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On the KR and WKR points of Orlicz spaces*

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Abstract

In this paper, we obtain criteria for KR and WKR points in Orlicz function spaces equipped with the Luxemburg norm.

I. Introduction

In this paper, we introduce the concept of KR and WKR points. Some results are obtained in Banach spaces, and then criteria for KR points and WKR points are given in Orlicz function spaces equipped with the Luxemburg norm.

Let X be a Banach space and X^* be the dual space of X. Let $S(X), B(X)$ be the unit sphere and the unit ball of X, respectively.

Definition 1 [1]. A point $x \in S(X)$ is called an UR point (WUR point) provided that for any $\{x_n\}$ of $S(X)$ such that $\|x_n + x\| \to 2$ with $n \to \infty$ implies $\|x_n - x\| \to 0$ $(x_n \xrightarrow{w} x)$ as $n \to \infty$. If every point on $S(X)$ is a UR point (WUR points), then X is called a LUR (WLUR) space.

Definition 2. A point $x \in S(X)$ is called a KR point (WKR point) provided that for any $\{x_n\} \subset S(X)$ and any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|(x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)} + x)/(k+1)\| \to 1$ with $n(1), n(2), \ldots, n(k) \to \infty$, there holds $\|x_n - x\| \to 0$ as $n \to \infty$ $(x_n \to x$ weakly). If every point on $S(X)$ is a KR point (WKR point), then X is said to be a LKR (WLKR) space.

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DEFINITION 3 [4]. A point $x \in S(X)$ is called a WM point if for any $\{x_n\} \subset S(X)$ such that $\|x_n + x\| \to 2$ with $n \to \infty$ there exists $f \in A(x)$ satisfying $f(x_n) \to 1$ as $n \to \infty$, where $A(x) = \{f \in S(X^*) : f(x) = 1\}$. X is said to be a WM space if every point on $S(X)$ is a WM point.

DEFINITION 4. A point $x \in S(X)$ is called an H point if for any $\{x_n\} \subset S(X)$ which is H convergent to x weakly, x_n is convergent to x in norm (see [5], [6]).

Throughout this paper, we denote by $\varphi\colon \mathbb{R} \to \mathbb{R}$ an even, convex and continuous function with $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty$ and $\varphi(u) = 0$ if and only if $u = 0$. By ψ we denote the complementary function of φ and by p (p_-) q (q_-) we denote the right (left) derivative of φ and ψ, respectively. The sequence $\{[a_n, b_n]\}_{n=1}^m$ (without loss of generality, we may assume that $m = \infty$) stands for affine intervals of ψ and $S_\psi^0 = \mathbb{R} \setminus \bigcup_{i=1}^\infty [a_i, b_i]$. Let (G, Σ, μ) be a nonnegative, finite, atomless and complete measure space.

For a measurable function x, let

$$R_\varphi(x) = \int_G \varphi\big(x(t)\big)d\mu.$$

The function R_φ is called a modular. We will write "$\varphi \in \Delta_2$" if φ satisfies the Δ_2-condition for large u. We define the Orlicz space $L_{(\varphi)}$ as the linear space

$$\Big\{x(t) \in L_0 : R_\varphi(\lambda x) < \infty, \quad \text{for some} \quad \lambda > 0\Big\}.$$

It is well known that $L_{(\varphi)}$ is a Banach space equipped with the Luxemburg norm

$$\|x\|_{(\varphi)} = \inf\Big\{c > 0 : R_\varphi(c^{-1}x) \le 1\Big\},$$

(see [8] and [9]).

Theorem 1

For any $x \in S(X)$ the following hold:

(1) *x is a* KR *point if and only if x is a* WKR *point and an H point.*

(2) *x is a* WUR *point if and only if is a* WKR *point and every a* WM *point.*

Proof. (1) *Necessity.* We only need to prove that every KR point is an H point. In fact, for any sequence $\{x_n\}$ of $S(X)$ with $x_n \to x$ weakly for any subsequence $\{x_{n(i)}\}$ $(i = 1, 2, \ldots, k)$ of $\{x_n\}$, we have

$$\big\|(x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)} + x)/(k+1)\big\| \to 1,$$

as $n(1), n(2), \ldots, n(k) \to \infty$. Since x is a KR point, we obtain that x_n tends to x strongly.

