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Nonlinear operators of integral type in some function spaces

C. Bardaro and G. Vinti

Dipartimento di Matematica, Universitá degli Studi, Via Vanvitelli 1, 06123 Perugia, Italy

E-mail: mategian@unipg.it

J. Musielak*

Faculty of Mathematics and Computer Science, Adam Mickiewicz University,

Matejki 48/49, 60-769 Poznań, Poland

Abstract

We give results about embeddings, approximation and convergence theorems for a class of general nonlinear operators of "integral type" in abstract modular function spaces. Thus we extend some previous result on the matter.

1. Introduction

In [5] there was investigated the problem of approximation of functions $f : G \to \overline{\mathbb{R}} = [-\infty, +\infty]$ over an abelian, locally compact topological group, belonging to some modular function spaces, by means of nonlinear integral operators

$$\big(T_w f\big)(t) = \int_G K_w\big(s, f(s+t)\big)\, ds,\ \ w \in \mathcal{W},$$

in the sense of modular convergence, where the kernel functions K_w satisfy the Lipschitz condition. The integration here was meant in the sense of Haar measure

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on G. This problem was treated in [1], in case of modulars in generalized Orlicz spaces, but applying a generalized Lipschitz condition to the family $(K_w)_{w\in\mathcal{W}}$.

Recently, we defined general moduli of continuity in case when G does not need to be a group, but it is closed under some operation, (see [2]). Going further in this direction, we shall give here results concerning embeddings, approximation and convergence at this degree of generality (see also [6]), and we shall show some new examples. In particular, our results generalize those of [4, 5].

2. Nonlinear operators of integral type

First, we recall some notation from [2]. We take a non empty set G with an operation "$\cdot$" : $G\times G\to G$ and we suppose $\mathcal{U}$ to be a filter of subsets of G with a basis $\mathcal{U}_o$. Moreover let μ be a σ-additive and σ-finite measure on a σ-algebra Σ of subsets of G. By $L^0(G)$ we denote the space of extended real valued functions on G, Σ-measurable and finite μ-a.e. with equality μ-a.e., and by $M^0(G)$ we denote the space of all Σ-measurable, extended real functions on G. Let

$$A_t^l = \{s\in G : ts\in A, s\notin A \text{ or } ts\notin A, s\in A\}$$

$$A_t^r = \{s\in G : st\in A, s\notin A, \text{ or } st\notin A, s\in A\},$$

for any $A\subset G$ and $t\in G$. The system $\mathcal{G}=\{G,\mathcal{U},\Sigma,\mu\}$ is called **left-correctly filtered** [resp. **right-correctly filtered**], if the following conditions are satisfied:

1. $\mathcal{U}$ contains a basis $\mathcal{U}_o\subset\Sigma$;
2. if $A\in\Sigma$ and $\mu(A)<+\infty$ then $A_t^l\in\Sigma$, [resp. $A_t^r\in\Sigma$] for every $t\in G$ and $\mu(A_t^l)\xrightarrow{\mathcal{U}}0$ $\big[$resp. $\mu\big(A_t^r\xrightarrow{\mathcal{U}}0\big)\big]$;
3. if $f\in L^0(G)$ then $f(t\cdot)\in L^0(G)$ and $f(\cdot t)\in L^0(G)$, for all $t\in G$.

Now, let χ_A be the characteristic function of a set $A\subset G$. We say that a linear subspace $\mathcal{F}\subset L^0(G)$ is a **correct subspace** of $L^0(G)$ if:

1. $A\in\Sigma$ and $\mu(A)<+\infty$ imply $\chi_A\in\mathcal{F}$;
2. $f\in\mathcal{F}$ and $A\in\Sigma$ imply $f\chi_A\in\mathcal{F}$.

It is easily seen that if $\mathcal{F}\subset L^0(G)$ and $f\in\mathcal{F}$ then $|f|\in\mathcal{F}$. Moreover, $L^0(G)$ is always a correct subspace of itself.

