spaces with a positive Radon measure and with values in a locally convex vector space.

In this paper, we examine the topological structure of the space $E(X)$ in case E is an ideal of L^0 and X is a real Banach space. It turns out that the notion of a locally solid topology defined in the theory of locally solid Riesz spaces can be in a natural way defined in $E(X)$.

In Section 1 the solid structure of the spaces $E(X)$ is considered. A subset H of $E(X)$ is said to be solid whenever $\|f_1(\omega)\|_X \leq \|f_2(\omega)\|_X$ μ-a.e. and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$. An equivalent of the Riesz decomposition property (see [1]) for $E(X)$, called here the solid decomposition property is obtained (see Lemma 1.1). This property is of a key importance for the study of the topological structure of $E(X)$.

In Section 2 we introduce locally solid topologies on $E(X)$ as linear topologies having a base of neighborhoods of 0 consisting of solid sets. It is shown that as in the theory of locally solid Riesz spaces (see [1, Theorem 6.3], [8, Proposition 2.2.C]) every locally solid topology on $E(X)$ can be generated by some family of solid pseudonorms on $E(X)$.

In Section 3 we examine the relationship between the topological structures of E and $E(X)$. It is shown that many of the topological properties of E can be lifted to $E(X)$. The first such results were obtained by A.V. Bukhvalov ([4, Theorem 2, Theorem 3]).

Section 4 deals with entire topologies on $E(X)$. It is proved that a locally solid topology τ on $E(X)$ is entire iff the embedding $(E(X)\tau) \hookrightarrow (L^0(X), \mathcal{T}_0(X))$ is continuous.

In Section 5 we examine locally solid topologies on $E(X)$ that are continuous with respect to natural order convergence in $E(X)$. Following the terminology of the theory of locally solid Riesz spaces we will call such topologies Lebesgue topologies. Finally, in Section 6 we describe the finest Lebesgue topology on Orlicz-Bochner spaces $L^\varphi(X)$.

For notation and terminology concerning locally solid Riesz spaces we refer to [1], [2]. As usual, $\mathbb{N}$ stands for the set of all natural numbers.

Throughout this paper let (Ω, Σ, μ) be a complete σ-finite measure space and let L^0 denote the corresponding linear space of equivalence classes of all Σ-measurable real valued functions. Then L^0 is a super Dedekind complete Riesz space under the ordering $u_1 \leq u_2$ whenever $u_1(\omega) \leq u_2(\omega)$ a.e. on Ω.

The Riesz F-norm

$$\|u\|_0 = \int_\Omega \frac{|u(\omega)|}{1+|u(\omega)|} w(\omega) d\mu$$

for $u \in L^0(\Omega)$, where $w: \Omega \to (0,\infty)$ is a Σ-measurable function with $\int_\Omega w(\omega)d\mu = 1$, determines the Lebesgue topology $\mathcal{T}_0$ on L^0, which generates the convergence in measure on subsets of finite measure.

Let $(X, \|\cdot\|_X)$ be a real Banach space. By $L^0(X)$ we denote the linear space of equivalence classes of all strongly Σ-measurable functions $f: \Omega \to X$. Then the F-norm

$$\|f\|_{L^0(X)} = \int_\Omega \frac{\|f(\omega)\|_X}{1+\|f(\omega)\|_X} w(\omega)d\mu$$

for $f \in L^0(X)$, generates the topology $\mathcal{T}_0(X)$ on $L^0(X)$ of convergence in measure on sets of finite measure.

For a function $f \in L^0(X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Throughout the paper E will be an ideal of L^0 with supp $E = \Omega$. The space

$$E(X) = \{f \in L^0(X): \widetilde{f} \in E\}$$

is called here a vector valued function space.

1. The solid structure of vector valued function spaces

In this section we examine the solid structure of $E(X)$.

Definition 1.1. (i) A subset H of $E(X)$ is said to be *solid* whenever $\|f(\omega)\|_X \leq \|g(\omega)\|_X$ μ-a.e. and $f \in E(X)$, $g \in H$ imply $f \in H$.

(ii) A linear subspace I of $E(X)$ is called an *ideal* of $E(X)$ if I is a solid subset of $E(X)$.

Note that $E(X)$ is an ideal of $L^0(X)$. Since the intersection of any family of solid subsets of $E(X)$ is solid, every subset A of $E(X)$ is contained in the smallest (with respect to the inclusion) solid set called the *solid hull* of A and denoted by $S(A)$. Note that

$$S(A) = \{g \in E(X): \|g(\omega)\|_X \leq \|f(\omega)\|_X \ \ \mu\text{-a.e. for some } f \in A\}.$$

One can easily verify that $S(\lambda A) = \lambda S(A)$ for $\lambda > 0$. The following lemma will be of a key importance for examination of the solid structure of $E(X)$.

Lemma 1.1 [The solid decomposition property]

Assume that in $L^0(X)$

$$\|f(\omega)\|_X \leq \|g_1(\omega) + g_2(\omega) + \cdots + g_n(\omega)\|_X \quad \mu\text{-a.e.}$$

Then there exist $f_1, \ldots, f_n \in L^0(X)$ satisfying

$$\|f_i(\omega)\|_X \le \|g_i(\omega)\|_X \quad \mu\text{-a.e.} \quad (i = 1, 2, \ldots, n) \quad \text{and} \quad f = f_1 + \cdots + f_n.$$

Proof. By using induction it is enough to establish the result for $n = 2$. Thus assume that $\|f(\omega)\|_X \le \|g_1(\omega) + g_2(\omega)\|_X$ μ-a.e.

Let us put (for $i = 1, 2$):

$$f_i(\omega) = \begin{cases} \dfrac{\widetilde{g}_i(\omega)}{\widetilde{g}_1(\omega) + \widetilde{g}_2(\omega)} \cdot f(\omega) & \text{if} \quad \widetilde{g}_1(\omega) + \widetilde{g}_2(\omega) > 0, \\ 0 & \text{if} \quad \widetilde{g}_1(\omega) + \widetilde{g}_2(\omega) = 0. \end{cases}$$

It is seen that $f_i \in L^0(X)$ and $f_1 + f_2 = f$.

To show that $\widetilde{f}_i \le \widetilde{g}_i$ for $i = 1, 2$, assume first that $\widetilde{g}_1(\omega_0) + \widetilde{g}_2(\omega_0) > 0$ for $w_0 \in \Omega$. Then

$$\begin{aligned} \widetilde{f}_i(\omega_0) &= \frac{\widetilde{g}_i(w_0)}{\widetilde{g}_1(w_0) + \widetilde{g}_2(w_0)} \widetilde{f}(\omega_0) \le \frac{\widetilde{g}_i(w_0)}{\widetilde{g}_1(\omega_0) + \widetilde{g}_2(\omega_0)} \big(\widetilde{g}_1(\omega_0) + \widetilde{g}_2)\big) \\ &= \widetilde{g}_i(\omega_0). \end{aligned}$$

Next, let $\widetilde{g}_1(\omega_0) + \widetilde{g}_2(\omega_0) = 0$ for some $\omega_0 \in \Omega$. Then $\widetilde{f}_i(\omega_0) = 0 = \widetilde{g}_i(\omega_0)$.

Thus the proof is complete. □

Theorem 1.2

The convex hull conv H *of a solid subset* H *of* $E(X)$ *is a solid set.*

Proof. Let H be a subset of $E(X)$, and let $\|f(\omega)\|_X \le \|g(\omega)\|_X$ μ-a.e., where $f \in E(X)$ and $g \in \operatorname{conv} H$. Then there exist $g_1, \ldots, g_n \in H$ and nonnegative numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $g = \sum_{i=1}^n \alpha_i g_i$. Hence by Lemma 1.1 there exist $f_1, \ldots, f_n \in L^0(X)$ such that $\|f_i(\omega)\|_X \le \alpha_i \|g_i(\omega)\|_X$ μ-a.e. for $i = 1, 2, \ldots, n$ and $f = \sum_{i=1}^n f_i$. Putting $h_i = \alpha_i^{-1} \cdot f_i$ we get $\|h_i(\omega)\|_X \le \|g_i(\omega)\|_X$ μ-a.e. for $i = 1, 2, \ldots, n$, so $h_i \in H$ for $i = 1, 2, \ldots$. But then $f = \sum_{i=1}^n f_i = \sum_{i=1}^n \alpha_i h_i \in \operatorname{conv} H$, so conv H is solid. □

2. Locally solid topologies on Banach-space valued function spaces

We define locally solid topologies on $E(X)$ that bind the solid and topological structures of the space $E(X)$ together.

Definition 1.2. A linear topology τ on $E(X)$ is said to be *locally solid* if it has a basis for neighborhoods of zero consisting of solid sets.

Theorem 2.1

Let τ be a locally solid topology on $E(X)$. Then the τ-closure $\overline{H}$ of a solid subset H of $E(X)$ is a solid set.

Proof. Let $\mathcal{B}_\tau$ be a basis at zero for τ consisting of solid sets. Then $\overline{H} = \bigcap\{H + V : V \in \mathcal{B}_\tau\}$. Let $\|f(\omega)\|_X \le \|g(\omega)\|_X$ μ-a.e. and $f \in E(X)$, $g \in \overline{H}$, and let $V_0 \in \mathcal{B}_\tau$. Then $g = g_1 + g_2$, where $g_1 \in H$ and $g_2 \in V_0$. Since $\|f(\omega)\|_X \le \|g_1(\omega) + g_2(\omega)\|_X$ μ-a.e., by Lemma 1.1 there exist $f_1, f_2 \in L^0(X)$ such that $f = f_1 + f_2$ and $\|f_1(\omega)\|_X \le \|g_1(\omega)\|_X$ and $\|f_2(\omega)\|_X \le \|g_2(\omega)\|_X$ μ-a.e. Since $E(X)$ is an ideal of $L^0(X)$, $f_1, f_2 \in E(X)$, so $f_1 \in H$ and $f_2 \in V_0$, because the sets H and V_0 are solid. Thus $f \in H + V$ for every $V \in \mathcal{B}_\tau$, so $f \in \overline{H}$; hence $\overline{H}$ is solid. □

Definition 2.2. A linear topology τ on $E(X)$ that is at the same time locally solid and locally convex will be called *a locally convex-solid topology* on $E(X)$.

