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The E and K functionals for the pair $(X(A), l_\infty(B))$

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Abstract

We prove some exact formulas for the E and K functionals for pairs of the type $(X(A), l_\infty(B))$ where X has the lattice property. These formulas are extensions of their well-known counterparts in the scalar valued case. In particular we generalize formulas by Pisier [4] and by the present author [2].

1. Introduction

If A and B are two quasi-normed spaces both linearly and continuously embedded in a Hausdorff topological vector space $\mathcal{X}$, then (A, B) is said to be a compatible pair. For $x \in A + B$, $t > 0$ the K and E functionals are defined as

$$K(t, f, A, B) := \inf\{\|g\|_A + t\,\|h\|_B : \ g \in A,\ h \in B \text{ and } f = g + h\},$$

$$E(t, f, A, B) := \inf\{\|f - g\|_A : \ f - g \in A,\ g \in B \text{ and } \|g\|_B \le t\}$$

respectively. By the definitions of the K and E functionals we obviously have

$$K(t, f, A, B) = \inf_{s>0} \{E(s, f, A, B) + st\} . \tag{1}$$

For the purpose of describing the connection between the K and E functionals we define the following transformations: For $f : (0, \infty) \to [0, \infty]$ and $t > 0$ let

$$f^\bullet(t) := \inf_{s>0} \{f(s) + st\} \tag{2}$$

and

$$f^{\circ}(t) := \sup_{s>0}\{f(s) - st\}\,. \tag{3}$$

These transformations are closely related to the Legendre transform. One can easily prove that $f^{\bullet\circ}$ is the greatest decreasing lower semicontinuous convex minorant of f, denoted by $f^{\vee}$, and that $f^{\circ\bullet}$ is the least concave majorant of f, denoted by $f^{\wedge}$. Formula (1) can now be written as $K = E^{\bullet}$. If we take the transform (3) of this formula we arrive at $E^{\vee} = K^{\circ}$. Since, in the normed space case, the E functional is convex we have $E = E^{\vee}$ with the possible exception at the point where the E functional jumps to infinity (since lower semicontinuity may be violated there). Hence, in the normed space case, $E(t, x, A, B) = K(t, x, A, B)^{\circ}$ with the possible exception where the E functional jumps to infinity.

In this paper we consider vector valued sequence spaces. The space $X(A)$, $A = \prod_{i=0}^{\infty} A_i$, is defined by

$$X(A) := \left\{ (a_0, a_1, \dots) \in \prod_{i=0}^{\infty} A_i : \ \left(\|a_0\|_{A_0}, \|a_1\|_{A_1}, \dots\right) \in X \right\}$$

where A_i are quasi-normed spaces and X is a normed real valued sequence space. The function

$$(a_0, a_1, \dots) := a \mapsto \|a\|_{X(A)} := \left\| \|a_i\|_{A_i} \right\|_X$$

is used as quasi-norm on this space. If $a \in X$ and, for $i \in \mathbb{N}$, $0 \le |b_i| \le |a_i|$ implies $b \in X$ and $\|b\|_X \le \|a\|_X$ then X is said to have the *lattice* property. For $a \notin X$ we define norm of a as infinity. By saying that (A, B), $A = \prod_{i=0}^{\infty} A_i$ and $\prod_{i=0}^{\infty} B_i$, is a compatible sequence pair we mean that (A_i, B_i) are compatible pairs of quasi-normed spaces for $i = 0, 1, \dots$.

For a normed real valued sequence space X we define, for a weight $\omega = (\omega_0, \omega_1, \dots)$ (a strictly positive sequence), the weighted space X^{ω} as the set of all $x = (x_0, x_1, \dots)$ for which $x\omega = (x_0\omega_0, x_1\omega_1, \dots) \in X$ with the norm $\|x\|_{X^{\omega}} := \|x\omega\|_X$. Finally we define $\omega^{-1} := (\omega_0^{-1}, \omega_1^{-1}, \dots)$ when $\omega = (\omega_0, \omega_1, \dots)$.