A linear subspace $\mathcal{F}\subset L^0(G)$ will be called **left-translation invariant** (l-τ-invariant) [resp. **right-translation invariant** (r-τ-invariant)] if $f\in\mathcal{F}$ implies $f(t\cdot)\in\mathcal{F}$ [resp. $f\in\mathcal{F}$ implies $f(\cdot t)\in\mathcal{F}$] for every $t\in G$. If $\mathcal{F}$ is both l-τ-invariant and r-τ-invariant, it will be called **translation invariant** (τ-invariant).

Suppose that $\mathcal{F}$ is a correct translation invariant subspace of $L^0(G)$. Let $\| \ \|$ be a monotone extended seminorm on $\mathcal{F}$, i.e. for all $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ there holds: $0 \le \|f\| \le +\infty$, $\|f+g\| \le \|f\| + \|g\|$, $\|cf\| = |c| \, \|f\|$, $|f| \le |g|$ implies $\|f\| \le \|g\|$. Then $\{\mathcal{F}, \| \ \|\}$ will be called a **seminormed subspace** of $L^0(G)$. Let $\mathcal{S} : \mathcal{F} \to \mathbb{R}$ be such that for all $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ there holds: $\mathcal{S}\,(f+g) \le \mathcal{S}f + \mathcal{S}g$, $|\mathcal{S}(cf)| = |c| \, |\mathcal{S}f|$ and $|f| \le |g|$ implies $\mathcal{S}|f| \le \mathcal{S}|g|$. The functional $\mathcal{S}$ will be called then a **functional of integral type** on $\mathcal{F}$. Let us remark that we have $\mathcal{S}|f| \ge 0$ and $|\mathcal{S}f| \le \mathcal{S}|f|$ for all $f \in \mathcal{F}$.

A functional $\mathcal{S}$ of integral type on a seminormed subspace $\{\mathcal{F}, \| \ \|\}$ of $L^0(G)$ is called continuous at $f = 0$ if the following property holds: for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $f \in \mathcal{F}$ with $\|f\| < \delta$ there holds $|\mathcal{S}f| < \varepsilon$; it is easily seen that this condition implies also the continuity at every $f \in \mathcal{F}$, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $g \in \mathcal{F}$ with $\|f - g\| < \delta$ there holds $|\mathcal{S}f - \mathcal{S}g| < \varepsilon$. It is also easily seen that for a continuous functional $\mathcal{S}$ of integral type, the set $\{|\mathcal{S}f| : f \in \mathcal{F}, \ \|f\| = 1\}$ is bounded. The **norm** of $\mathcal{S}$ will be defined then by

$$\|\mathcal{S}\| = \sup \left\{ |\mathcal{S}f| : f \in \mathcal{F}, \ \|f\| = 1 \right\}.$$

Obviously we have also

$$\|\mathcal{S}\| = \sup \left\{ \frac{|\mathcal{S}f|}{\|f\|} : f \in \mathcal{F}, \ f \ne 0 \right\}, \quad \text{and} \quad |\mathcal{S}f| \le \|\mathcal{S}\| \, \|f\|.$$

We recall now the notion of an (L, ψ)-Lipschitz kernel.

Let $\{\mathcal{F}, \| \ \|\}$ be a seminormed subspace of $L^0(G)$ and let $L : G \to \mathbb{R}_0^+ = [0, +\infty[$ be such that $L \in \mathcal{F}$ and $0 < \|L\| < +\infty$. We shall use also the notation $p(t) = L(t)/\|L\|$; evidently, $\|p\| = 1$. Moreover let $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function such that $\psi(\cdot, u)$ is a Σ-measurable function for all $u \ge 0$ and $\psi(t, 0) = 0$, $\psi(t, u) > 0$ for $u > 0$, $\psi(t, u)$ is a nondecreasing and continuous function of $u \ge 0$, for every $t \in G$. The class of all such functions ψ will be denoted by Ψ.