Sufficiency. Suppose x is a WKR point and an H point. For any subsequence $\{x_{n(i)}\}$ $(i = 1, 2, \ldots, k)$ of $\{x_n\}$ such that

$$\big\|(x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)} + x)/(k+1)\big\| \to 1,$$

as $n(1), n(2), \ldots, n(k) \to \infty$, we get first that $x_n \to x$ weakly and then, by the assumption that x is an H point, we get that $\{x_n\}$ tends to x strongly. This means that x is a KR point.

(2) We only need to prove the sufficiency. For any $\{x_n\}$ of $S(X)$ such that $\|x_n + x\} \to 2$, by the assumption that x is a WM point, there exists $f \in A(x)$ that satisfies $f(x_n) \to 1$ as $n \to \infty$. Hence for any subsequence $\{x_{n(i)}\}$ $(i = 1, 2, \ldots, k)$ of $\{x_n\}$ we have

$$\big\|(x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)} + x)/(k+1)\big\| \to 1,$$

as $n(1), n(2), \ldots, n(k) \to \infty$.

Since x is a WKR point we get that $\{x_n\}$ tends to x weakly. This show that x is a WUR point. □

It is obvious that x is a UR point if and if x is a WUR point and an H point. So we have the following result.

Corollary 1

A point $x \in S(X)$ is a UR *point if and only if x is a KR point and a* WM *point.*

Corollary 2

For any Banach space X the following hold:

(1) *X is a* LUR *space if and only if X is a* LKR *and a* WM *space.*
(2) *X is a* LKR *space if and only if X is a* WLKR *space with the H property.*
(3) *X is a* WLUR *space if and only if X is a* WLKR *space and a* WM *space.*

Theorem 2

For any $x \in S(L_\varphi)$ the following statements are equivalent:

(1) X *is a* UR *point.*
(2) x *is a* WUR *point.*
(3) x *is a* KR *point.*
(4) x *is a* WKR *point.*
(5) (i) $\varphi \in \Delta_2$,
(ii) $x(t) \in S_\varphi$ *for* μ*-a.e.* $t \in G$,
(iii) $\mu\{t \in G : x(t) = b\} = 0$ *or* $\psi \in \Delta_2$ *and* $\mu\{t \in G : x(t) = a\} = 0$,
where $[a, b]$ *is an affine interval of* φ.

Proof. By the definitions and the facts that (1) is equivalent to (5) (see [1]), we only need to prove that (4) implies (5). First, we will prove that (4) implies (i) in (5).

Take D large enough with $\mu G_0 > 0$, where $G_0 = \{t \in G : |x(t) \leq D\}$. If $\varphi \notin \Delta_2$, then there exists $z \in S(l_{(\varphi)})$ such that $R_{(\varphi)}(\lambda z) = \infty$ for any $\lambda > 1$. Hence there exists a singular functional φ that satisfies $\varphi(z) \neq 0$.

If $G_n = \{t \in G_0 : |z(t)| > n\}$, then $\mu G_n \to 0$ as $n \to \infty$ and $G_1 \supseteq G_2 \supseteq \ldots$. Put $x_n = x\chi_{G\setminus G_n} + z\chi_{G_n}$, $n = 1, 2, \ldots$. Then

$$\begin{aligned}\lim_{n\to\infty} \|x_n\|_{(\varphi)} &= \lim_{n\to\infty} \|x\chi_{G\setminus G_n} + z\chi_{G_n}\|_{(\varphi)} \\ &\leq \lim_{n\to\infty} \|x\chi_{G\setminus G_n}\|_{\varphi)} + \lim_{n\to\infty} \|z\chi_{G_n}\|_{(\varphi)} \\ &= \lim_{n\to\infty} \|x\chi_{G\setminus G_n}\|_{(\varphi)} \leq \|x\|_{(\varphi)} = 1\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \|x_n\|_{(\varphi)} &= \lim_{n\to\infty} \|x\chi_{G\setminus G_n} + z\chi_{G_n}\|_{(\varphi)} \\ &\geq \lim_{n\to\infty} \|x\chi_{G\setminus G_n}\|_{(\varphi)} = \|x\|_{(\varphi)} = 1.\end{aligned}$$

This means that $\lim_{n\to\infty} \|x_n\|_{(\varphi)} = 1$.