2. Formulas for the K and E functionals

If X has the lattice property then the E functional for the pair (X, l_{∞}) can easily be seen to be

$$E(t, f, X, l_{\infty}) = \|(|f| - t)_+\|_X \,, \tag{4}$$

see e.g. [1] and [3]. We note that this formula can be rewritten as

$$E(t, f, X, l_{\infty}) = \|E(t, f(\cdot), \mathbb{C}, \mathbb{C})\|_X \,.$$

Therefore, our next theorem may be regarded as a generalization of (4) to the case of vector valued sequence spaces.

Theorem 1

Let X be a normed sequence space with the lattice property and let (A, B) be a compatible sequence pair. If X contains a strictly positive sequence, then

$$E\Big(t, x, X(A), l_\infty(B)\Big) = \|E(t, x_i, A_i, B_i)\|_X \ . \tag{5}$$

Proof. We begin to prove $E(t, x, X(A), l_\infty(B)) \geq \|E(t, x_i, A_i, B_i)\|_X$. We may assume that $E(t, x, X(A), l_\infty(B)) < \infty$ since it holds trivially otherwise. Choose $y \in l_\infty(B)$ with $\|y\|_{l_\infty(B)} \leq t$ such that $x - y \in X(A)$ arbitrarily. This implies that

$$\|x_i - y_i\|_{A_i} \geq E(t, x_i, A_i, B_i)$$

and, by the lattice property, it yields that

$$E\Big(t, x, X(A), l_\infty(B)\Big) \geq \|E(t, x_i, A_i, B_i)\|_X$$

follows. In order to prove $E(t, x, X(A), l_\infty(B)) \leq \|E(t, x_i, A_i, B_i)\|_X$ we may assume that $\|E(t, x_i, A_i, B_i)\|_X < \infty$. In particular this means that $E(t, x_i, A_i, B_i) < \infty$ for all i since X is a real valued sequence space. Let ρ be a strictly positive sequence with norm less or equal to one. For every $\varepsilon > 0$ we may choose y with $\|y\|_{l_\infty(B)} \leq t$ such that

$$E(t, x_i, A_i, B_i) \geq \|x_i - y_i\|_{A_i} - \varepsilon\rho_i$$

for all $i \in \mathbb{N}$. By using the lattice property we obtain that

$$\|E(t, x_i, A_i, B_i)\|_X \geq \|x - y\|_{X(A)} - \varepsilon \geq E\Big(t, x, X(A), l_\infty(B)\Big) - \varepsilon\,,$$

and the theorem follows. □

By using formula (1) and the previous theorem we immediately get:

Corollary 2

With the same assumptions as in the previous theorem the following formula for the K functional holds:

$$K\Big(t, x, X(A), l_\infty(B)\Big) = \inf_{\lambda>0} \{\|E(\lambda, x_i, A_i, B_i)\|_X + \lambda t\} \ . \tag{6}$$

Remark 1. The formulas (5) and (6) can be generalized to the weighted case. We have

$$E\Big(t, x, X(A), l_\infty^\omega(B)\Big) = \|E(t, \omega_i x_i, A_i, B_i)\|_{X^{\omega^{-1}}}$$

and

$$K\Big(t, x, X(A), l_\infty^\omega(B)\Big) = \inf_{\lambda>0} \left\{\|E(\lambda, \omega_i x_i, A_i, B_i)\|_{X^{\omega^{-1}}} + \lambda t\right\} \,,$$

where ω is an arbitrary weight.

Next we state the following description of $K(t, x, l_p(A), l_\infty(B))$:

Theorem 3

Let (A,B) be a compatible sequence pair and $1 \le p < \infty$. Then the K functional $K(t,x,l_p(A),l_\infty(B))$, for an $x \in l_p(A)+l_\infty(B)$, is equivalent to

$$\sup\left\{\left(\sum_{i=0}^{\infty} K(t_i,x_i,A_i,B_i)^p\right)^{1/p} : t_i>0,\, \sum t_i^p \le t^p\right\}, \tag{7}$$

with the equivalence constants being $2^{1-p/p}$ and 1, with expression (7) as the smaller one.

