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Dual action of asymptotically isometric copies of $l_p (1 \leq p < \infty)$ and c_0

Shutao Chen* and Bor-Luh Lin

Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

Abstract

P.N. Dowling and C.J. Lennard proved that if a Banach space contains an asymptotically isometric copy of l_1, then it fails the fixed point property. In this paper, necessary and sufficient conditions for a Banach space to contain an asymptotically isometric copy of $l_p (1 \leq p < \infty)$ or c_0 are given by the dual action. In particular, it is shown that a Banach space contains an asymptotically isometric copy of l_1 if its dual space contains an isometric copy of l_∞, and if a Banach space contains an asymptotically isometric copy of c_0, then its dual space contains an asymptotically isometric copy of l_1.

§1. Preliminaries

A Banach space X is said to have an asymptotically isometric copy of l_1 [2], if for every sequence (ϵ_n) $(0 < \epsilon_n < 1)$ decreasing to 0, there exists a norm-one sequence (x_n) in X such that

$$\sum_n (1-\epsilon_n)|\alpha_n| \leq \left\| \sum_n \alpha_n x_n \right\|, \qquad (\alpha_n) \in l_1. \tag{1}$$

P.N. Dowling and C.J. Lennard [2] have shown that if a Banach space contains an asymptotically isometric copy of l_1, then it fails to have the fixed point property, i.e., there exists a nonexpansive self-mapping on a bounded closed convex subset of X which has no fixed point.

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A natural question is that: what does the dual space X^* behave if a Banach space X contains an asymptotically isometric copy of l_1? The following theorem answers this question.

Theorem 1

A Banach space X contains an asymptotically isometric copy of l_1 if and only if for any sequence $\delta_n \downarrow 0$ $(0 < \delta_n \leq 1)$, there exists a subspace X_0 in X such that X_0^* contains a norm-one sequence (x_m^*) satisfying

$$\Big\| \sum_m \pm(1-\delta_m)x_m^* \Big\|_{X_0^*} \leq 1\,. \tag{2}$$

Proof. Necessity. Let $\delta_n \downarrow 0$ $(0 < \delta_n \leq 1)$. By assumption, there exists a sequence (x_n) in $S(X)$, the unit sphere of X, such that

$$\sum_n (1-\delta_n)|\alpha_n| \leq \Big\| \sum_n \alpha_n x_n \Big\|, \qquad (\alpha_n) \in l_1\,. \tag{3}$$

Let $X_0 = \operatorname{span}\{x_n\}$. For each fixed $m \in \mathbb{N} = \{1, 2, ...\}$ and for any $(\alpha_n) \in l_1$ with $\alpha_m = -1$, by (3),

$$\Big\| x_m - \sum_{n\neq m} \alpha_n x_n \Big\| = \Big\| \sum_{n=1}^{\infty} \alpha_n x_n \Big\| \geq \sum_{n=1}^{\infty} (1-\delta_n)|\alpha_n| \geq 1-\delta_m\,.$$

Since for any $x \in X_0$, x has the form $x = \sum_n \alpha_n x_n$, the above inequality in fact implies that $dist(x_m,\ \operatorname{span}\{x_n\}_{n\neq m}) \geq 1-\delta_m$ for all $m \in \mathbb{N}$. Whence by Hahn-Banach Theorem, for each $m \in \mathbb{N}$, there exists $x_m^* \in S(X_0^*)$ such that

$$< x_m^*,\, x_m > \geq 1-\delta_m \text{ and } < x_m^*,\, x_n >= 0, \ \ (n \neq m)\,.$$

Therefore, for any $x = \sum_n \alpha_n x_n \in X_0$, by (3),

$$\begin{aligned} \Big\langle \sum_m \pm(1-\delta_m)x_m^*,\ \sum_n \alpha_n x_n \Big\rangle &= \sum_n \pm(1-\delta_n)\alpha_n < x_n^*,\, x_n > \\ &\leq \sum_n (1-\delta_n)|\alpha_n| \leq \Big\| \sum_n \alpha_n x_n \Big\| = \|x\|\,. \end{aligned}$$