For all subsequences $\{x_{n(i)}\}$ $(i = 1, 2, \ldots, k)$ of $\{x_n\}$ we have

$$\left\|(x + \Sigma x_{n(i)})/(k+1)\right\|_{(\varphi)} \geq \left\|x\chi_{G\setminus G_{(m_i)}}\right\|_{(\varphi)} \to 1,$$

where $m_i = \min\{n(i) : i = 1, 2, \ldots, k\}$. But

$$\varphi(x_n - x) = \varphi(z\chi_{G_n}) - \varphi(x\chi_{G_n}) = \varphi(z\chi_{G_n}) \neq 0.$$

This contradiction shows that (i) holds true.

Next we will prove that (4) implies (ii) in (5). Otherwise, there exist an interval (a, b) and $\varepsilon > 0$ such that φ is an affine function on $[a, b]$, i.e., $\varphi(u) = Au + B$ whenever $u \in [a, b]$ and $\mu G_0 > 0$, where $G_0 = \{t \in G : x(t) \in [a + \varepsilon, b - \varepsilon]\}$.

Take two subsets G_1 and G_2 of G_0 such that $G_1 \cap G_2 = \varphi$, (φ stand for the empty set), $G_1 \cup G_2 = G_0$ and $\mu G_1 = \mu G_2$. Put

$$x_n = x\chi_{G\setminus G_0} + (x - \varepsilon)\chi_{G\setminus G_1} + (x + \varepsilon)\chi_{G\setminus G_2},$$

$n = 1, 2, \dots$. Then $R_\varphi(x_n) = R_\varphi(x) = 1$ and for all subsequences $\{x_{n(i)}\}$ $(i = 1, 2, \dots, k)$ of $\{x_n\}$, we have

$$\begin{aligned}
&R_\varphi\big((x + \Sigma x_{n(i)})/(k+1)\big) \\
&\quad = R_\varphi(x\chi_{G\setminus G_0}) + \int_{G_1} \big(Ax(t) + B - kA\varepsilon/(k+1)\big)dt \\
&\quad\quad + \int_{G_2} \big(Ax(t) + B + kA\varepsilon/(k+1)\big)dt \\
&\quad = R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi(x\chi_{G_1}) + R_\varphi(x\chi_{G_2}) = R_\varphi(x) = 1.
\end{aligned}$$

Taking $y = \chi_{G_2} - \chi_{G_1}$, we have $\langle x, y\rangle = \varepsilon\mu(G_0) > 0$. This contradiction shows that condition (ii) holds true.

Finally, we will prove that (4) implies (iii) in (5). We will divide the proof into two parts.

(1) If $\psi \notin \Delta_2$ and $\mu G_0 > 0$, where $G_0 = \{t \in G : x(t) = b\}$. Take $\varepsilon > 0$ small enough such that $\varphi(u) = Au + B$ for any $u \in [b - \varepsilon, b]$ and $kb \geq (k+1)\varepsilon$.

Since $\psi \notin \Delta_2$, there exist $u_n \nearrow \infty$ satisfying

$$\varphi\big(u_n/(k+1)\big) > (1 - 1/2^n)\varphi(u_n)/(k+1), \quad n = 1, 2, \dots .$$

Choose a subset G^0 of G_0 with $\mu G^0 = \mu G_0/2$ and $G_n \subseteq G_0\setminus G^0$ such that $G_i \cap G_j = \phi$ if $i \neq j$ and $\varphi(u_n)\mu G_n = A\varepsilon\mu G_0$ $(n = 1, 2, \dots)$. It is obvious that $\mu G_n \to 0$ as $n \to \infty$. Put

$$x_n(t) = x(t)\chi_{G\setminus G_0}(t) + (b - \varepsilon)\chi_{G_0\setminus G_n}(t) + u_n\chi_{G_n}(t).$$

Then

$$\begin{aligned}
R_\varphi(x_n) &= R_\varphi(x\chi_{G\setminus G_0}) + (Ab + B - A\varepsilon)\mu(G \setminus G_n) + \varphi(u_n)\mu(G_n) \\
&\leq R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi(b\chi_{G\setminus G_n}) - A\varepsilon\mu(G_0 \setminus G_n) + A\varepsilon\mu(G_0) \\
&= R_\varphi(x\chi_{G\setminus G_n}) + A\varepsilon\mu(G_n) \to R_\varphi(x) = 1.
\end{aligned} \tag{1}$$