In view of Theorem 1.2 and Theorem 2.1 we see that for a locally convex-solid topology τ on $E(X)$ the collection of all τ-closed, convex and solid τ-neighborhoods of zero forms a basis at zero for τ.

Definition 2.3. A pseudonorm (resp. a seminorm) on $E(X)$ is said to be *solid*, whenever $\rho(f_1) \le \rho(f_2)$ if $f_1, f_2 \in E(X)$ and $\|f_1(\omega)\|_X \le \|f_2(\omega)\|_X$ μ-a.e.

The next two theorems tell us that as in the theory of locally solid Riesz spaces every locally solid topology (resp. locally convex-solid topology) τ on $E(X)$ can be generated by some family of solid pseudonorms (resp. solid seminorms).

Theorem 2.2

For a linear topology τ on $E(X)$ the following statements are equivalent:

(i) τ is generated by some family of solid seminorms defined on $E(X)$.

(ii) τ is a locally convex-solid topology.

Proof. (i) ⇒ (ii) Obvious.

(ii) ⇒ (i) Let $\mathcal{B}_\tau = \{V_\alpha : \alpha \in \{\alpha\}\}$ be a basis of 0 for τ consisting of τ-closed, solid and convex sets. Let ρ_α denote the Minkowski functional generated by V_α, that is

$$\rho_\alpha(f) = \inf\left\{\lambda > 0 : f \in \lambda V_\alpha\right\} \quad \text{for} \quad f \in E(X).$$

Then ρ_α is a solid τ-continuous seminorm and

$$\left\{f \in E(X) : \rho_\alpha(f) < 1\right\} \subset V_\alpha = \left\{f \in E(X) : \rho_\alpha(f) \le 1\right\}.$$

This means that the family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ generates the topology τ. □

Theorem 2.3

For a linear topology τ on $E(X)$ the following statements are equivalent:

(i) τ *is generated by some family of solid pseudonorms defined on* $E(X)$*.*

(ii) τ *is a locally solid topology.*

Proof. (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) Let $\mathcal{V}$ be a τ-neighborhood of 0. Choose a sequence of solid τ-neighborhoods of 0 such that $V_0 \subset \mathcal{V}$ and $V_{n+1} + V_{n+1} + V_{n+1} \subset V_n$ for $n = 0, 1, 2, \ldots$.

Define a function $d: E(X) \to \mathbb{R}$ by

$$d(f) = \begin{cases} 1 & \text{if } f \notin V_0 \\ 2^{-n} & \text{if } f \in V_n \setminus V_{n+1}, \quad n = 0, 1, 2, \ldots \\ 0 & \text{if } f \in \bigcap_{n=1}^{\infty} V_n. \end{cases}$$

Then $\|f_1(\omega)\|_X \leq \|f_2(\omega)\|_X$ μ-a.e. implies $d(f_1) \leq d(f_2)$, because V_n for $n = 0, 1, 2, \ldots$ are solid.

Define ρ: $E(X) \to \mathbb{R}$ by

$$\rho(f) = \inf \left\{ \sum_{i=1}^{n} d(f_i) \colon f = \sum_{i=1}^{n} f_i, \quad n \in \mathbb{N} \right\}.$$

Using Lemma 1.1 and arguing as in the proof of [8, Proposition 2.2.C] one can check that ρ is a τ-continuous solid pseudonorm on $E(X)$ and $\{f \in E(X) \colon \rho(f) < 1\} \subset \mathcal{V}$. It follows that τ is generated by some family of solid pseudonorms. □

3. The relationship between topological structures of E and $E(X)$

In this section we examine the relationship between topological structures of E and $E(X)$. It is shown that some topological properties of E are inherited by $E(X)$.

Let E be an ideal of L^0 with supp $E = \Omega$. Given a Riesz pseudonorm (resp. a Riesz seminorm) p on E, let us set

$$\overline{p}(f) = p(\widetilde{f}) \quad \text{for all} \quad f \in E(X).$$

It is easy to check that $\overline{p}$ is a solid pseudonorm (resp. a solid seminorm) on $E(X)$.

Let $x \in S_X$. Given $u \in E$ let us put $\overline{u}(\omega) = u(\omega) \cdot x$ for $\omega \in \Omega$. Then $\overline{u} \in L^0(X)$ and $\|\overline{u}(\omega)\|_X = |u(\omega)|$ for $\omega \in \Omega$, so $\overline{u} \in E(X)$. Given a solid pseudonorm (resp. a solid seminorm) ρ on $E(X)$, let us set

$$\widetilde{\rho}(u) = \rho(\overline{u}) \quad \text{for} \quad u \in E.$$

It is seen that $\widetilde{\rho}$ is well defined, because $\rho(\overline{u})$ does not depend on $x \in S_X$ due to the solidness of ρ. It is easy to verify that $\widetilde{\rho}$ is a Riesz pseudonorm (resp. a Riesz seminorm) on E.

Lemma 3.1

(i) If ρ is a solid pseudonorm on $E(X)$, then $\bar{\widetilde{\rho}}(f) = \rho(f)$ for $f \in E(X)$.

(ii) If p is a Riesz pseudonorm on E, then $\widetilde{\overline{p}}(u) = p(u)$ for $u \in E$.

Proof. (i) For $f \in E(X)$ we have $\bar{\widetilde{\rho}}(f) = \widetilde{\rho}(\widetilde{f}) = \rho(\bar{\widetilde{f}})$, when $\|\bar{\widetilde{f}}(\omega)\|_X = \|f(\omega)\|_X$ for $\omega \in \Omega$. Hence by the solidness of ρ we see that $\rho(\bar{\widetilde{f}}) = \rho(f)$.

(ii) For $u \in E$ we have $\widetilde{\overline{p}}(u) = \overline{p}(\overline{u}) = p(\widetilde{\overline{u}})$, where $\widetilde{\overline{u}}(\omega) = \|\overline{u}(\omega)\|_X = |u(\omega)|$ for $\omega \in \Omega$. Since p is a Riesz pseudonorm, $p(\widetilde{\overline{u}}) = p(u)$. □

Let τ be a locally solid topology (resp. a locally convex-solid topology) on $E(X)$. Then in view of Theorem 2.3 (resp. Theorem 2.2) τ is generated by some family $\{\rho_\alpha \colon \alpha \in \{\alpha\}\}$ of solid pseudonorms (resp. solid seminorms) on $E(X)$.

By $\widetilde{\tau}$ we will denote the locally solid topology (resp. locally convex-solid topology) on E generated by the family $\{\widetilde{\rho}_\alpha \colon \alpha \in \{\alpha\}\}$ of Riesz pseudonorm (resp. Riesz seminorms) on E. It is seen that if τ is a Hausdorff topology, then so is $\widetilde{\tau}$.

In turn, let ξ be a locally solid topology (resp. a locally convex-solid topology) on E. Then ξ is generated by some family $\{p_\alpha \colon \alpha \in \{\alpha\}\}$ of Riesz pseudonorms (resp. Riesz seminorms) on E ([1, Theorem 6.1, Theorem 6.3]).

By $\overline{\xi}$ we will denote the locally solid topology (resp. locally convex-solid topology) on $E(X)$ generated by the family $\{\overline{p}_\alpha \colon \alpha \in \{\alpha\}\}$ of solid pseudonorms (resp. solid seminorms) on $E(X)$. It is seen that if ξ is a Hausdorff topology, then so is $\overline{\xi}$.

By applying Lemma 3.1 we have the following:

Theorem 3.2

Let E be an ideal of L^0 with supp $E = \Omega$.

(i) For a locally solid topology τ on $E(X)$ we have: $\bar{\widetilde{\tau}} = \tau$.

(ii) For a locally solid topology ξ on E we have: $\widetilde{\overline{\xi}} = \xi$.

Theorem 3.3

Let τ_1 and τ_2 be locally solid topologies on $E(X)$, and let ξ_1, ξ_2 be locally solid topologies on E. Then:

(i) If $\tau_1 \subset \tau_2$, *then* $\widetilde{\tau}_1 \subset \widetilde{\tau}_2$
(ii) If $\xi_1 \subset \xi_2$, *then* $\overline{\xi}_1 \subset \overline{\xi}_2$.

Proof. (i) Let $\{\rho_\alpha \colon \alpha \in \{\alpha\}\}$ and $\{\rho_\beta \colon \beta \in \{\beta\}\}$ be families of solid pseudonorms on $E(X)$ that generate τ_1 and τ_2 resp. Then the topologies $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ on E are generated by the families $\{\widetilde{\rho}_\alpha \colon \alpha \in \{\alpha\}\}$ and $\{\widetilde{\rho}_\beta \colon \beta \in \{\beta\}\}$ of Riesz pseudonorms on E resp. To prove that $\widetilde{\tau}_1 \subset \widetilde{\tau}_2$, let $u_\sigma \xrightarrow{\widetilde{\tau}_2} 0$ for a net (u_σ) in E. This means that $\widetilde{\rho}_\beta(u_\sigma) \underset{\sigma}{\longrightarrow} 0$ for each $\beta \in \{\beta\}$. Since $\widetilde{\rho}_\beta(u_\sigma) = \rho_\beta(\overline{u}_\sigma)$, we get $\overline{u}_\sigma \xrightarrow{\tau_2} 0$, so $\overline{u}_\sigma \xrightarrow{\tau_1} 0$ because $\tau_1 \subset \tau_2$. Hence $\rho_\alpha(\overline{u}_\sigma) \underset{\sigma}{\longrightarrow} 0$ for each $\alpha \in \{\alpha\}$, and since $\widetilde{\rho}_\alpha(u_\sigma) = \rho_\alpha(\overline{u}_\sigma)$ we get $\widetilde{\rho}_\alpha(u_\sigma) \to 0$ for each $\alpha \in \{\alpha\}$, so $u_\sigma \xrightarrow{\widetilde{\tau}_1} 0$. This means that $\widetilde{\tau}_1 \subset \widetilde{\tau}_2$.