This implies that

$$\Big\| \sum_m \pm(1-\delta_m)x_m^* \Big\|_{X_0^*} \le 1 .$$

Sufficiency. For any $\epsilon_n \downarrow 0$ $(0 < \epsilon_n < 1)$, let $0 < \delta_m < 1$ satisfy $1 - \epsilon_m = 2(1-\delta_m)^2 - 1$, $(\delta_m = 1 - \sqrt{1 - \epsilon_m/2})$. By assumption, X has a subspace X_0 such that $S(X_0^*)$ contains a sequence (x_m^*) satisfying (2). For every $m \in \mathbb{N}$, pick $x_m \in S(X_0)$ such that $< x_m^*, x_m >> 1 - \delta_m$. We shall show that (x_n) satisfies (1).

For each fixed $n \in \mathbb{N}$, let $\sigma_m = \operatorname{sign} < x_m^*, x_n >$ $(m \in \mathbb{N})$. By (2),

$$\begin{aligned}\sum_{m\ne n}(1-\delta_m)\big| < x_m^*, x_n > \big| &= \sum_{m\ne n} \sigma_m(1-\delta_m) < x_m^*, x_n > \\ &= \Big\langle \sum_{m=1}^{\infty} \sigma_m(1-\delta_m)x_m^*, x_n \Big\rangle - \sigma_n(1-\delta_n) < x_n^*, x_n > \\ &\le \|x_n\| - (1-\delta_n)^2 = 1 - (1-\delta_n)^2 .\end{aligned}$$

Whence, for any $(\alpha_n) \in l_1$, if we set $\sigma_m = \operatorname{sign}\alpha_m$, then

$$\begin{aligned}\Big\| \sum_n \alpha_n x_n \Big\| &\ge \Big\langle \sum_m \sigma_m(1-\delta_m)x_m^*, \sum_n \alpha_n x_n \Big\rangle \\ &= \sum_n \Big[(1-\delta_n)|\alpha_n| < x_n^*, x_n > + \sum_{m\ne n} \sigma_m(1-\delta_m)\alpha_n < x_m^*, x_n > \Big] \\ &\ge \sum_n \Big[(1-\delta_n)^2|\alpha_n| - |\alpha_n| \sum_{m\ne n}(1-\delta_m)| < x_m^*, x_n > | \Big] \\ &\ge \sum_n \Big[(1-\delta_n)^2|\alpha_n| - |\alpha_n|\big(1-(1-\delta_n)^2\big) \Big] \\ &= \sum_n (1-\epsilon_n)|\alpha_n| .\end{aligned}$$

Therefore, (1) holds. The proof is completed. □

Corollary 2

(i) A Banach space X *contains no asymptotically isometric copies of* l_1*, if for any infinite dimensional subspace* X_0*, there exists a positive integer* n *and a positive constant* δ *such that for any* $x_1^*, x_2^*, ..., x_n^* \in S(X_0^*)$*, there exist* $\epsilon_i = \pm 1, i = 1, 2, ..., n$ *satisfying*

$$\Big\| \sum_{i=1}^n \epsilon_i x_i^* \Big\|_{X_0^*} \ge 1 + \delta .$$

Especially, if for any infinite dimensional subspace X_0, X_0^* is uniformly non-square (i.e., above condition holds for $n = 2$), then X contains no asymptotically isometric copies of l_1.
(ii) A Banach space contains an asymptotically isometric copy of l_1, if its dual space contains an isometric copy of l_∞.

Proof. We only need to show (i) since (ii) follows directly from the proof of Theorem 1.

Pick natural numbers $1 = k_1 < k_2 < k_3 < ...$ such that $k_{j+1} - k_j \uparrow \infty$ as $j \uparrow \infty$. Define $\delta_m = 1/k_{j+1}$ for $k_j \leq m < k_{j+1}, j = 1, 2, ...$. If X contains an asymptotically isometric copy of l_1, then by Theorem 1, for the sequence $\{\delta_m\}$, X has a subspace X_0 such that X_0^* contains a norm-one sequence (x_m^*) satisfying (2). For any $n \in \mathbb{N}$ and $\delta > 0$, pick $t \geq 1$ such that $k_{t+1} - k_t \geq n$ and that $k_{t+1}/(k_{t+1} - 1) < 1 + \delta$.