Moreover, for any subsequence $\{x_{n(i)}\}$ $(i = 1, 2, \dots, k)$ of $\{x_n\}$ we have

$$
\begin{aligned}
(2)\qquad & R_\varphi((x + x_{n(1)} + \cdots + x_{n(k)})/(k+1)) \\
&= R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi((x\chi_{G_0} + (b-\varepsilon)\chi_{G\setminus G_{n(1)}} + u_{n(1)}\chi_{G_{n(1)}} \\
&\quad + \cdots + (b-\varepsilon)\chi_{G\setminus G_{n(k)}} + u_{n(k)}\chi_{G_{n(k)}})/(k+1)) \\
&= R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi((b\chi_{G_0} + k(b-\varepsilon)\chi_{G_0} - (b-\varepsilon)\chi_{G_{n(1)}} \\
&\quad - \cdots - (b-\varepsilon)\chi_{G_{n(k)}} + u_{n(1)}\chi_{G_{n(1)}} + \cdots + u_{n(k)}\chi_{G_{n(k)}})/(k+1)) \\
&= R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi((((k+1)b\chi_{G_0} - k\varepsilon\chi_{G_0} + (u_{n(1)} - (b-\varepsilon))\chi_{G_{n(1)}} \\
&\quad + \cdots + (u_{n(k)} - (b-\varepsilon))\chi_{G_{n(k)}})/(k+1)) \\
&= R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi((k+1)b\chi_{G_0\setminus G_{n(1)}\cup\ldots\cup G_{n(k)}} - k\varepsilon\chi_{G_0\setminus G_{n(1)}\cup\ldots\cup G_{n(k)}} \\
&\quad + ((k+1) - k\varepsilon + u_{n(1)} - (b-\varepsilon))\chi_{G_{n(1)}} \\
&\quad + \cdots + (k+1) - k\varepsilon + u_{n(k)} - (b-\varepsilon))\chi_{G_{n(k)}})/(k+1)) \\
&= R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi((((k+1)b - k\varepsilon)\chi_{G_0\setminus G_{n(1)}\cup\ldots\cup G_{n(k)}}/(k+1) \\
&\quad + R_\varphi((kb - (k+1)\varepsilon)\chi_{G_{n(1)}}/(k+1)) \\
&\quad + \ldots + R_\varphi((kb - (k+1)\varepsilon)\chi_{G_{n(k)}}/(k+1)) \\
&\geq R_\varphi(x\chi_{G\setminus G_0}) + R_\varphi(b\chi_{G_0\setminus G_{n(1)}\cup\ldots\cup G_{n(k)}}) \\
&\quad - kA\varepsilon\mu(G_0 \setminus G_{n(1)} \cup \ldots \cup G_{n(k)}) \\
&\quad + R_\varphi(y_{n(1)}\chi_{G_{n(1)}}/(k+1)) + \cdots + R_\varphi(u_{n(k)}\chi_{G_{n(k)}}/(k+1)) \\
&= R_\varphi(x\chi_{G\setminus G_{n(1)}\cup\ldots\cup G_{n(k)}}) - kA\varepsilon\mu(G_0 \setminus G_{n(1)} \cup \ldots \cup G_{n(k)})/(k+1) \\
&\quad + \varphi(u_{n(1)}/(k+1))\mu(G_{n(1)}) + \cdots + \varphi(u_{n(k)}/(k+1))\mu(G_{n(k)}) \\
&\geq R_\varphi(x\chi_{G\setminus G_{n(1)}\cup\ldots\cup G_{n(k)}}) - kA\varepsilon\mu(G_0)/(k+1) \\
&\quad + kA\varepsilon\mu(G_{n(1)} \cup \ldots \cup G_{n(k)})/(k+1) \\
&\quad + (1 - 1/2^{n(1)})\varphi(u_{n(1)})\mu(G_{n(1)})/(k+1) \\
&\quad + \ldots + (1 - 1/2^{n(k)})\varphi(u_{n(k)})\mu(G_{n(k)})/(k+1) \\
&\geq R_\varphi(x\chi_{G\setminus G_{n(1)}\cup\ldots\cup G_{n(k)}}) + kA\varepsilon\mu(G_{n(1)} \cup \ldots \cup G_{n(k)})/(k+1) \\
&\quad - (1/2^{n(1)} + \ldots + 1/2^{n(k)})A\varepsilon\mu(G_0)/(k+1) \to R_\varphi(x) = 1.
\end{aligned}
$$