A function $K : G \times \mathbb{R} \to \mathbb{R}$ is called a **kernel function** if $K(\cdot, u) \in \mathcal{F}$ with $\|K(\cdot, u)\| < +\infty$, for all $u \ge 0$ and $K(t, 0) = 0$ for each $t \in G$. A kernel function is called (L, ψ)-**Lipschitz** if

$$\big| K(t, u) - K(t, v) \big| \le L(t) \; \psi(t, |u - v|)$$

for all $t \in G$ and $u, v \in \mathbb{R}$. Taking $v = 0$ we get the inequality $|K(t, u)| \le L(t) \; \psi(t, |u|)$ for all $t \in G$, $u \in \mathbb{R}$. It is easily seen that if $\mathcal{F}$ is a correct left-translation invariant [resp. right-translation invariant] subspace of $L^0(G)$, $f \in \mathcal{F}$ and K is an

(L,ψ)-Lipschitz kernel, then for every $t \in G$ the function $K\big(\cdot, f(t\cdot)\big) : G \to \mathbb{R}$ [resp. the function $K\big(\cdot, f(\cdot t)\big) : G \to \mathbb{R}$] is Σ-measurable and belongs to $L^0(G)$ (see [1]).

Let $\{\mathcal{F}, \|\ \|\}$ be a seminormed correct, left-translation invariant [resp. right-translation invariant] subspace of $L^0(G)$ and let $\mathcal{S} : \mathcal{F} \to \mathbb{R}$ be a functional of integral type on $\mathcal{F}$. Let K be a kernel function. We define operators $T^l : \operatorname{Dom} T^l \to M^0(G)$ and $T^r : \operatorname{Dom} T^r \to M^0(G)$ by formulae

$$\big(T^l f\big)(t) = \mathcal{S}K\big(\cdot, f(t\cdot)\big), \quad \big(T^r f\big)(t) = \mathcal{S}K\big(\cdot, f(\cdot t)\big) \tag{1}$$

for μ a.e. $t \in G$, where $\operatorname{Dom} T^l$ [resp. Dom T^r] is the set of functions $f \in L^0(G)$ such that $K\big(\cdot, f(t\cdot)\big) \in \mathcal{F}$ [resp. $K\big(\cdot, f(\cdot t)\big) \in \mathcal{F}$] and that the function $T^l f : G \to \mathbb{R}$ [resp. $T^r f : G \to \mathbb{R}$] is Σ-measurable. If all constant functions on G belong to $\operatorname{Dom} T^l$ [resp. $\operatorname{Dom} T^r$], we call T^l [resp. T^r] a **nonlinear operator of integral type**. In the following T° will mean always any of the operators T^l and T^r.

EXAMPLE: Let T° be a nonlinear operator of integral type. Suppose f to be a constant function, i.e. $f(s) = C$ for all $s \in G$. Then $f \in \operatorname{Dom} T^\circ$ and

$$\big(T^\circ C\big)(t) = \mathcal{S}K\big(\cdot, C\big)$$

for $t \in G$. In particular taking $f \in \operatorname{Dom} T^\circ$ arbitrarily and choosing a fixed $t_o \in G$ such that $|f(t_o)| < +\infty$, we have $(T^\circ f(t_o))(t) = \mathcal{S}K\big(\cdot, f(t_o)\big)$ for $t \in G$. Taking $t = t_o$, we get

$$\big(T^\circ f(t_o)\big)\ (t_o) = \mathcal{S}K\big(\cdot, f(t_o)\big). \tag{2}$$

Thus, if $f \in \operatorname{Dom} T^\circ \cap L^0(G)$ then (2) holds for μ-a.e. $t \in G$.

3. Embedding theorem

In order to study problems of embeddings and of approximation by means of nonlinear operators of integral type, we have to recall some notions concerning modulars and modular spaces in a correct subspace $\mathcal{F}$ of $L^0(G)$ (see [2, 5]).