Observe that $\|x \pm y\| \leq 1$ implies

$$\|x\| = \left\| \frac{(x+y) + (x-y)}{2} \right\| \leq \frac{\|x+y\| + \|x-y\|}{2} \leq 1\,,$$

it follows from (2) that

$$1 \geq \left\| \sum_{m=k_t+1}^{k_t+n} \pm (1-\delta_m) x_m^* \right\|_{X_0^*} = \left\| \sum_{m=k_t+1}^{k_t+n} \pm (1-1/k_{t+1}) x_m^* \right\|_{X_0^*}.$$

Therefore, by the choice of t,

$$\left\| \sum_{m=k_t+1}^{k_t+n} \pm x_m^* \right\|_{X_0^*} \leq \frac{1}{1-1/k_{t+1}} = \frac{k_{t+1}}{k_{t+1}-1} < 1+\delta\,.$$

This proves (i).

Next, we investigate the dual action of asymptotically isometric copy of c_0. □

DEFINITION 3. We say that a Banach space X contains an asymptotically isometric copy of c_0, if for any $\delta_n \downarrow 0$ $(0 < \delta_n \leq 1)$, X contains a norm-one sequence (x_n) such that

$$\sup_n (1-\delta_n)|\beta_n| \leq \left\| \sum_n \beta_n x_n \right\| \leq \sup_n (1+\delta_n)|\beta_n|, \quad (\beta_n) \in c_0. \tag{4}$$

Theorem 4

A Banach space X contains an asymptotically isometric copy of c_0 if and only if for any $\epsilon_n \downarrow 0$ $(0 < \epsilon_n < 1)$, there exists a norm-one shrinking basic sequence (x_n) in X such that the coefficient functionals $\{x_n^*\}$ on $X_0 = \operatorname{span}\{x_n\}$ have the properties that $\|x_n^*\|_{X_0^*} \le 1 + \epsilon_n$ and

$$\sum_n (1-\epsilon_n)|\alpha_n| \le \left\| \sum_n \alpha_n x_n^* \right\|_{X_0^*}, \quad \textit{for all} \quad \sum_n \alpha_n x_n^* \in X_0^*. \tag{5}$$

Proof. Sufficiency. For any $\delta_n \downarrow 0$ $(0 < \delta_n \le 1)$, let $\epsilon_n \downarrow 0$ $(0 < \epsilon_n < 1)$ satisfy

$$(1-\delta_n)(1+\epsilon_n) \le 1 \le (1+\delta_n)(1-\epsilon_n).$$

By assumption, X has a norm-one shrinking basic sequence (x_n) such that the coefficient functionals $\{x_n^*\}$ on $X_0 = \operatorname{span}\{x_n\}$ satisfy (5) and $\|x_n^*\|_{X_0^*} \le 1 + \epsilon_n$ for all $n \in \mathbb{N}$. We shall prove that (x_n) satisfies (4).

Since (x_n) is shrinking, $\{x_n^*\}$ is a basis of X_0^*. Whence, for any $x^* \in X_0^*$, x^* has the form $x^* = \sum_n \alpha_n x_n^*$, and (5) indicates that $(\alpha_n) \in l_1$. Therefore, for any $(\beta_n) \in c_0$, by (5) and the choice of (δ_n),

$$\begin{aligned}
\left\langle x^*, \sum_n \beta_n x_n \right\rangle &= \left\langle \sum_n \alpha_n x_n^*, \sum_n \beta_n x_n \right\rangle = \sum_n \alpha_n \beta_n \\
&= \sum_n \frac{1}{1-\epsilon_n} \beta_n (1-\epsilon_n)\alpha_n \le \sup_n \frac{1}{1-\epsilon_n} |\beta_n| \sum_n (1-\epsilon_n)|\alpha_n| \\
&\le \sup_n (1+\delta_n)|\beta_n| \left\| \sum_n \alpha_n x_n^* \right\|_{X_0^*} = \sup_n (1+\delta_n)|\beta_n| \|x^*\|_{X_0^*}.
\end{aligned}$$