(ii) Let $\{p_\alpha \colon \alpha \in \{\alpha\}\}$ and $\{p_\beta \colon \beta \in \{\beta\}\}$ be families of Riesz pseudonorms on E that generate ξ_1 and ξ_2 resp. Then the topologies $\overline{\xi}_1$ and $\overline{\xi}_2$ on $E(X)$ are generated by the families $\{\overline{p}_\alpha \colon \alpha \in \{\alpha\}\}$ and $\{\overline{p}_\beta \colon \beta \in \{\beta\}\}$ of solid pseudonorms on $E(X)$ resp. To prove that $\overline{\xi}_1 \subset \overline{\xi}_2$, let $f_\sigma \xrightarrow{\overline{\xi}_2} 0$ for a net (f_σ) in $E(X)$. This means that $\overline{p}_\beta(f_\sigma) \underset{\sigma}{\longrightarrow} 0$ for each $\beta \in \{\beta\}$. Since $\overline{p}_\beta(f_\sigma) = p_\beta(\widetilde{f}_\sigma)$, $\widetilde{f}_\sigma \xrightarrow{\xi_2} 0$, so $\widetilde{f}_\sigma \xrightarrow{\xi_1} 0$ because $\xi_1 \subset \xi_2$. Thus $p_\alpha(\widetilde{f}_\sigma) \underset{\sigma}{\longrightarrow} 0$ for each $\alpha \in \{\alpha\}$, and since $\overline{p}_\alpha(f_\sigma) = p_\alpha(\widetilde{f}_\sigma)$ we get $\overline{p}_\alpha(f_\sigma) \underset{\sigma}{\longrightarrow} 0$ for each $\alpha \in \{\alpha\}$, so $f_\sigma \xrightarrow{\overline{\xi}_1} 0$. Thus $\overline{\xi}_1 \subset \overline{\xi}_2$. □

In case E is endowed with a Hausdorff locally convex-solid topology ξ the topological properties of $(E(X), \overline{\xi})$ were studied by A.V. Buchvalov [4]. It is shown that if (E, ξ) is a sequentially complete space (resp. a complete space) and ξ is a Fatou topology then the space $(E(X), \overline{\xi})$ is sequentially complete (resp. complete) (see [4, Theorem 2, Theorem 3]).

Let us recall that a Hausdorff locally solid topology ξ on a Riesz space L is called *minimal*, if ξ is coarser than any other Hausdorff locally solid topology on L (see [3]).

Theorem 3.4

If ξ_0 is a minimal topology on E then $\overline{\xi}_0$ is a minimal topology on $E(X)$.

Proof. Let τ be a Hausdorff locally solid topology on $E(X)$. Then $\xi_0 \subset \widetilde{\tau}$, and by Theorem 3.3 and Theorem 3.2 we have $\overline{\xi}_0 \subset \overline{\widetilde{\tau}} = \tau$. This means that $\overline{\xi}_0$ is a minimal topology on $E(X)$. □

Theorem 3.5

If E has no minimal topology, then $E(X)$ has no minimal topology.

Proof. Assume, on the contrary, that τ_0 is a minimal topology on $E(X)$. Let ξ be a Hausdorff locally solid topology on $E(X)$. Then $\overline{\xi} \supset \tau_0$, so by Theorem 3.2 and Theorem 3.3 $\xi = \widetilde{\overline{\xi}} \supset \widetilde{\tau}_0$, and this means that $\widetilde{\tau}_0$ is a minimal topology on E, which is impossible. □

Corollary 3.6

(i) The topology $\mathcal{T}_0(X)$ is a minimal topology on $L^0(X)$.
(ii) If φ is a finite valued Orlicz function, than $\mathcal{T}_0(X)|_{L^\varphi(X)}$ is a minimal topology on $L^\varphi(X)$.
(iii) Let (Ω, Σ, μ) a be a σ-finite atomless measure space. Then $L^\infty(X)$ has no minimal topology.

Proof. (i) It is well known that the topology $\mathcal{T}_0$ of the Riesz F-norm $\|\cdot\|_0$ is a minimal topology on L^0 (see [3]). Since $\|f\|_{L^0(X)} = \|\widetilde{f}\|_0$ for $f \in L^0(X)$, the identity $\mathcal{T}_0(X) = \overline{\mathcal{T}}_0$ holds, so by Theorem 3.4 $\mathcal{T}_0(X)$ is a minimal topology on $L^0(X)$. □

(ii) It follows from Theorem 3.4, because $\mathcal{T}_0|_{L^\varphi}$ is a minimal topology on L^φ (see [3], [18, Corollary 1.5]).

(iii) It follows from Theorem 3.5, because L^∞ has no minimal topology (see [3, Theorem 8]). □

Theorem 3.7

If η_0 is a finest locally solid topology on E, then $\overline{\eta}_0$ is the finest locally solid topology on $E(X)$.

Proof. Let τ be a locally solid topology on $E(X)$. Then $\widetilde{\tau} \subset \eta_0$, and by Theorem 3.2 and Theorem 3.3, $\tau = \overline{\widetilde{\tau}} \subset \overline{\eta}_0$. This means that $\overline{\eta}_0$ is the finest locally solid topology on $E(X)$. □

Corollary 3.8

Let $(E, \|\cdot\|_E)$ be a complete F-normed function space. Then the topology $\mathcal{T}_E(X)$ of the F-norm $\|\cdot\|_{E(X)}$ is the finest locally solid topology on $E(X)$.

Proof. It is known that the topology $\mathcal{T}_E$ of the F-norm $\|\cdot\|_E$ is the finest locally solid topology on E (see [1, Theorem 16.7]). Since $\|f\|_{E(X)} = \|\widetilde{f}\|_E$ for $f \in E(X)$, the identity $\mathcal{T}_E(X) = \overline{\mathcal{T}}_E$ holds, so by Theorem 3.7, $\mathcal{T}_E(X)$ is the finest locally solid topology on $E(X)$. □

Corollary 3.9

The topology $\mathcal{T}_0(X)$ of the F-norm $\|\cdot\|_{L^0(X)}$ is the only Hausdorff locally solid topology on $L^0(X)$.

Proof. It follows from Corollary 3.6 and Corollary 3.8. □

4. Entire topologies on $E(X)$

Let us recall that a locally solid topology ξ on a Riesz space E is said to be *entire* if its carrier C_ξ is an order dense ideal of E. Entire topologies are always Hausdorff, and Hausdorff Fatou topologies (and therefore also Hausdorff Lebesgue topologies) are entire (see [2], [3]).

In case E is an ideal of L^0 and (Ω, Σ, μ) is a σ-finite measure space ξ is entire if and only if $\operatorname{supp} C_\xi = \Omega$. W. Wnuk [17] showed that a locally solid topology ξ on E is entire iff the embedding $(E, \xi) \hookrightarrow (L^0, \mathcal{T}_0)$ is continuous.

In this section we consider entire topologies on $E(X)$. For a solid pseudonorm ρ on $E(X)$ let

$$N_\rho = \{h \in E(X)\colon \rho(h) = 0\} \quad \text{and} \quad N_\rho^d = \{f \in E(X)\colon \widetilde{f} \wedge \widetilde{h} = 0 \ \text{ for all } \ h \in N_\rho\}.$$

Then both N_ρ and N_ρ^d are ideals of $E(X)$.

DEFINITION 4.1. The carrier C_τ of a locally solid topology τ on $E(X)$ is defined by

$$C_\tau = \bigcup \{N_\rho^d\colon \rho \ \text{ is a } \ \tau\text{-continuous solid pseudonorm}\}.$$

Theorem 4.1

The carrier C_τ of a locally solid topology τ on $E(X)$ is an ideal of $E(X)$.

Proof. It is seen that C_τ is a solid subset of $E(X)$. To prove that C_τ is a linear subspace of $E(X)$, let $f_1, f_2 \in C_2$. Then $f_1 \in N_{\rho_1}^d, f_2 \in N_{\rho_2}^d$ for some solid τ-continuous pseudonorms ρ_1, ρ_2 on $E(X)$. Let $\rho(f) = \rho_1(f) \vee \rho_2(f)$ for $f \in E(X)$. Then ρ is a solid τ-continuous pseudonorm on $E(X)$ and $N_\rho = N_{\rho_1} \cap N_{\rho_2}$. Moreover, for each $h \in N_\rho = N_{\rho_1} \cap N_{\rho_2}$ we have $0 \le \widetilde{f_1 + f_2} \wedge \widetilde{h} \le (\widetilde{f_1} + \widetilde{f_2}) \wedge \widetilde{h} \le (\widetilde{f_1} \wedge \widetilde{h}) + (\widetilde{f_2} \wedge \widetilde{h}) = 0$, so $f_1 + f_2 \in N_\rho^d \subset C_\tau$. Since $f \in C_\tau$ and $\lambda \in \mathbb{R}$ implies $\lambda f \in C_\tau$, the proof is complete. $\square$

Note that

$$\begin{aligned} \operatorname{supp} C_\tau &= \bigcup \{\operatorname{supp} f\colon f \in C_\tau\} \\ &= \bigcup \{\operatorname{supp} N_\rho^d\colon \rho \ \text{ is } \tau\text{-continuous solid pseudonorm}\}. \end{aligned}$$

DEFINITION 4.2. A locally solid topology τ on $E(X)$ is said to be *entire* if $\operatorname{supp} C_\tau = \Omega$.