Remark 2. For the case $p=1$ this result can be found in [4] as Corollary 3. Our proof below is completely different.

Proof. First we note that, according to Corollary 2, it yields that

$$K\Big(t,x,l_p(A),l_\infty(B)\Big) = \inf_{\lambda>0}\left\{\left(\sum_{i=0}^{\infty} E(\lambda,x_i,A_i,B_i)^p\right)^{1/p} + \lambda t\right\}. \tag{8}$$

We begin to prove that expression (7) is greater than or equal to a constant times the right hand side of (8). Since

$$2^{(1-p)/p}\inf_{\lambda>0}\left\{\left(\sum_{i=0}^{\infty} E(\lambda,x_i)^p\right)^{1/p} + \lambda t\right\} \le \left\{\inf_{\lambda>0}\sum_{i=0}^{\infty} E(\lambda,x_i)^p + \lambda^p t^p\right\}^{1/p},$$

we have to show that

$$\inf_{\lambda>0}\sum_{i=0}^{\infty} E(\lambda,x_i)^p + \lambda^p t^p \le \sup_{t_i}\sum_{i=0}^{\infty} K(t_i,x_i)^p\,.$$

But $\sum_{i=0}^{\infty} E(\lambda,x_i)^p + \lambda^p t^p$ is convex as a function of λ and converges to infinity at infinity. Since the E functional may be infinite on an initial segment $\sum_{i=0}^{\infty} E(\lambda,x_i)^p + \lambda^p t^p$ is infinite on $(0,c)$ and finite on (c,∞). By (8) this c is finite since $x \in l_p(A)+l_\infty(B)$. Hence, the infimum is attained in an interior point of (c,∞) (case I) or it is the limit as $\lambda \searrow c$ (case II).

Case I: Let λ_0 be the point where the infimum is attained. We have

$$\frac{1}{p\lambda_0^{p-1}}\left(\sum_{i=0}^{\infty} E\,(\lambda_0, x_i)^p\right)_l' \le -t^p \le \frac{1}{p\lambda_0^{p-1}}\left(\sum_{i=0}^{\infty} E(\lambda_0, x_i)^p\right)_r', \tag{9}$$

where f_l' (f_r') is the left (right) derivative of f. Choose α such that

$$\frac{\alpha}{p\lambda_0^{p-1}}\left(\sum_{i=0}^{\infty} E(\lambda_0, x_i)^p\right)_l' + \frac{1-\alpha}{p\lambda_0^{p-1}}\left(\sum_{i=0}^{\infty} E(\lambda_0, x_i)^p\right)_r' = -t^p\,.$$

Define t_i via

$$-t_i^p = \frac{\alpha}{p\lambda_0^{p-1}}\,\Big(E(\lambda_0, x_i)^p\Big)_l' + \frac{1-\alpha}{p\lambda_0^{p-1}}\,\Big(E(\lambda_0, x_i)^p\Big)_r'\,.$$

By the fact that the difference quotient of a convex function increases it follows, by uniform convergence, that we may differentiate termwise in (9). This implies that $\sum t_i^p = t^p$ and that the infimum of $E(\lambda, x_i)^p + \lambda^p t_i^p$ is attained at $\lambda = \lambda_0$. Hence

$$\begin{aligned}
&\inf_{\lambda>0}\sum_{i=0}^{\infty} E(\lambda, x_i)^p + \lambda^p t^p \\
&= \sum_{i=0}^{\infty} E(\lambda_0, x_i)^p + \lambda_0^p t^p = \sum_{i=0}^{\infty}\Big(E(\lambda_0, x_i)^p + \lambda_0^p t_i^p\Big) \\
&= \sum_{i=0}^{\infty}\inf_{\mu_i}\Big(E(\mu_i, x_i)^p + \mu_i^p t_i^p\Big) \le \sum_{i=0}^{\infty}\Big(\inf_{\mu_i} E(\mu_i, x_i) + \mu_i t_i\Big)^p \\
&= \sum_{i=0}^{\infty} K(t_i+, x_i)^p \le \sup_{t_i}\sum_{i=0}^{\infty} K(t_i, x_i)^p\,,
\end{aligned}$$

note that we need have t_i+ since t_i may be zero.