Since $x^* \in X_0^*$ is arbitrary, this inequality implies $\sum_n \beta_n x_n \in X_0$ and

$$\left\| \sum_n \beta_n x_n \right\| \le \sup_n (1+\delta_n)|\beta_n|.$$

Next, we prove the other part of (4). For any $m \in \mathbb{N}$, by the choice of (δ_m) and $\|x_m^*\|_{X_0^*} \le 1 + \epsilon_m$,

$$\left\| \sum_n \beta_n x_n \right\| \ge \left| \left\langle \Sigma_n \beta_n x_n, \frac{x_m^*}{\|x_m^*\|_{X_0^*}} \right\rangle \right| = \frac{|\beta_m|}{\|x_m^*\|_{X_0^*}} \ge \frac{|\beta_m|}{1+\epsilon_m} \ge (1-\delta_m)|\beta_m|.$$

Since $m \in \mathbb{N}$ is arbitrary, this implies

$$\left\| \sum_n \beta_n x_n \right\| \geq \sup_n (1-\delta_n)|\beta_n| \,.$$

Necessity. For any $\epsilon_n \downarrow 0$ $(0 < \epsilon_n < 1)$, let $\delta_n \downarrow 0$ $(0 < \delta_n \leq 1)$ satisfy

$$(1-\epsilon_n)(1+\delta_n) \leq 1 \leq (1+\epsilon_n)(1-\delta_n)\,.$$

By Definition 3, X contains a norm-one sequence (x_n) satisfying (4). Let $\{e_n\}$ be the natural basis of c_0. Then by (4), the mapping: $e_n \longmapsto x_n$ induces an isomorphism from c_0 to $X_0 = \operatorname{span}\{x_n\}$. Therefore, $\sum_n \beta_n x_n \in X_0$ if and only if $(\beta_n) \in c_0$. Moreover, since $\{e_n\}$ is shrinking, so is $\{x_n\}$.

Let $\{x_m^*\}$ be the coefficient functionals on X_0. Then for any $m \in \mathbb{N}$ and any $\sum_n \beta_n x_n \in X_0$, by the choice of (δ_n) and (4),

$$\begin{aligned}\left|\left\langle x_m^*, \sum_n \beta_n x_n \right\rangle\right| &= |\beta_m| = \frac{1}{1-\delta_m}(1-\delta_m)|\beta_m| \\ &\leq (1+\epsilon_m)\sup_n (1-\delta_n)|\beta_n| \leq (1+\epsilon_m)\left\| \sum_n \beta_n x_n \right\|.\end{aligned}$$

This implies $\|x_m^*\|_{X_0^*} \leq 1+\epsilon_m$. To prove (5), for any $\sum_n \alpha_n x_n^* \in X_0^*$, we set $\beta_n = (1+\delta_n)^{-1}\operatorname{sign}\alpha_n$ $(n \in \mathbb{N})$. For any $m \in \mathbb{N}$, by (4),

$$\left\| \sum_{n=1}^m \beta_n x_n \right\| \leq \sup_n (1+\delta_n)|\beta_n| = 1\,.$$

Whence, by the choice of (β_n) and (δ_n),

$$\begin{aligned}\left\| \sum_n \alpha_n x_n^* \right\|_{X_0^*} &\geq \left\langle \sum_n \alpha_n x_n^*, \sum_{n=1}^m \beta_n x_n \right\rangle = \sum_{n=1}^m \alpha_n \beta_n \\ &= \sum_{n=1}^m \frac{1}{1+\delta_n}|\alpha_n| \geq \sum_{n=1}^m (1-\epsilon_n)|\alpha_n|\end{aligned}$$

which implies (5) since $m \in \mathbb{N}$ is arbitrary. □

Theorem 5

If a Banach space X contains an asymptotically isometric copy of c_0, then X^* contains an asymptotically isometric copy of l_1.