We are going to show that $\operatorname{supp} C_\tau = \operatorname{supp} C_{\underset{\tau}{\sim}}$. For this purpose, for a solid pseudonorm ρ on $E(X)$ let

$$N_{\underset{\rho}{\sim}} = \{u \in E\colon \widetilde{\rho}(u) = 0\} \quad \text{and} \quad N^d_{\underset{\rho}{\sim}} = \{v \in E\colon |u| \wedge |v| = 0 \ \text{ for all } \ u \in N_{\underset{\rho}{\sim}}\}.$$

We shall need the following lemmas.

Lemma 4.2

For a solid pseudonorm ρ on $E(X)$

$$N_{\underset{\rho}{\sim}} = \big\{u \in E\colon |u(\omega)| \le \|f(\omega)\|_X \ \ \mu\text{–a.e. for some } \ f \in N_\rho\big\}.$$

Proof. Let $u \in N_{\underset{\rho}{\sim}}$, i.e. $\widetilde{\rho}(u) = \rho(\overline{u}) = 0$. Then $|u(\omega)| = \|\overline{u}(\omega)\|_X$ for $\omega \in \Omega$ and $\overline{u} \in N_\rho$. Next let $u \in E$ and $|u(\omega)| \le \|f(\omega)\|_X$ μ-a.e. for some $f \in N_\rho$. Then $\widetilde{\rho}(u) = \rho(\overline{u}) \le \rho(f)$, so $\widetilde{\rho}(u) = 0$, that is $u \in N_{\underset{\rho}{\sim}}$. □

Lemma 4.3

For a solid pseudonorm ρ on $E(X)$, $\operatorname{supp} N^d_\rho = \operatorname{supp} N^d_{\underset{\rho}{\sim}}$.

Proof. To prove that $\operatorname{supp} N^d_\rho \subset \operatorname{supp} N^d_{\underset{\rho}{\sim}}$, let $\omega \in \operatorname{supp} N^d_\rho = \bigcup\{\operatorname{supp} f\colon f \in N^d_\rho\}$. Then $f(\omega) \ne 0$ for some $f \in N^d_\rho$. It is enough to show that $\widetilde{f} \in N^d_{\underset{\rho}{\sim}}$. Indeed, let $u \in N_{\underset{\rho}{\sim}}$. Then by Lemma 4.1, $|u(\omega)| \le \|h(\omega)\|_X$ μ-a.e. for some $h \in N_\rho$. Then $0 \le \widetilde{f} \wedge |u| \le \widetilde{f} \wedge \widetilde{h} = 0$, so $\widetilde{f} \in N^d_{\underset{\rho}{\sim}}$, as desired. Since $\widetilde{f}(\omega) > 0$ we see that $\omega \in \operatorname{supp} N^d_{\underset{\rho}{\sim}}$.

Next, let $\omega \in \operatorname{supp} N^d_{\underset{\rho}{\sim}} = \bigcup\{\operatorname{supp} u\colon u \in N^d_{\underset{\rho}{\sim}}\}$. Then $u(\omega) \ne 0$ for some $u \in N^d_{\underset{\rho}{\sim}}$. Hence $\overline{u}(\omega) \ne 0$ and it is enough to show that $\overline{u} \in N^d_\rho$. Indeed, let $h \in N^d_\rho$. Then by Lemma 4.1, $\widetilde{h} \in N_{\underset{\rho}{\sim}}$ and $\widetilde{\overline{u}} \wedge \widetilde{h} = |u| \wedge \widetilde{h} = 0$, so $\overline{u} \in N^d_\rho$, as desired. Since $\overline{u}(\omega) = 0$ it means that $\omega \in \operatorname{supp} N^d_\rho$. □

Lemma 4.4

Let τ be a locally solid topology on $E(X)$.

(i) If ρ is a solid τ-continuous pseudonorm on $E(X)$, then $\widetilde{\rho}$ is a solid $\widetilde{\tau}$-continuous pseudonorm on E.

(ii) If p is $\widetilde{\tau}$-continuous solid pseudonorm on E, then $\overline{p}$ is a solid τ-continuous pseudonorm on $E(X)$.

Proof. Let τ be generated by some family $\{\rho_\alpha\colon \alpha \in \{\alpha\}\}$ of solid pseudonorms on $E(X)$. Then $\widetilde{\tau}$ is generated by the family $\{\widetilde{\rho}_\alpha\colon \alpha \in \{\alpha\}\}$.

(i) Assume that ρ is τ-continuous. To prove that $\widetilde{\rho}$ is $\widetilde{\tau}$-continuous let $u_\sigma \xrightarrow[\sigma]{\widetilde{\tau}} 0$ for a net (u_σ) in E. It means that $\widetilde{\rho}_\alpha(u_\sigma) \xrightarrow[\sigma]{} 0$ for each $\alpha \in \{\alpha\}$. But $\widetilde{\rho}_\alpha(u_\sigma) = \rho_\alpha(\overline{u}_\sigma)$, so $\rho_\alpha(\overline{u}_\sigma) \xrightarrow[\sigma]{} 0$ for each $\alpha \in \{\alpha\}$. But $\widetilde{\rho}_\alpha(u_\sigma) = \rho_\alpha(\overline{u}_\sigma)$, so $\rho_\alpha(\overline{u}_\sigma) \xrightarrow[\sigma]{} 0$ for each $\alpha \in \{\alpha\}$. Hence $\rho(\overline{u}_\sigma) \xrightarrow[\sigma]{} 0$, because ρ is τ-continuous. Since $\widetilde{\rho}(u_\sigma) = \rho(\overline{u}_\sigma)$ it means that $\widetilde{\rho}$ is $\widetilde{\tau}$-continuous.

(ii) Assume that p is a $\widetilde{\tau}$-continuous. To prove that $\overline{p}$ is τ-continuous let $f_\sigma \xrightarrow[\sigma]{\tau} 0$ for a net (f_σ) in $E(X)$. It means that $\rho_\alpha(f_\sigma) \xrightarrow[\sigma]{} 0$ for each $\alpha \in \{\alpha\}$. But $\widetilde{\rho}_\alpha(\widetilde{f}_\sigma) = \rho_\alpha(\overline{\widetilde{f}}_\sigma) = \rho_\alpha(f_\sigma)$, so $\widetilde{\rho}_\alpha(\widetilde{f}_\sigma) \xrightarrow[\sigma]{} 0$ for each $\alpha \in \{\alpha\}$. Hence $p(\widetilde{f}_\sigma) \to 0$ because p is $\widetilde{\tau}$-continuous. Since $\overline{p}(f_\sigma) = p(\widetilde{f}_\sigma)$, it means that $\overline{p}$ is τ-continuous. □

Denote by $\mathcal{P}_\tau(E(X))$ the family of all τ-continuous solid pseudonorms on $E(X)$, and by $\mathcal{P}_{\widetilde{\tau}}(E)$ the family of all $\widetilde{\tau}$-continuous solid pseudonorms on E.

Lemma 4.5

We have the following identity: $\mathcal{P}_{\widetilde{\tau}}(E) = \{\widetilde{\rho}\colon \rho \in \mathcal{P}_\tau(E(X))\}$.

Proof. Let $p \in \mathcal{P}_{\widetilde{\tau}}(E)$. Then by Lemma 3.1 $p = \widetilde{\overline{p}}$. Thus $p = \widetilde{\rho}$ where $\rho = \overline{p} \in \mathcal{P}_\tau(E(X))$ by (ii) of Lemma 4.4.

Now, let $p = \widetilde{\rho}$ where $\rho \in \mathcal{P}_\tau(E(X))$. By (i) of Lemma 4.4 p is $\widetilde{\tau}$-continuous, and we are done. □

Now we are in position to prove our desired result.

Theorem 4.6

For a locally solid topology τ on $E(X)$, $\quad \operatorname{supp} C_\tau = \operatorname{supp} C_{\widetilde{\tau}}$.

Proof. Since $\operatorname{supp} C_\tau = \bigcup\{\operatorname{supp} N_\rho^d\colon \rho \in \mathcal{P}_\tau(E(X)\}$, we have $\operatorname{supp} C_{\widetilde{\tau}} = \bigcup\{\operatorname{supp} N_p^d\colon p \in \mathcal{P}_{\widetilde{\tau}}(E)\}$. Using Lemma 4.3 and Lemma 4.5 we get $\operatorname{supp} C_\tau = \operatorname{supp} C_{\widetilde{\tau}}$. □

As an application of Theorem 4.6 we have the following two theorems.

Theorem 4.7

For a locally solid topology τ on $E(X)$ the following statements are equivalent:

(i) τ *is entire.*
(ii) $\widetilde{\tau}$ *is entire.*
(iii) The embedding $(E, \widetilde{\tau}) \hookrightarrow (L^0, \mathcal{T}_0)$ *is continuous.*
(iv) The embedding $(E(X), \tau) \hookrightarrow (L^0(X), \mathcal{T}_0(X))$ *is continuous.*

Proof. (i) $\Leftrightarrow$ (ii) It follows from Theorem 4.6.

(ii) $\Leftrightarrow$ (iii) See [17].

(iii) $\Rightarrow$ (iv) Assume that $\widetilde{\tau} \supset \mathcal{T}_0|_E$. Then by Theorem 3.3 and Theorem 3.2 $\tau = \overline{\widetilde{\tau}} \supset \mathcal{T}_0|_{E(X)}$.