A **modular** $\eta : \mathcal{F} \to \overline{\mathbb{R}_0^+} = [0, +\infty]$ is a functional satisfying 1° $\eta(f) = 0$ iff $f = 0$, 2° $\eta(-f) = \eta(f)$, 3° $\eta(\alpha f + \beta g) \le \eta(f) + \eta(g)$, for $\alpha, \beta \ge 0, \alpha + \beta = 1$, $f, g \in \mathcal{F}$. A **modular space** generated by η is defined by $\mathcal{F}_\eta = \{f \in \mathcal{F} : \eta(\lambda f) \to 0$ as $\lambda \to 0^+\}$. η is called **monotone** if $f, g \in \mathcal{F}$ and $|f| \le |g|$ imply $\eta(f) \le \eta(g)$; η is called **finite** if $A \in \Sigma$ and $\mu(A) < +\infty$ imply $\chi_A \in \mathcal{F}_\eta$; η is called **absolutely finite** if it is finite and if for every $\varepsilon > 0$ and every $\lambda_o > 0$ there exists a $\delta > 0$ such that every set $B \in \Sigma$ with $\mu(B) < \delta$ satisfies $\eta(\lambda_o\, \chi_B) < \varepsilon$; η is called **absolutely**

continuous if there exists an $\alpha > 0$ such that for any $f \in \mathcal{F}$ with $\eta(f) < +\infty$, there hold the following conditions: 1. for every $\varepsilon > 0$ there exists a set $A \in \Sigma$ with $\mu(A) < +\infty$ such that $\eta(\alpha f \chi_{G \setminus A}) < \varepsilon$, 2. for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $B \in \Sigma$ with $\mu(B) < \delta$, then $\eta(\alpha f \chi_B) < \varepsilon$.

Supposing $\mathcal{F}$ to be left-translation invariant [resp. right-translation invariant] we say that η is **left-τ-bounded** [resp. **right-τ-bounded**] if there exist a number $c \geq 1$ and $h : G \to \overline{\mathbb{R}_0^+}$ such that $h \in M^0(G)$, with $h(t) \xrightarrow{\mathcal{U}} 0$ such that for every $f \in \mathcal{F}$ with $\eta(f) < +\infty$ there holds the inequality $\eta(f(t\cdot)) \leq \eta(cf) + h(t)$ [resp. $\eta(f(\cdot t)) \leq \eta(cf) + h(t)$] for all $t \in G$. Let η and ρ be two monotone modulars on $\mathcal{F}$, and let the function $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined as before. We say that the triple $\{\rho, \psi, \eta\}$ is **properly directed**, if there is a set $G_o \in \Sigma$ with $\mu(G \setminus G_o) = 0$ such that for every $\lambda \in]0,1]$ there exists a $C_\lambda \in]0,1]$ such that

$$\rho[C_\lambda \ \psi(t, f(:))] \leq \eta[\lambda \ f(:)]$$

for all $t \in G_o$ and $f \in L^0(G)$. Here " : " means the variable concerning the modulars. If $(F_t(:))_{t \in G}$ is a family of functions $F_t \in L^0(G)$, then the above condition implies the inequality $\rho\big[C_\lambda \ \psi(t, F_t(:))\big] \leq \eta\big[\lambda \ F_t(:)\big]$ for every $t \in G_o$. One may always suppose C_λ to be a nondecreasing function of $\lambda \in]0,1]$. As regard the above relation, see [1].

Let $\| \ \|$ be a monotone, extended seminorm on $L^0(G)$. We shall say that a modular ρ on $L^0(G)$ is **norm convex**, if for every Σ-measurable function $p : G \to \mathbb{R}_0^+$ such that $p \in L^0(G)$ and $\|p\| = 1$ and for every function $F : G \times G \to \mathbb{R}$ measurable on $G \times G$, there holds the inequality

$$\rho\big[\|p(\cdot) \ F(\cdot, :)\|\big] \leq \big\|p(\cdot) \ \rho(F(\cdot, :))\big\|;$$

here " $\cdot$ " means the variable concerning the seminorm $\| \ \|$ and as before " : " denotes the variable concerning the modular ρ.