Proof. For any $\epsilon_n \downarrow 0$ $(0 < \epsilon_n < 1)$, let (δ_n) satisfy $\frac{1-\delta_n}{1+\delta_n} = 1 - \epsilon_n$. Then $\delta_n \downarrow 0$ $(0 < \delta_n \le 1)$. By Definition 3, there exists a norm-one sequence (x_n) in X satisfying (4). By (4), $\sum_n \alpha_n x_n \in X_0 = \text{span}\{x_n\}$ if and only if $(\alpha_n) \in c_0$. For each $m \in \mathbb{N}$ and any $(\alpha_n) \in c_0$ with $\alpha_m = -1$, by (4),

$$\left\| x_m - \sum_{n \ne m} \alpha_n x_n \right\| = \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| \ge (1-\delta_m)|\alpha_m| = 1 - \delta_m$$

which implies that $dist(x_m, \ \text{span}\{x_n\}_{n \ne m}) \ge 1 - \delta_m$. Whence by Hahn-Banach Theorem, for each $m \in \mathbb{N}$, there exists $x_m^* \in S(X^*)$ such that

$$< x_m^*, x_m > \ge 1 - \delta_m \quad \text{and} \quad < x_m^*, x_n > = 0, \quad (n \ne m)\,.$$

For any $(\beta_n) \in l_1$, let $\alpha_n = (1+\delta_n)^{-1} \text{sign}\, \beta_n$ $(n \in \mathbb{N})$. Then for any $m \in \mathbb{N}$, by (4),

$$\left\| \sum_{n=1}^{m} \alpha_n x_n \right\| \le \sup_n |\alpha_n|(1+\delta_n) = 1\,.$$

Whence,

$$\begin{aligned} \left\| \sum_n \beta_n x_n^* \right\| &\ge \Big\langle \sum_n \beta_n x_n^*, \sum_{n=1}^{m} \alpha_n x_n \Big\rangle = \sum_{n=1}^{m} \beta_n \alpha_n < x_n^*, x_n > \\ &\ge \Big\langle \sum_{n=1}^{m} |\beta_n| \frac{1}{1+\delta_n}(1-\delta_n) \Big\rangle = \sum_{n=1}^{m} (1-\epsilon_n)|\beta_n|\,. \end{aligned}$$

Since $m \in \mathbb{N}$ is arbitrary, we have

$$\left\| \sum_n \beta_n x_n^* \right\| \ge \sum_{n=1}^{\infty} (1-\epsilon_n)|\beta_n|$$

which shows that X^* has an asymptotically isometric copy of l_1. □

Finally, we discuss the dual action of asymptotically isometric copy of $l_p\,(1 < p < \infty)$.

DEFINITION 6. We say that a Banach space X contains an asymptotically isometric copy of $l_p\,(1 < p < \infty)$, if for any $\delta_n \downarrow 0$ $(0 < \delta_n \le 1)$, X contains a norm-one sequence (x_n) such that

$$\begin{aligned} \left(\sum_n (1-\delta_n)^p |\alpha_n|^p \right)^{1/p} &\le \left\| \sum_n \alpha_n x_n \right\| \\ &\le \left(\sum_n (1+\delta_n)^p |\alpha_n|^p \right)^{1/p}, \ (\alpha_n) \in l_p\,. \qquad (6) \end{aligned}$$

Theorem 7

A Banach space X contains an asymptotically isometric copy of l_p $(1 < p < \infty)$ if and only if for any $\epsilon_n \downarrow 0$ $(0 < \epsilon_n < 1)$, X contains a subspace X_0 such that X_0^* has a normalized basis $\{x_n^*\}$ satisfying

$$\Big(\sum_n (1-\epsilon_n)^q |\beta_n|^q\Big)^{1/q} \le \Big\|\sum_n \beta_n x_n^*\Big\|_{X_0^*} \le \Big(\sum_n (1+\epsilon_n)^q |\beta_n|^q\Big)^{1/q}, \quad (\beta_n) \in l_q \quad (7)$$

where $1/p + 1/q = 1$.