(iv) $\Rightarrow$ (iii) Assume that $\tau \supset \mathcal{T}_0(X)|_{E(X)}$. Then by Theorem 3.3 $\widetilde{\tau} \supset \widetilde{\mathcal{T}_0(X)|_{E(X)}} = \mathcal{T}_0|_E$. □

Theorem 4.8

For a locally solid topology ξ on E the following statements are equivalent:

(i) ξ *is entire.*
(ii) $\overline{\xi}$ *is entire.*
(iii) The embedding $(E, \xi) \hookrightarrow (L^0, \mathcal{T}_0)$ *is continuous.*
(iv) The embedding $(E(X), \overline{\xi}) \hookrightarrow (L^0(X), \mathcal{T}_0(X))$ *is continuous.*

Proof. Let $\tau = \overline{\xi}$. Then by Theorem 3.2 $\xi = \widetilde{\overline{\xi}} = \widetilde{\tau}$, and by Theorem 4.7 the proof is complete. □

Remark. A.V. Bukhvalov [4] showed that if ξ is a locally convex-solid topology on E with the Fatou property (so E is entire), then the embedding $(E(X), \overline{\xi}) \hookrightarrow (L^0(X), \mathcal{T}(X))$ is continuous.

Corollary 4.9

Every entire topology τ on $E(X)$ is Hausdorff.

Proof. Assume that τ is entire. Then by Theorem 4.7 $\widetilde{\tau}$ is entire on E, so $\widetilde{\tau}$ is Hausdorff. Since $\tau = \overline{\widetilde{\tau}}$ (see Theorem 3.2), τ is Hausdorff. □

5. Lebesgue topologies on $E(X)$

In this section, following the concept from the theory of locally solid Riesz spaces, we defined some class of locally solid topologies on $E(X)$ (called Lebesgue topologies) connecting the solid structure of $E(X)$ and topological continuity.

Definition 5.1. A sequence (f_n) in $E(X)$ is said to be order convergent to 0 in $E(X)$, in symbols $f_n \xrightarrow{(0)} 0$, if $\widetilde{f_n} \xrightarrow{(0)} 0$ in E; i.e., $\|f_n(\omega)\|_X \to 0$ μ-a.e. and $\|f_n(\omega)\|_X \leq u(\omega)$ μ-a.e. for some $u \in E$ $(n = 1, 2, \ldots)$.

DEFINITION 5.2. A solid pseudonorm ρ on $E(X)$ is said to be *order continuous* whenever for $f_n \in E(X)$, $f_n \xrightarrow{(0)} 0$ implies $\rho(f_n) \to 0$.

Let us recall that for a sequence (A_n) in Σ we write $A_n \searrow \emptyset$ whenever (A_n) is a decreasing sequence and $\mu(A_n \cap A) \to 0$ for every $A \in \Sigma$ with $\mu(A) < \infty$.

DEFINITION 5.3. A solid pseudonorm ρ on $E(X)$ is said to be *absolutely continuous*, whenever $\rho(\chi_{A_n} f) \to 0$ as $f \in E(X)$ and $A_n \searrow \emptyset$.

By a standard argument (see [14]) one can prove the following:

Theorem 5.1

For a solid pseudonorm ρ on $E(X)$ the following statements are equivalent:

(i) ρ is absolutely continuous.
(ii) For every $f \in E(X)$ and $\varepsilon > 0$ there exist $\delta > 0$ and a set $A_0 \in \Sigma$ with $\mu(A_0) < \infty$ such that $\rho(\chi_{\Omega \setminus A_0} f) \le \varepsilon$ and $\rho(\chi_A f) \le \varepsilon$ whenever $\mu(A) \le \delta$.

Theorem 5.2

For a solid pseudonorm ρ on $E(X)$ and a subset D of $E(X)$ the following statements are equivalent:

(i) D is of uniformly absolute continuous pseudonorm ρ, i.e.,

$$\sup_{f \in D} \rho(\chi_{A_n} f) \to 0 \quad \text{as} \quad A_n \searrow \emptyset .$$

(ii) For every $\varepsilon > 0$ there exist $\delta > 0$ and a set $A_0 \in \Sigma$ with $\mu(A_0) < \infty$ such that $\rho(\chi_{\Omega \setminus A_0} f) \le \varepsilon$ and $\sup_{f \in D} \rho(\chi_A f) \le \varepsilon$ whenever $\mu(A) \le \delta$.

Theorem 5.3

For a solid pseudonorm ρ on $E(X)$ the following statements are equivalent:

(i) $\|f_n\|_{L^0(X)} \to 0$ and $\|f_n(\omega)\|_X \le u(\omega)$ μ-a.e. for some $u \in E$ $(n = 1, 2, \ldots)$ imply $\rho(f_n) \to 0$.
(ii) ρ is order continuous.
(iii) $\|f_n(\omega)\|_X \downarrow_n 0$ μ-a.e. implies $\rho(f_n) \to 0$.
(iv) ρ is absolutely continuous.

Proof. (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) Assume that $\|f_n\|_{L^0(X)} \to 0$ μ-a.e. and $\|f_n(\omega)\|_X \le u(\omega)$ μ-a.e. for some $u \in E$. Then $\widetilde{f}_n \to 0$ (μ) in L^0, so by the Riesz theorem for every subsequence $(\widetilde{f}_{k_n})$ of $(\widetilde{f}_n)$ there exists a subsequence $(\widetilde{f}_{l_{k_n}})$ of $(\widetilde{f}_{k_n})$ such that $\widetilde{f}_{l_{k_n}}(\omega) \to 0$ μ-a.e., i.e. $\|f_{l_{k_n}}(\omega)\|_X \to 0$ μ-a.e. Hence by (ii), $\rho(f_{l_{k_n}}) \to 0$; hence $\rho(f_n) \to 0$.

(iii) $\Rightarrow$ (ii) Assume that $\|f_n(\omega)\|_X \to 0$ μ-a.e. and $\|f_n(\omega)\|_X \le u(\omega)$ μ-a.e. for some $u \in E$. Let $u_n(\omega) = \sup_{k\ge n} \|f_k(\omega)\|_X$ for $\omega \in \Omega, n = 1, 2, \ldots$, and let $h_n(\omega) = u_n(\omega)x$ for some $x \in S_X$. Then $h_n \in E(X)$ and $\|h_n(\omega)\|_X \downarrow_n 0$ for $\omega \in \Omega$. To see that $\|h_n(\omega)\|_X \to 0$ for $\omega \in \Omega$, let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that for $k \ge n_0$, $\|f_k(\omega)\|_X \le \varepsilon$ for $\omega \in \Omega$. Then for $n \ge n_0$, $u_n(\omega) \le \varepsilon$, i.e., $\|h_n(\omega)\|_X \le \varepsilon$ for $\omega \in \Omega$. Thus $\|h_n(\omega)\|_X \downarrow_n 0$ for $\omega \in \Omega$, so $\rho(h_n) \to 0$ by (iii). Since $\|f_n(\omega)\|_X \le \|h_n(\omega)\|_X$ μ-a.e., $\rho(f_n) \le \rho(h_n)$, so $\rho(f_n) \to 0$.

(i) $\Rightarrow$ (iv) Let $f \in E(X)$ and $A_n \searrow \emptyset$, and let us put $f_n(\omega) = \chi_{A_n}(\omega)f(\omega)$ for $\omega \in \Omega$, $n = 1, 2, \ldots$. Let $\varepsilon > 0$ be given and let $A \in \Sigma$ with $\mu(A) < \infty$. Since $\{\omega \in A\colon \|f_n(\omega)\|_X > \varepsilon\} \subset A \cap A_n$ and $\mu(A\cap A_n) \to 0$ it follows that $\|f_n\|_{L^0(X)} \to 0$. But $\|f_n(\omega)\|_X \le \widetilde{f}(\omega)$ μ-a.e. $(n = 1, 2, \ldots)$, so by (i) $\rho(\chi_{A_n} f) \to 0$, as desired.

(iv) $\Rightarrow$ (i) Assume that $\|f_n\|_{L^0(X)} \to 0$ and $\|f_n(\omega)\|_X \le u(\omega)$ μ-a.e. for some $u \in E$ $(n = 1, 2, \ldots)$, and let $\varepsilon > 0$ be given. Since $\operatorname{supp} E = \Omega$, there exists an increasing sequence (Ω_n) in Σ such that $\bigcup_{n=1}^\infty \Omega_n = \Omega, \mu(\Omega_n) < \infty$ and $\chi_{\Omega_n} \in E$ for $n = 1, 2, \ldots$ (see [26, Theorem 86.2]).

Let $A_n = \Omega - \Omega_n$ $(n = 1, 2, \ldots)$. Then $A_n \searrow \emptyset$. Let $h_0(\omega) = u(\omega)x_0$, where $x_0 \in S_X$. Since $\rho(\chi_{A_n} h_0) \to 0$ and $\rho(\chi_{A_n} f_m) \le \rho(\chi_{A_n} h_0)$ for $m = 1, 2, \ldots$, it follows that $\sup_m \rho(\chi_{A_n} f_m) \to 0$.