Now, we are able to formulate the following embedding theorem:

Theorem 1

Let $\mathcal{U}_o \subset \Sigma$ a basis of $\mathcal{U}$, let the space $L^0(G)$ be translation invariant and let $\| \ \|$ be a monotone, extended seminorm on $L^0(G)$. Let $\psi \in \Psi$ and let η, ρ be two monotone modulars on $L^0(G)$ such that η is right-τ-bounded [resp. left-τ-bounded], ρ is norm convex and the triple $\{\rho, \psi, \eta\}$ is properly directed. Let $K : G \times \mathbb{R} \to \mathbb{R}$ be an (L, ψ)-Lipschitz kernel function and denote $p(t) = L(t)/\|L\|$ for $t \in G$. Let $\mathcal{S}$ be a functional of integral type with $\|\mathcal{S}\| \neq 0$ and let T^l [resp. T^r] be a nonlinear operator of integral type defined by (1). Then, supposing the function h from the *definition of τ-boundedness of η to be bounded by a constant $h_o < +\infty$ on G, the following conditions are satisfied:*

(a) *for every $\varepsilon > 0$ there exists a $U \in \mathcal{U} \cap \Sigma$ such that*

$$\rho\left(\frac{C_\lambda}{\|L\| \ \|\mathcal{S}\|} T^\circ f\right) \leq \eta(c\lambda f) + h_o \ \|p\chi_{G\setminus U}\| + \varepsilon \,,$$

(b) $\rho\left(\frac{C_\lambda}{\|L\| \ \|\mathcal{S}\|} T^\circ f\right) \leq \eta(c\lambda f) + h_o$, *for sufficiently small $\lambda \in]0,1[$ and $f \in \mathrm{Dom}\, T^\circ \cap (L^0(G))_\eta$,*

(c) *if moreover ρ is a convex modular or if η is such that $\eta(f(: t)) = \eta(f)$ [resp. $\eta(f(t :)) = \eta(f)$] for all $f \in L^0(G)$ with $\eta(f) < +\infty$ and all $t \in G$, then*

$$T^\circ : \mathrm{Dom}\, T^\circ \cap (L^0(G))_\eta \rightarrow (L^0(G))_\rho \,.$$

Proof. (In case of T^l).

(a) Let $f \in \mathrm{Dom}\, T^l \cap (L^0(G))_\eta$, then $K\big(\cdot, f(t\cdot)\big) \in L^0(G)$ for μ-a.e. $t \in G$. Applying properties of $\mathcal{S}$ and the (L, ψ)-Lipschitz condition for K, we easily obtain

$$\left| \frac{C_\lambda}{\|L\| \ \|\mathcal{S}\|} (T^l f)(t) \right| \leq \big\| p(\cdot) C_\lambda \psi(\cdot, f(t\cdot)) \big\|$$

for all $\lambda \in]0,1[$ and μ-a.e. $t \in G$. Now, we apply monotony and norm convexity of ρ and after that, monotony of η and the fact that $\{\rho, \psi, \eta\}$ is properly directed. This leads to the inequality

$$\rho\left(\frac{C_\lambda}{\|L\| \ \|\mathcal{S}\|} (T^l f)(:)\right) \leq \big\| p(\cdot) \ \eta(\lambda f(: \cdot)) \big\| \,; \tag{3}$$

as before, $\| \ \|$ acts on the variable "$\cdot$" and ρ, η on the variable "$:$". Now, let $\lambda_o \in]0,1[$ be so small that $\eta\ (\lambda_o f) < +\infty$ and let $\lambda \in]0, \lambda_o[$. Then we have, by right-τ-boundedness of η, the inequality $\eta\ (\lambda f(: t)) \leq \eta\ (c\lambda f) + h(t)$ for μ-a.e. $t \in G$. Hence, by monotony of $\| \ \|$ and other its properties, we get for any $U \in \mathcal{U} \cap \Sigma$

$$\|p(\cdot) \ \eta(\lambda f(: \cdot))\| \leq \eta(c\lambda f) + \|p \ h\chi_U\| + h_o \ \|p\chi_{G\setminus U}\| \,. \tag{4}$$

But $h(t) \xrightarrow{\mathcal{U}} 0$, so for any $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $h(t) < \varepsilon$ for $t \in U$. Hence, by monotony of $\| \ \|$, $\|ph\chi_U\| \leq \varepsilon \ \|p\| = \varepsilon$. Thus, by (3) and (4), we obtain (a).