Proof. Necessity. Let $\epsilon_n \downarrow 0$ $(0 < \epsilon_n < 1)$. Set

$$\delta_n = \frac{(1-\epsilon_1)^{q-1}\epsilon_n}{1+2^{p-1}} \le \frac{\epsilon_n}{1+\epsilon_n}\,.$$

Then $\delta_n \downarrow 0$ $(0 < \delta_n \le 1)$. By Definition 6, X contains a norm-one sequence (x_n) satisfying (6). Let $X_0 = \mathrm{span}\{x_n\}$. By (6), X_0 is isomorphic to l_p with the mapping induced by $x_n \longmapsto e_n$ $(n = 1, 2, ...)$, where $\{e_n\}$ is the natural basis of l_p. Therefore, X_0 is reflexive and $\sum_n \alpha_n x_n \in X_0$ if and only if $(\alpha_n) \in l_p$. For any fixed $m \in \mathbb{N}$ and any $(\alpha_n) \in l_p$ with $\alpha_m = -1$, by (6),

$$\Big\|x_m - \sum_{n\ne m} \alpha_n x_n\Big\| = \Big\|\sum_{n=1}^{\infty} \alpha_n x_n\Big\| \ge \Big(\sum_n (1-\delta_n)^p |\alpha_n|^p\Big)^{1/p} \ge 1 - \delta_m.$$

This implies that $dist(x_m,\ \mathrm{span}\{x_n\}_{n\ne m}) \ge 1 - \delta_m$. By Hahn-Banach Theorem, for each $m \in \mathbb{N}$, there exists $x_m^* \in S(X^*)$ such that

$$< x_m^*, x_m > \ge 1 - \delta_m \quad \text{and} \quad < x_m^*, x_n > = 0, \ \ (n \ne m)\,.$$

Since $\{x_m^*\}$ is orthogonal to $\{x_n\}$ and the reflexivity of X_0 implies that $\{x_n\}$ is a shrinking basis, $\{x_m^*\}$ in fact is a basis of X_0^*. Let $(\beta_n) \in l_q$, it remains to show (7).

First, for any $(\alpha_n) \in l_p$, by (6) and the choice of (δ_n),

$$\begin{aligned}\Big\langle \sum_n \beta_n x_n^*, \sum_n \alpha_n x_n \Big\rangle &= \sum_n \alpha_n\beta_n < x_n^*, x_n > \le \sum_n |\alpha_n\beta_n| \\ &= \sum_n (1-\delta_n)|\alpha_n| \frac{1}{1-\delta_n} |\beta_n| \\ &\le \Big(\sum_n (1-\delta_n)^p |\alpha_n|^p\Big)^{1/p} \Big(\sum_n \big(\frac{1}{1-\delta_n}\big)^q |\beta_n|^q\Big)^{1/q} \\ &\le \Big\|\sum_n \alpha_n x_n\Big\| \Big(\sum_n (1+\epsilon_n)^q |\beta_n|^q\Big)^{1/q}\end{aligned}$$

which implies

$$\Big\|\sum_n \beta_n x_n^*\Big\|_{X_0^*} \le \Big(\sum_n (1+\epsilon_n)^q |\beta_n|^q\Big)^{1/q}.$$

To prove the other part of (7), we denote $\alpha_n = |\beta_n|^{q-1}\operatorname{sign}\beta_n$. Then,

$$\Big\langle \sum_n \beta_n x_n^*, \sum_n \alpha_n x_n \Big\rangle = \sum_n |\beta_n||\beta_n|^{q-1} < x_n^*, x_n > \ge \sum_n |\beta_n|^q (1-\delta_n).$$

We shall show that

$$\sum_n |\beta_n|^q (1-\delta_n) \ge \Big(\sum_n (1+\delta_n)^p |\alpha_n|^p\Big)^{1/p} \Big(\sum_n (1-\epsilon_n)^q |\beta_n|^q\Big)^{1/q}. \tag{8}$$

Then by (6),

$$\Big\langle \sum_n \beta_n x_n^*, \sum_n \alpha_n x_n \Big\rangle \ge \Big\|\sum_n \alpha_n x_n\Big\| \Big(\sum_n (1-\epsilon_n)^q |\beta_n|^q\Big)^{1/q}$$

which implies

$$\Big\|\sum_n \beta_n x_n^*\Big\|_{X_0^*} \ge \Big(\sum_n (1-\epsilon_n)^q |\beta_n|^q\Big)^{1/q}$$

completing the proof of the necessity.