Choose $n_0 \in \mathbb{N}$ such that

$$\sup_m \rho(\chi_{A_{n_0}} f_m) \le \frac{\varepsilon}{3}. \tag{1}$$

Let $f_0(\omega) = x_0$ for all $\omega \in \Omega$. Then $\chi_{A_{n_0}} f_0 \in E(X)$ and choose $\eta_0 > 0$ such that

$$\rho(\eta_0 \chi_{\Omega_{n_0}} f_0) \le \frac{\varepsilon}{3}. \tag{2}$$

Write $C_n = \{\omega \in \Omega_{n_0}\colon \|f_n(\omega)\|_X > \eta_0\}$, $B_n = \{\omega \in \Omega_{n_0}\colon \|f_n(\omega)\|_X \le \eta_0\}$ $(n = 1, 2, \ldots)$. Since $\|f_n\|_{L^0(X)} \to 0$, we get $\mu(C_n) \to 0$, so by (ii) there exists $n_1 \in \mathbb{N}$ such that for $n \ge n_1$

$$\rho(\chi_{C_n} h_0) \le \frac{\varepsilon}{3}. \tag{3}$$

But $\|\chi_{B_n}(\omega)f_n(\omega)\|_X \leq \|\eta_0\chi_{\Omega_{n_0}}(\omega)f_0(\omega)\|_X$ μ-a.e., hence by (1), (2) and (3) we get for $n \geq n_1$:

$$\begin{aligned}\rho(f_n) &= \rho(\chi_{C_n}f_n + \chi_{B_n}f_n + \chi_{A_{n_0}}f_n)\\ &\leq \rho(\chi_{C_n}f_n) + \rho(\chi_{B_n}f_n) + \rho(\chi_{A_{n_0}}f_n)\\ &\leq \rho(\chi_{C_n}h_0) + \rho(\eta_0\chi_{\Omega_{n_0}}f_0) + \rho(\chi_{A_{n_0}}f_n) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus the proof is complete. □

Remark. Since the measure space (Ω, Σ, μ) is σ-finite, the space E has the countable sup property. It follows that every σ-Lebesgue topology ξ on E is a Lebesgue topology. Therefore in the below definition of a Lebesgue topology on $E(X)$ one can also take sequences instead of nets.

DEFINITION 5.4. A locally solid topology τ on $E(X)$ is said to be a *Lebesgue topology* whenever for $f_n \in E(X)$, $f_n \xrightarrow{(0)} 0$ implies $f_n \xrightarrow{\tau} 0$.

Applying Theorem 2.3 and Theorem 5.3 we obtain some general characterizations of Lebesgue topologies on $E(X)$.

Theorem 5.4

For a locally solid topology τ on $E(X)$ the following statements are equivalent:
(i) τ is a Lebesgue topology.
(ii) $\|f_n\|_{L^0(X)} \to 0$ and $\|f_n(\omega)\|_X \leq u(\omega)$ μ-a.e. for some $u \in E$ $(n = 1, 2, \ldots)$ imply $f_n \xrightarrow{\tau} 0$.
(iii) $\|f_n(\omega)\|_X \downarrow 0$ μ-a.e. implies $f_n \xrightarrow{\tau} 0$.
(iv) τ is generated by some family of absolutely continuous pseudonorms.

Theorem 5.5

Let ρ be an absolutely continuous pseudonorm on $E(X)$, and let $f \in E(X), f_n \in E(X)$ $(n = 1, 2, \ldots)$ with $f_n \to f$ for $\mathcal{T}_0(X)$ and $\sup_m \rho(\chi_{A_n}f_m) \xrightarrow[n]{} 0$ as $A_n \searrow \emptyset$. Then $\rho(f_n - f) \to 0$.

Proof. Let $\varepsilon > 0$ be given. Since $\operatorname{supp} E = \Omega$ there exists a sequence (Ω_n) in Σ such that $\Omega_n \uparrow \Omega, \chi_{\Omega_n} \in E$ $(n = 1, 2, \ldots)$, and let $A_n = \Omega - \Omega_n$ $(n = 1, 2, \ldots)$. Then $A_n \searrow \emptyset$, so there exists $n_0 \in \mathbb{N}$ such that $\sup_m \rho(\chi_{A_{n_0}}f_m) \leq \varepsilon/6$ and $\rho(\chi_{A_{n_0}}f) \leq \varepsilon/6$. Since $\rho(\chi_{A_{n_0}}(f_m - f)) \leq \rho(\chi_{A_{n_0}}f_m) + \rho(\chi_{A_{n_0}}f)$ for $m \in \mathbb{N}$, we see that

(1) $$\sup_m \rho(\chi_{A_{n_0}}(f_m - f) \leq \frac{\varepsilon}{3}\,.$$

Let $f_0(\omega) = x_0$ for $\omega \in \Omega$, where $x_0 \in S_X$. Then $\chi_{\Omega_{n_0}} f_0 \in E(X)$, and choose $\eta_0 > 0$ such that

$$\rho(\eta_0 \chi_{\Omega_{n_0}} f_0) \le \frac{\varepsilon}{3}\,. \tag{2}$$

Write $(n = 1, 2, \dots)$ $C_n = \{\omega \in \Omega_{n_0}\colon \|f_n(\omega) - f(\omega)\|_X > \eta_0\}$, $B_n = \{\omega \in \Omega_{n_0}\colon \|f_n(\omega) - f(\omega)\|_X \le \eta_0\}$. Since $f_n \to f$ for $\mathcal{T}_0(X)$, we get $\mu(C_n) \to 0$; so by Theorem 5.2 there exists $n_1 \in \mathbb{N}$ such that for $n \ge n_1$, $\sup_m \rho(\chi_{C_n} f_n) \le \varepsilon/6$ and $\rho(\chi_{C_n} f) \le \varepsilon/6$. Hence for $n \ge n_1$

$$\sup_m \big(\chi_{C_n}(f_m - f)\big) \le \frac{\varepsilon}{3}\,. \tag{3}$$

Since $\|\chi_{B_n}(\omega)(f_n(\omega) - f(\omega))\|_X \le \chi_{\Omega_{n_0}}(\omega) \cdot \eta_0 = \|\eta_0 \chi_{\Omega_{n_0}} f_0(\omega)\|_X$, by (2) we have for $n = 1, 2, \dots$

$$\rho\big(\chi_{B_n}(f_n - f)\big) \le \frac{\varepsilon}{3}. \tag{4}$$

Hence by (1), (3) and (4) for $n \ge n_1$ we have:

$$\begin{aligned}\rho(f_n - f) &\le \rho\big(\chi_{C_n}(f_n - f)\big) + \rho\big(\chi_{B_n}(f_n - f)\big) + \rho\big(\chi_{\Omega - \Omega_{n_0}}(f_n - f)\big)\\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus the proof is complete. □

The next theorem describes sequential convergence in $E(X)$ endowed with a Lebesgue topology.

Theorem 5.6

Let τ be a Lebesgue topology on $E(X)$ generated by a family $\{\rho_\alpha\colon \alpha \in \{\alpha\}\}$ of absolutely continuous pseudonorms on $E(X)$, and assume that $\mathcal{T}_0(X)|_{E(X)} \subset \tau$. Then for $f \in E(X)$, $f_n \in E (n = 1, 2, \dots)$ the following statements are equivalent:

(i) $f_n \to f$ for τ.

(ii) $f_n \to f$ for $\mathcal{T}_0(X)$, and for every $\alpha \in \{\alpha\}$, $\sup_m \rho_\alpha(\chi_{A_n} f_m) \xrightarrow[n]{} 0$ as $A_n \searrow \emptyset$.

Proof. (i) $\Rightarrow$ (ii) Let $\alpha \in \{\alpha\}$, and let $\varepsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $\rho_\alpha(f_n - f) \le \varepsilon/2$ for $n > n_0$. Since ρ_α in absolutely continuous, there exists $\delta > 0$ and $B_0 \in \Sigma$ with $\mu(B_0) < \infty$ such that $\rho_\alpha(\chi_B f) \le \varepsilon/2$ for $B \in \Sigma,\ \mu(B) \le \delta$ and $\rho(\chi_{\Omega - B_0} f) \le \varepsilon/2$. It follows that for $B \in \Sigma$ with $\mu(B) \le \delta$,

$$\rho(\chi_B f_n) \le \varepsilon \quad \text{and} \quad \rho(\chi_{\Omega - B_0} f_n) \le \varepsilon \quad \text{for} \ \ n > n_0.$$

Moreover, for every $n = 1, \dots, n_0$ there exist $\delta_n > 0$ and $C_n \in \Sigma$ with $\mu(C_n) < \infty$ such that $\rho(\chi_C f_n) \le \varepsilon$ for $C \in \Sigma$ with $\mu(C) \le \delta$ and $\rho(\chi_{\Omega - C_n} f_n) \le \varepsilon$. Then for $C_0 = \bigcup_{n=1}^{n_0} C_n, \delta_0 = \min(\delta_1, \dots, \delta_n)$ we have $\mu(A_0) < \infty$ and

$$\sup_{1 \le n \le n_0} \rho_\alpha(\chi_C f_n) \le \varepsilon \ \text{ for } \ C \in \Sigma,\ \mu(C) \le \delta_0 \ \text{ and } \ \sup_{1 \le n \le n_0} \rho_\alpha(\chi_{\Omega - C_0} f_n) \le \varepsilon.$$

Putting $\delta' = \min(\delta, \delta_0)$ and $A_0 = B_0 \cup C_0$ we get

$$\sup_n \rho_\alpha(\chi_A f_n) \le \varepsilon \ \text{ for } \ A \in \Sigma \ \text{ with } \ \mu(A) \le \delta', \ \text{ and } \ \sup_n \rho_\alpha(\chi_{\Omega - A_0} f_n) \le \varepsilon.$$

By Theorem 5.2 it follows that $\sup_m \rho_\alpha(\chi_{A_n} f_m) \to 0$ as $A_n \searrow \emptyset$.

(ii) $\Rightarrow$ (i) It follows from Theorem 5.5. □

The relationship between Lebesgue topologies on E and $E(X)$ is explained by the next theorem.

Theorem 5.7

(i) If ξ is a Lebesgue topology on E, then $\overline{\xi}$ Lebesgue topology on $E(X)$.
(ii) If τ is a Lebesgue topology on $E(X)$, then $\widetilde{\tau}$ is a Lebesgue topology on E.
(iii) If ξ is the finest Lebesgue topology on E, then $\overline{\xi}$ is the finest Lebesgue topology on $E(X)$.