(b) follows from (a) immediately estimating $\|p\ \chi_{G\setminus U}\|$ by $\|p\| = 1$, and then taking $\varepsilon \to 0^+$ at the right-hand side of the inequality in (a).

(c) Suppose ρ to be convex and let $\varepsilon > 0$ be so small that $\varepsilon < 2\ h_o$. We choose $\lambda_1 \in]0, \lambda_o[$ so small that $\eta\ (c\lambda_1 f) < \varepsilon/2$ and that (b) holds. We may take C_λ in such a manner that $C_\lambda \searrow 0$ as $\lambda \searrow 0$. Then we obtain

$$\rho\left(\frac{\varepsilon}{2h_o}\ \frac{C_\lambda}{\|L\|\ \|\mathcal{S}\|}\ T^l f\right) \leq \frac{\varepsilon}{2h_o}\ \rho\left(\frac{C_\lambda}{\|L\|\ \|\mathcal{S}\|}\ T^l f\right) \leq \eta(c\lambda f) + \frac{\varepsilon}{2}$$

for $\lambda \in]0, \lambda_1[$. But $\lambda \searrow 0$ is equivalent to $C_\lambda \searrow 0$, and since $f \in (L^0(G))_\eta$, we may deduce that $\rho(\delta\ T^l f) \to 0$ as $\delta \to 0^+$. Hence $T^l f \in (L^0(G))_\rho$.

Supposing η to satisfy the condition $\eta(f(:t)) = \eta(f)$ for $f \in L^0(G)$ with $\eta(f) < +\infty$ and for $t \in G$, we may take in the definition of left-τ-boundedness of η, $c = 1$ and $h(t) = 0$ identically in G. Thus we get on the right-hand side of the inequality in (b), only the term $\eta(\lambda f)$. This shows that $T^l f \in (L^0(G))_\rho$. $\square$

4. Approximation theorem

We shall estimate the ρ-error of approximation of f by T°, i.e. $\rho[a(T^\circ f - f)]$, for $f \in \operatorname{Dom} T^\circ \cap (L^0(G))_{\eta+\rho}$ and sufficiently small $a > 0$. The following auxiliary notations will be needed (see also [1]):

$$A_k = \{t \in G : |f(t)| > k\}, \quad B_k = \left\{t \in G : |f(t)| < \frac{1}{k}\right\}, \quad C_k = G \setminus (A_k \cup B_k),$$

$$r_k = \sup_{\frac{1}{k} \leq |u| \leq k} \left|\frac{1}{u}\ \mathcal{S}K(\cdot, u) - 1\right|$$

for $k = 1, 2, \ldots$ and

$$r_o = \sup_k r_k = \sup_{u \neq 0} \left|\frac{1}{u}\ \mathcal{S}K(\cdot, u) - 1\right|.$$

We shall need also of the notions of left and right η-moduli of continuity $\omega^l_\eta,\ \omega^r_\eta : L^0(G) \times \mathcal{U} \to [0, +\infty]$, defined by $\omega^l_\eta(f, U) = \sup_{t \in U} \eta(f(t:) - f(:))$ and $\omega^r_\eta(f, U) = \sup_{t \in U} \eta\ (f(:t) - f(:))$. Here in case of ω^l_η we assume $L^0(G)$ to be left-translation invariant, and in case of ω^r_η we assume $L^0(G)$ to be right-translation invariant. For the definitions and properties, see [2].