Case II: Now we thus assume that $\sum_{i=0}^{\infty} E(\lambda, x_i)^p + \lambda^p t^p$ increases on (c, ∞). Let J be the set if i for which c is in the interior of the set where $E(\cdot, x_i)$ is finite. Choose t_i as

$$-t_i^p = \frac{1}{pc^{p-1}}\,\Big(E(c, x_i)^p\Big)_r', \quad \text{if} \quad i \in J$$

and

$$t_i^p = \xi_i\left(t^p - \sum_{i\in J} t_i^p\right), \quad \text{if} \quad i \notin J\,,$$

where

$$\xi_i = \frac{\Big(E(c+, x_i)^p\Big)'_r}{\sum_{i\notin J}\Big(E(c+, x_i)^p\Big)'_r}\,,$$

if not all $(E(c+, x_i)^p)'_r = 0$ otherwise we choose $\xi_i > 0$ and such that $\sum_{i\notin J} \xi_i = 1$.

Obviously, $\sum t_i^p = t^p$ and a simple calculation shows that $E(\lambda, x_i)^p + \lambda^p t_i^p$ decreases to its infimum as $\lambda \searrow c$, and

$$\inf_{\lambda>0} \sum_{i=0}^{\infty} E(\lambda, x_i)^p + \lambda^p t^p \le \sup_{t_i} \sum_{i=0}^{\infty} K(t_i, x_i)^p\,,$$

follows as in case I.

Conversely, choose $\varepsilon > 0$ arbitrary. We can find $t_i' > 0$ such that $\sum (t_i')^p \le t^p$ and

$$\sup \left(\sum_{i=0}^{\infty} K(t_i, x_i)^p\right)^{1/p} \le \varepsilon + \left(\sum_{i=0}^{\infty} K(t_i', x_i)^p\right)^{1/p}.$$

For an arbitrary λ' we have, in view of formula (1), that

$$\begin{aligned}\sup \left(\sum_{i=0}^{\infty} K(t_i, x_i)^p\right)^{1/p} &\le \varepsilon + \left(\sum_{i=0}^{\infty} \Big(E(\lambda', x_i) + \lambda' t_i'\Big)^p\right)^{1/p} \\ &\le \varepsilon + \left(\sum_{i=0}^{\infty} E(\lambda', x_i)^p\right)^{1/p} + \lambda' t\,.\end{aligned}$$

Since λ' is arbitrary we can assume that it satisfies

$$\left(\sum_{i=0}^{\infty} E(\lambda', x_i)^p\right)^{1/p} + \lambda' t \le \varepsilon + \inf_{\lambda>0} \left\{ \left(\sum_{i=0}^{\infty} E(\lambda, x_i)^p\right)^{1/p} + \lambda t \right\}.$$

Thus we have proved

$$\sup \left(\sum_{i=0}^{\infty} K(t_i, x_i)^p\right)^{1/p} \le 2\varepsilon + \inf_{\lambda>0} \left\{ \left(\sum_{i=0}^{\infty} E(\lambda, x_i)^p\right)^{1/p} + \lambda t \right\},$$

and the proof is complete. □

Before we state the next theorem we need same notations. Let $L_{p,q}$, where $p = (p_0, p_1, \ldots)$ and $0 < p_n < \infty$, denote the sequence $(L_{p_0,q}, L_{p_1,q}, \ldots)$. The spaces $L_{p_n,q}$ are the *Lorentz* spaces (one star definition). By $l_\infty(L_\infty)$ we mean $l_\infty(L_\infty, L_\infty, \ldots)$. For $f = (f_n) \in l_q(L_{p,q}) + l_\infty(L_\infty)$ we define