Proof. (i) Let ξ be generated by a family $\{p_\alpha\colon \alpha \in \{\alpha\}\}$ of solid pseudonorms on E. Let $f_n \in E(X)$ $(n = 1, 2, \dots)$ and $\|f_n(\omega)\|_X \to 0$ μ-a.e. and $\|f_n(\omega)\|_X \le u(\omega)$ μ-a.e. for some $u \in E$ $(n = 1, 2, \dots)$. This means that $\widetilde{f}_n \xrightarrow{(0)} 0$ in E, so $\widetilde{f}_n \xrightarrow{\xi} 0$, i.e. for each $\alpha \in \{\alpha\}, p_\alpha(\widetilde{f}_n) \xrightarrow[n]{} 0$. Then for each $\alpha \in \{\alpha\}$, $\overline{p}_\alpha(f_n) \to 0$, i.e. $f_n \to 0$ for $\overline{\xi}$, and in view of Theorem 5.4 this means that $\overline{\xi}$ is a Lebesgue topology.

(ii) Let τ be generated by a family $\{\rho_\alpha\colon \alpha \in \{\alpha\}\}$ of solid pseudonorms on $E(X)$ and assume $u_n \xrightarrow{(0)} 0$ in E, i.e. $u_n(\omega) \to 0$ μ-a.e. and $|u_n(\omega)| \le u(\omega)$ μ-a.e. for some $u \in E$ $(n = 1, 2, \dots)$. Let $\overline{u}_n(\omega) = u_n(\omega) \cdot x$ for $\omega \in \Omega$ and some $x \in S_X$.

Then by Theorem 5.4 $\overline{u}_n \xrightarrow{\tau} 0$, i.e., for each $\alpha \in \{\alpha\}, \rho_\alpha(\overline{u}_n) \underset{n}{\longrightarrow} 0$. Thus for each $\alpha \in \{\alpha\}, \widetilde{\rho}_\alpha(u_n) \to 0$, i.e., $u_n \to 0$ for $\widetilde{\tau}$. It means that $\widetilde{\tau}$ is a Lebesgue topology.

(iii) Let τ be a Lebesgue topology on $E(X)$. Then $\widetilde{\tau}$ is a Lebesgue topology on E, so $\widetilde{\tau} \subset \xi$. Hence $\tau = \overline{\widetilde{\tau}} \subset \overline{\xi}$ (see Theorem 3.2 and Theorem 3.3), so $\overline{\xi}$ is the finest Lebesgue topology on $E(X)$. □

6. The finest Lebesgue topology on Orlicz-Bochner spaces

In this section we describe the finest Lebesgue topology on Orlicz-Bochner spaces $L^\varphi(X)$.

First we recall some terminology concerning the theory of Orlicz spaces and Orlicz-Bochner spaces (see [13], [23]). By an Orlicz function we mean a function φ: $[0,\infty) \to [0,\infty]$ which is non decreasing, left continuous at zero and $\varphi(u) = 0$ iff $u = 0$. A convex Orlicz function will be called a Young function.

An Orlicz function φ defines two functionals m_φ: $L^0 \to [0,\infty]$ and M_φ: $L^0(X) \to [0,\infty]$ by

$$m_\varphi(u) = \int_\Omega \varphi\big(|u(\omega)|\big)d\mu, \qquad M_\varphi(f) = \int_\Omega \varphi\big(\|f(\omega)\|_X\big)d\mu .$$

The space

$$L^\varphi(X) = \big\{f \in L^0(X)\colon \widetilde{f} \in L^\varphi\big\} = \big\{f\colon L^0(X)\colon M_\varphi(\lambda f) < \infty \ \text{ for some } \ \lambda > 0\big\}$$

will be called *an Orlicz-Bochner space* (here L^φ is the Orlicz space defined by φ). The functional M_φ restricted to $L^\varphi(X)$ is a modular (see [21], [25]). $L^\varphi(X)$ can be equipped with the complete metrizable linear topology $\mathcal{T}_\varphi(X)$ of the solid F-norm $|f|_{L^\varphi(X)} = \inf\{\lambda > 0\colon M_\varphi(f/\lambda) \le \lambda\}$. Moreover, when φ is a Young function, the topology $\mathcal{T}_\varphi(X)$ can be generated by the so-called Luxemburg norm $|||f|||_{L^\varphi(X)} = \inf\{\lambda > 0\colon M_\varphi(f/\lambda) \le 1\}$.

Definition 6.1. A sequence (f_n) in $L^\varphi(X)$ is said to be *modular convergent* to $f \in L^\varphi(X)$ (in symbols $f_n \xrightarrow{M_\varphi} f$) whenever $M_\varphi(\lambda(f_n - f)) \to 0$ for some $\lambda > 0$.

For $\varepsilon > 0$ let $U_\varphi(\varepsilon) = \{f \in L^\varphi(X)\colon M_\varphi(f) \le \varepsilon\}$. Then the family of all sets of the form:

$$(*) \qquad \bigcup_{N=1}^\infty \Big(\sum_{n=1}^N U_\varphi(\varepsilon_n)\Big),$$

where (ε_n) is a sequence of positive numbers, is a basis at 0 for a linear topology on $L^\varphi(X)$, that will be called *the modular topology* on $L^\varphi(X)$ and will be denoted by $\mathcal{T}_\varphi^\wedge(X)$. Using Lemma 1.1 it is easy to show that the sets of the form $(*)$ are solid, so $\mathcal{T}_\varphi^\wedge(X)$ is a locally solid topology.

The modular topology $\mathcal{T}_\varphi^\wedge$ on Orlicz spaces L^φ has been examined in [19], [20], [22].

Arguing as in the proofs of [19, Theorem 1.2, Theorem 1.3] we obtain:

Theorem 6.1

(i) $\mathcal{T}_\varphi^\wedge(X)$ *is the finest of all linear topologies* τ *on* $L^\varphi(X)$ *for which* $f_n \xrightarrow{M_\varphi} 0$ *implies* $f_n \xrightarrow{\tau} 0$.
(ii) $\mathcal{T}_\varphi^\wedge(X) \subset \mathcal{T}_\varphi(X)$ *and the identity* $\mathcal{T}_\varphi^\wedge(X) = \mathcal{T}_\varphi(X)$ *holds whenever* $\varphi \in \Delta_2$, *i.e.* $\limsup \varphi(2u)/\varphi(u) < \infty$ *as* $u \to 0$ *and* $u \to \infty$.

Theorem 6.2

For a locally solid topology τ *on* $L^\varphi(X)$ *the following statements are equivalent:*
(i) τ *is a Lebesgue topology.*
(ii) $f_n \xrightarrow{M_\varphi} 0$ *implies* $f_n \xrightarrow{\tau} 0$.

Proof. (i) $\Rightarrow$ (ii) Let $f_n \xrightarrow{M_\varphi} 0$, i.e. $\widetilde{f_n} \xrightarrow{m_\varphi} 0$. Hence by [21, Corollary 2.4] $\widetilde{f} \xrightarrow{(0)^*} 0$. Thus for any subsequence $(\widetilde{f}_{k_n})$ of $(\widetilde{f_n})$ there exists a subsequence $(\widetilde{f}_{l_{k_n}})$ of $(\widetilde{f}_{k_n})$ such that $\widetilde{f}_{l_{k_n}} \xrightarrow{(0)} 0$ in L^φ, i.e. $\widetilde{f}_{l_{k_n}}(\omega) \to 0$ μ-a.e. and $|\widetilde{f}_{l_{k_n}}(\omega)| \le u(\omega)$ μ-a.e. for some $u \in L^\varphi$. Since τ is a Lebesgue topology, $f_{l_{k_n}} \xrightarrow{\tau} 0$. This means that $f_n \xrightarrow{(\tau)^*} 0$, so $f_n \xrightarrow{\tau} 0$, as desired.

(ii) $\Rightarrow$ (i) Let $f_n \in L^\varphi(X)$ and $\|f_n(\omega)\|_X \to 0$ and $\|f_n(\omega)\|_X \le u(\omega)$ μ-a.e. for some $u \in L^\varphi$, i.e. $\widetilde{f_n} \xrightarrow{(0)} 0$ in L^φ. Then by [21, Theorem 1.3] $\widetilde{f_n} \xrightarrow{m_\varphi} 0$, i.e. $f_n \xrightarrow{M_\varphi} 0$. By (ii) $f_n \xrightarrow{\tau} 0$, and it means that τ is a Lebesgue topology. □

Theorem 6.3

(i) $\mathcal{T}_\varphi^\wedge(X)$ *is the finest Lebesgue topology on* $L^\varphi(X)$.
(ii) $\mathcal{T}_\varphi^\wedge(X)$ *is generated by the family of all absolutely continuous solid pseudonorms on* $L^\varphi(X)$.

Proof. (i) It follows from Theorem 6.1 and Theorem 6.2.

(ii) It follows from (i) and Theorem 2.3. □

To present a basic description of the modular topology $\mathcal{T}_\varphi^\wedge(X)$ we recall some relations among Orlicz functions and next, distinguish some classes of Orlicz and Young functions.

We shall say that an Orlicz function ψ is *completely weaker* than another φ for all u (resp. for small u; resp. for large u), in symbols $\psi \overset{a}{\triangleleft} \varphi$ (resp. $\psi \overset{s}{\triangleleft} \varphi$; resp. $\psi \overset{l}{\triangleleft} \varphi$), if for arbitrary $c > 1$ there exists $d > 1$ such that $\psi(cu) \leq d\varphi(u)$ for $u \geq 0$ (res. for $0 \leq u \leq u_0$; resp. for $u \geq u_0 > 0$) (see [19], [20]). It is seen that φ satisfies the so called Δ_2-condition for all u (resp. for small u; resp. for large u) if and only if $\varphi \overset{a}{\triangleleft} \varphi$ (resp. $\varphi \overset{s}{\triangleleft} \varphi$; resp. $\varphi \overset{l}{\triangleleft} \varphi$).