Since $(\alpha_n) \in l_p$ and $(\beta_n) \in l_q$, to show (8), it suffices to show that

$$\sum_{n=1}^k |\beta_n|^q (1-\delta_n) \ge \Big(\sum_{n=1}^k (1+\delta_n)^p |\alpha_n|^p\Big)^{1/p} \Big(\sum_{n=1}^k (1-\epsilon_n)^q |\beta_n|^q\Big)^{1/q} \tag{9}$$

holds for each $k \in \mathbb{N}$.

Denote $d_{kn} = |\beta_n|^q / \sum_{m=1}^k |\beta_m|^q$, then $d_{kn} \ge 0$ and $\sum_{n=1}^k d_{kn} = 1$. Notice that $|\alpha_n|^p = |\beta_n|^{(q-1)p} = |\beta_n|^q$, divided by $\sum_{m=1}^k |\beta_m|^q$, (9) becomes

$$1 - \sum_{n=1}^k d_{kn}\delta_n \ge \Big(\sum_{n=1}^k (1+\delta_n)^p d_{kn}\Big)^{1/p} \Big(\sum_{n=1}^k (1-\epsilon_n)^q d_{kn}\Big)^{1/q} := D_k. \tag{10}$$

By mean value theorem,

$$(1+\delta_n)^p = 1 + \xi_n \delta_n \quad \text{and} \quad (1-\epsilon_n)^q = 1 - \eta_n \epsilon_n$$

where

$$\xi_n = p(1+x')^{p-1} \le p2^{p-1} \quad \text{and} \quad -\eta_n = -q(1-x'')^{q-1} \le -q(1-\epsilon_1)^{q-1}$$

for some $x' \in (0,\,1)$ and $x'' \in (0,\,\epsilon_1)$. Whence,

$$\begin{aligned} D_k &= \left(\sum_{n=1}^{k}(1+\xi_n\delta_n)d_{kn}\right)^{1/p}\left(\sum_{n=1}^{k}(1-\eta_n\epsilon_n)d_{kn}\right)^{1/q} \\ &= \left(1+\sum_{n=1}^{k}\xi_n d_{kn}\delta_n\right)^{1/p}\left(1-\sum_{n=1}^{k}\eta_n d_{kn}\epsilon_n\right)^{1/q} \\ &\le \left(1+\frac{1}{p}\sum_{n=1}^{k}\xi_n d_{kn}\delta_n\right)\left(1-\frac{1}{q}\sum_{n=1}^{k}\eta_n d_{kn}\epsilon_n\right) \\ &\le 1+\frac{1}{p}\sum_{n=1}^{k}\xi_n d_{kn}\delta_n - \frac{1}{q}\sum_{n=1}^{k}\eta_n d_{kn}\epsilon_n. \end{aligned}$$

Whence,

$$\begin{aligned} D_k - \left(1-\sum_{n=1}^{k}d_{kn}\delta_n\right) &\le \sum_{n=1}^{k}\left[\left(\frac{1}{p}\xi_n+1\right)\delta_n - \frac{1}{q}\eta_n\epsilon_n\right]d_{kn} \\ &\le \sum_{n=1}^{k}\left[(2^{p-1}+1)\delta_n-(1-\epsilon_1)^{q-1}\epsilon_n\right]d_{kn}=0\,. \end{aligned}$$

This verifies (10).

Sufficiency. For any $\delta_n \downarrow 0$ $(0<\delta_n\le 1)$, let $\epsilon_n = \frac{(1-\delta_1)^{p-1}\delta_n}{1+2^{q-1}}$. Then $\epsilon_n \downarrow 0$ $(0<\epsilon_n<1)$. By assumption, X contains a subspace X_0 such that X_0^* has a normalized basis $\{x_n^*\}$ satisfying (7). Therefore, X_0^* is reflexive and hence so is X_0. Exactly as in the proof of the necessary part, we can find a normalized basis $\{x_n\}$ of $X_0^{**}=X_0$ such that (6) holds for all $(\alpha_n)\in l_p$. □

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