$$T(\lambda) := \frac{\left(\sum_{n=0}^{\infty} \|(|f_n| - \lambda)_+\|_{p_n(q-1)/q, q-1}^{q-1}\right)^{1/(q-1)}}{\left(\sum_{n=0}^{\infty} \|(|f_n| - \lambda)_+\|_{p_n,q}^{q}\right)^{1/q}},$$

for $\underline{\lambda} < \lambda < \|f\|_{l_\infty(L_\infty)}$ where

$$\underline{\lambda} := \inf\left\{\lambda \geq 0 : \sum_{n=0}^{\infty} \|(|f_n| - \lambda)_+\|_{p_n,q}^{q} < \infty\right\}.$$

Theorem 4

Assume that $1 < q < \infty$ *and* $f = (f_n) \in l_q(L_{p,q}) + l_\infty(L_\infty)$. *If* $\underline{\lambda} < \|f\|_{l_\infty(L_\infty)}$, *then*

$$K\Big(t, f, l_q(L_{p,q}), l_\infty(L_\infty)\Big)$$

$$= \begin{cases} \left(\sum_{n=0}^{\infty} \|(|f_n| - \underline{\lambda})_+\|_{p_n,q}^{q}\right)^{1/q} + \underline{\lambda} t, & t^{1/(q-1)} > T(\lambda) \text{ for all } \underline{\lambda} < \lambda < \|f\|_{l_\infty(L_\infty)}, \\ \left(\sum_{n=0}^{\infty} \|(|f_n| - \lambda)_+\|_{p_n,q}^{q}\right)^{1/q} + \lambda t, & t^{1/(q-1)} = T(\lambda), \\ t\|f\|_{l_\infty(L_\infty)}, & t^{1/(q-1)} < T(\lambda) \text{ for all } \underline{\lambda} < \lambda < \|f\|_{l_\infty(L_\infty)}. \end{cases}$$

In the remaining case, i.e. when $\underline{\lambda} = \|f\|_{l_\infty(L_\infty)}$, *we have that the K functional equals* $t\|f\|_{l_\infty(L_\infty)}$.

Proof. To prove this we use Corollary 2 together with the well-known formula

$$E(t, f_n, L_{p_n,q}, L_\infty) = \|(|f_n| - t)_+\|_{p_n,q},$$

see e.g. [1] and [3]. The rest of the proof follows the proof in the scalar valued case, see [2]. □

We end this paper by stating an exact version of the following well-known equivalence formula

$$\|K(t,x_i,A,B)\|_\infty \le K\Big(t,x,l_\infty(A),l_\infty(B)\Big) \le 2\,\|K(t,x_i,A,B)\|_\infty \ .$$

Proposition 5

Let (A,B) be a compatible sequence pair of normed spaces. Then

$$K\Big(t,x,l_\infty(A),l_\infty(B)\Big) = \|K(t,x_i,A_i,B_i)\|_\infty^\wedge \ .$$

Proof. To prove this we use the connection between the K and E functionals. Since there are at most a countable set of points where the formula $E = K^\circ$ doesn't hold it can be used when we are taking infimum over all $\lambda > 0$. It yields that

$$\begin{aligned} K\Big(t,x,l_\infty(A),l_\infty(B)\Big) &= \inf_{\lambda>0}\,\{\sup_i E(\lambda,x_i,A_i,B_i) + \lambda t\} \\ &= \inf_{\lambda>0}\ \sup_i\ \sup_{s>0}\{K(s,x_i,A_i,B_i) - s\lambda + \lambda t\} \\ &= \inf_{\lambda>0}\ \sup_{s>0}\ \sup_i\{K(s,x_i,A_i,B_i) - s\lambda + \lambda t\} \\ &= \|\,K(t,x_i,A_i,B_i)\|_\infty^\wedge \ , \end{aligned}$$

and the proof is complete. □

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