An Orlicz function φ continuous for all $u \geq 0$, taking only finite values and such that $\varphi(u) \to \infty$ as $u \to \infty$ is usually called a φ-function. We will denote by Φ the collection of all φ-functions. A Young function φ taking only finite values is called an *N-function* whenever $\lim_{u\to 0} \varphi(u)/u = 0$ and $\lim_{u\to\infty} \varphi(u)/u = \infty$. We will denote by Φ_N the collection of all N-functions.

Let Φ_0 be the collection of all Orlicz functions such that $\varphi(u) \to \infty$ as $u \to \infty$. Let

$$\Phi_{01} = \big\{\varphi \in \Phi_0 \colon \varphi(u) < \infty \ \text{ for } \ u \geq 0\big\},$$
$$\Phi_{02} = \big\{\varphi \in \Phi_0 \colon \varphi \ \text{ jumps to } \ \infty, \text{ i.e., } \ \varphi(u) = \infty \ \text{ for } \ u > u_0 > 0\big\}.$$

The next two theorems give a basic characterization of the modular topology $\mathcal{T}_\varphi^\wedge(X)$ in terms of some family of solid norms on $L^\varphi(X)$ defined by Orlicz functions.

Theorem 6.4

Let $\varphi \in \Phi_{0i}$ $(i = 1, 2)$. Then the modular topology $\mathcal{T}_\varphi^\wedge(X)$ on $L^\varphi(X)$ is generated by the family of solid F-norms $\{|\!\cdot\!|_{L^\psi(X)} \colon \psi \in \Psi_{0i}^\varphi\}$, where $\Psi_{01}^\varphi = \{\psi \in \Phi \colon \psi \overset{a}{\triangleleft} \varphi\}$, $\quad \Psi_{02}^\varphi = \{\psi \in \Phi \colon \psi \overset{s}{\triangleleft} \varphi\}$.

Proof. Let $\varphi \in \Phi_{0i}$ $(i = 1, 2)$. Then $\mathcal{T}_\varphi^\wedge$ is the finest Lebesgue topology on L^φ and is generated by the family $\{|\!\cdot\!|_\psi \colon \psi \in \psi_{0i}^\varphi\}$ of F-norms (see [22, Theorem 1.1, Theorem 1.2]). Then the topology $\overline{\mathcal{T}_\varphi^\wedge}$ on $L^\varphi(X)$ is generated by the family $\{|\!\cdot\!|_{L^\psi(X)} \colon \psi \in \Psi_{0i}^\varphi\}$ of solid F-norms and by Theorem 5.7. $\overline{\mathcal{T}_\varphi^\wedge}$ is the finest Lebesgue topology on $L^\varphi(X)$. By Theorem 6.3 $\mathcal{T}_\varphi^\wedge(X) = \overline{\mathcal{T}_\varphi^\wedge}$, and we are done. □

Now let Φ_0^C be the collection of all Young functions φ and such that $\lim_{u\to\infty}\varphi(u)/u=\infty$. Let us write:

$$\Phi_{01}^C=\Big\{\varphi\in\Phi_0^C\colon \varphi(u)<\infty \ \text{ for all } \ u\geq 0 \ \text{ and } \ \lim_{u\to 0}\frac{\varphi(u)}{u}=0\Big\},$$
$$\Phi_{02}^C=\Big\{\varphi\in\Phi_0^C\colon \varphi \ \text{ jumps to } \ \infty \ \text{ and } \ \lim_{u\to 0}\frac{\varphi(u)}{u}=0\Big\},$$
$$\Phi_{03}^C=\Big\{\varphi\in\Phi_0^C\colon \varphi(u)<\infty \ \text{ for all } \ u\geq 0 \ \text{ and } \ \lim_{u\to 0}\frac{\varphi(u)}{u}>0\Big\},$$
$$\Phi_{04}^C=\Big\{\varphi\in\Phi_0^C\colon \varphi \ \text{ jumps to } \ \infty \ \text{ and } \ \lim_{u\to 0}\frac{\varphi(u)}{u}>0\Big\}.$$

Then $\Phi_0^C=\bigcup_{i=1}^4\Phi_{0i}^C$, and the sets Φ_{0i}^C are pairwise disjoint. It is seen that $\Phi_{01}^C=\Phi_N$. Denote by

$$\Psi_{01}^{\varphi}(C)=\big\{\psi\in\Phi_N\colon \psi\overset{a}{\triangleleft}\varphi\big\} \ \text{ whenever } \ \varphi\in\Phi_{01}^C\,,$$
$$\Psi_{02}^{\varphi}(C)=\big\{\psi\in\Phi_N\colon \psi\overset{s}{\triangleleft}\varphi\big\} \ \text{ whenever } \ \varphi\in\Phi_{02}^C\,,$$
$$\Psi_{03}^{\varphi}(C)=\big\{\psi\in\Phi_N\colon \psi\overset{l}{\triangleleft}\varphi\big\} \ \text{ whenever } \ \varphi\in\Phi_{03}^C\,,$$
$$\Psi_{04}^{\varphi}(C)=\Phi_{03}^C, \ \text{ whenever } \ \varphi\in\Phi_{04}^C\,.$$

Theorem 6.5

Let $\varphi\in\Phi_{0i}^C$ $(i=1,2,3,4)$. Then the modular topology $\mathcal{T}_\varphi^\wedge(X)$ on $L^\varphi(X)$ is generated by the family of solid norms: $\{|||\cdot|||_{L^\psi(X)}\colon \psi\in\Psi_{0i}^\varphi(C)\}$.

Proof. Let $\varphi\in\Phi_{0i}^C$ $(i=1,2,3,4)$. Then $\mathcal{T}_\varphi^\wedge$ is the finest Lebesgue topology on L^φ and is generated by the family $\{|||\cdot|||_\psi\colon \psi\in\Psi_{0i}^\varphi(C)\}$ of norms (see [22, Theorem 1.1, Theorem 1.5]). Then the topology $\overline{\mathcal{T}_\varphi^\wedge}$ on $L^\varphi(X)$ is generated by the family $\{|||\cdot|||_{L^\psi(X)}\colon \psi\in\Psi_{0i}^\varphi(C)\}$ of solid norms on $L^\varphi(X)$, and by Theorem 5.7 $\overline{\mathcal{T}_\varphi^\wedge}$ is the finest Lebesgue topology on $L^\varphi(X)$. In view of Theorem 6.3 the identity $\mathcal{T}_\varphi^\wedge(X)=\overline{\mathcal{T}_\varphi^\wedge}$ holds. □

Corollary 6.6

Let $\varphi\in\Phi_{0i}^C$ $(i=1,2,3,4)$. Then the space $(L^\phi(X),\ \mathcal{T}_\varphi^\wedge(X))$ is complete.

Proof. The modular topology $\mathcal{T}_\varphi^\wedge$ on L^φ has the Fatou property (see [1, p. 80]), so $\mathcal{T}_\varphi^\wedge$ is generated by same family $\{p_\alpha\colon \alpha \in \{\alpha\}\}$ of Fatou seminorms (i.e. p_α satisfy the condition C in the paper [4]). Since the space $(L^\varphi, \mathcal{T}_\varphi^\wedge)$ is complete (see [20, Theorem 1.3]), by [4, Theorem 3] the space $(L^\varphi(X), \mathcal{T}_\varphi^\wedge(X))$ is complete, because $\mathcal{T}_\varphi^\wedge(X) = \overline{\mathcal{T}_\varphi^\wedge}$ (see the proof of Theorem 6.5). □

As an application of Theorem 6.5 we obtain the following characterization of absolutely continuous seminorms on $L^\varphi(X)$.

Corollary 6.7

Let $\varphi \in \Phi_{0i}^C$ $(i = 1, 2, 3, 4)$. Then for a solid seminorm ρ on $L^\varphi(X)$ the following statements are equivalent:

(i) ρ is absolutely continuous on $L^\varphi(X)$.
(ii) There exist $\psi \in \Psi_{0i}^\varphi(C)$ and a number $a > 0$ such that

$$\rho(f) \le a|||f|||_{L^\psi(X)} \quad \text{for all} \quad f \in L^\varphi(X).$$

Proof. (i) $\Rightarrow$ (ii) Let $\varphi \in \Phi_{0i}^C$ $(i = 1, 2, 3, 4)$. Since $\mathcal{T}_\varphi^\wedge(X)$ is the finest Lebesgue topology on $L^\varphi(X)$, by Theorem 6.5 and [12, Ch. 4, § 18,(4)] there exist $\psi_1, \ldots, \psi_n \in \Psi_{0i}^\varphi(C)$ and a number $a > 0$ such that

$$\rho(f) \le a \max(|||f|||_{L^{\psi_1}(X)}, \ldots, |||f|||_{L^{\psi_n}(X)}) \text{ for all } f \in L^\varphi(X).$$

Let $\psi(u) = \max(\psi_1(u), \ldots, \psi_n(u))$ for $u \ge 0$. Then $\psi \in \Psi_{0i}^\varphi(C)$ and $|||f|||_{L^{\psi_j}(X)} \le |||f|||_{L^\psi(X)}$ for $j = 1, 2, \ldots, n$, so

$$\rho(f) \le a|||f|||_{L^\psi(X)} \quad \text{for all} \quad f \in L^\varphi(X).$$

(ii) $\Rightarrow$ (i) By Theorem 6.3 and Theorem 6.5 for each $\psi \in \Psi_{0i}^\varphi(C)$, $|||\cdot|||_{L^\psi(X)}$ is an continuous norm on $L^\varphi(X)$, so ρ is also absolutely continuous. □

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