Combining (1) and (2), we obtain

$$\|x_n\| \to 1 \quad \text{and} \quad \big\|(x + x_{n(1)} + \ldots + x_{n(k)})/(k+1)\big\| \to 1$$

as $n \to \infty$ and $n(1), \dots, n(k) \to \infty$, respectively. But

$$\int_G \big(x(t) - x_n(t)\big)\chi_{G^0}(t)dt = \varepsilon\mu(G_0)/2 > 0,$$

which contradicts the fact that $\{x_n\}$ tends to x weakly.

Now we will show that (4) implies (ii) in (5). Otherwise, denote $G_0 = \{t \in G : x(t) = b\}$ and $G_1 = \{t \in G : x(t) = a\}$. Then $\mu G_0 > 0$ and $\mu G_1 > 0$. For a convenience, we may assume that $\mu G_0 \le \mu G_1$.

Take a subset G_2 of G_1 and numbers $A, B, A_1, B_1, \varepsilon, \varepsilon_1, \delta, \delta_1$ such that $\mu G_2 = \mu G_0$, $\varphi(u) = Au + B$ for $u \in (a, a+\varepsilon)$, $\varphi(u) = A_1 u + B_1$ for $u \in (b - \varepsilon_1, b)$, $0 < \delta < \varepsilon$, $0 < \delta_1 < \varepsilon_1$ and $A\delta = A_1\delta_1$. Put

$$x_n = x\chi_{G\setminus G_0\cup G_2} + (a+\delta)\chi_{G_2} + (b-\delta_1)\chi_{G_0} \quad n = 1, 2, \dots .$$

Then

$$\begin{aligned} R_\varphi(x_n) &= R_\varphi(x\chi_{G\setminus G_0\cup G_2}) + \big[(a+\delta)A + B\big]\mu G_2 + \big[(b-\delta_1)A_1 + B_1\big]\mu G_0 \\ &= R_\varphi(x\chi_{G\setminus G_0\cup G_2}) + R_\varphi(x\chi_{G_2}) + R_\varphi(x\chi_G) = R_\varphi(x) = 1, \end{aligned}$$

i.e., $\|x_n\| = 1$. Moreover, for all subsequences $\{x_{n(i)}\}$ ($i = 1, 2, \dots, k$; $n(i) \in \{1, 2, \dots\}$) of $\{x_n\}$, we have

$$\begin{aligned} &R_\varphi\big((x + x_{n(1)} + x_{n(2)} + \ldots + x_{n(k)})/(k+1)\big) \\ &= R_\varphi(x\chi_{G\setminus G_0\cup G_2}) + k\delta A/(k+1)\mu G_2 + R_\varphi(x\chi_{G_2}) \\ &\quad - k\delta_1 A_1/(k+1)\mu G_0 + R_\varphi(x\chi_{G_0}) \\ &= R_\varphi(x) = 1, \end{aligned}$$

i.e., $\|(x + x_{n(1)} + x_{n(2)} + \ldots + x_{n(k)})/(k+1))\| = 1$.

Take $y(t) = \chi_{G_2}(t) - \chi_{G_0}(t)$. Then

$$\langle x_n, y\rangle = \int_G \big(x_n(t) - x(t)\big)dt = \delta\mu G_2 + \delta_1\mu G_0 > 0.$$

This contradiction shows that the condition holds, so the proof is finished. □

Corollary 3

For any Orlicz space $L_{(\varphi)}$ the following statements are equivalent:

(1) $L_{(\varphi)}$ *is* LUR*;*
(2) $L_{(\varphi)}$ *is* WLUR*;*
(3) $L_{(\varphi)}$ *is* KR*;*
(4) $L_{(\varphi)}$ *is* WKR*;*
(5) $L_{(\varphi)}$ *is* LRK*;*
(6) $L_{(\varphi)}$ *is* WLKR*;*
(7) $\varphi \in \Delta_2$ *and* φ *is a strictly convex function.*

Proof. This follows from Theorem 3 and the result that $L_{(\varphi)}$ is LUR if and only if $\varphi \in \Delta_2$ and φ is a strictly convex function (see [1]).

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