Theorem 2

Let us suppose that $\mathcal{U}_o \subset \Sigma$. Let the space $L^0(G)$ be translation invariant and let $\| \|$ be a monotone, extended seminorm on $L^0(G)$.

Let $\psi \in \Psi$ and let η, ρ be two monotone modulars on $L^0(G)$ such that η is right τ-bounded [resp. left τ-bounded], ρ is norm convex and the triple $\{\rho, \psi, \eta\}$ is properly directed. Let $K : G \times \mathbb{R} \to \mathbb{R}$ be an (L, ψ)-Lipschitz kernel function and denote $p(t) = L(t)/\|L\|$ for $t \in G$. Let $\mathcal{S}$ be a functional of integral type with $\|\mathcal{S}\| \neq 0$ and let T^l [resp. T^r] be a nonlinear operator of integral type, defined by (1). Suppose the function h from the definition of τ-boundedness of η to be bounded by a constant h_o on G. Let $f \in \mathrm{Dom}\, T^\circ \cap (L^0(G))_{\eta+\rho}$ and let $\lambda \in]0,1[$ and $a > 0$ be so small that $\eta(2c\lambda f) < +\infty$, $\rho(16af) < +\infty$ and $16a\|L\|\ \|\mathcal{S}\| < C_\lambda$. Then for an arbitrary set $A \in \Sigma$ and arbitrary $U \in \mathcal{U} \cap \Sigma$ there holds the inequality

$$\rho\big[a(T^\circ f - f)\big] \leq \omega_\eta^o\big(\lambda f, U\big) + \big[2\eta\ (2c\lambda f) + h_o\big]\ \big\|p\chi_{G\setminus U}\big\| + R_k, \tag{5}$$

where

$$\begin{aligned} R_o &= \rho\big(2ar_o f\big), \\ R_k &= \eta\big(\lambda f \chi_{G\setminus A}\big) + \rho\big(16af\chi_{G\setminus A}\big) + \eta\big(\lambda f\chi_{A\cap A_k}\big) + \rho\big(16af\chi_{A\cap A_k}\big) \\ &\quad + \eta\big(\lambda f\chi_{A\cap B_k}\big) + \rho\big(16af\chi_{A\cap B_k}\big) + \rho\big(8ar_k f\big), \quad \textit{for} \ \ k = 1, 2, \ldots, \end{aligned}$$

with $T^\circ = T^l$ and $\omega_\eta^o = \omega_\eta^r$ [resp. $T^\circ = T^r$ and $\omega_\eta^o = \omega_\eta^l$].

Proof. (In case of T^l). Applying properties of $\mathcal{S}$ we obtain

$$\big|(T^l f)(t) - f(t)\big| \leq \mathcal{S}\big|K\big(\cdot, f(t\cdot)\big) - K(\cdot, f(t))\big| + \big|\mathcal{S}K(\cdot, f(t)) - f(t)\big|.$$

Hence we get, by monotony of ρ,

$$\rho\big[a(T^l f - f)\big] \leq J_1 + J_2,$$

where

$$J_1 = \rho\big[2a\mathcal{S}|K(\cdot, f(:\cdot)) - K(\cdot, f(:))|\big], \quad J_2 = \rho\big[2a|\mathcal{S}K(\cdot, f(:)) - f(:)\big]. \tag{6}$$

(L, ψ)-Lipschitz condition yields

$$\big|K(\cdot, f(:\cdot)) - K(\cdot, f(:))\big| \leq L(\cdot)\ \psi\big(\cdot, |f(:\cdot) - f(:)|\big).$$

Thus, by monotony of ρ and properties of $\mathcal{S}$ we have

$$J_1 \le \rho\big[\|p(\cdot)2a\|L\|\ \|\mathcal{S}\|\ \psi(\cdot, |f(:\cdot) - f(:)|)\|\big]\,.$$

Now we apply the inequality $16a\|L\|\ \|\mathcal{S}\|\ \le C_\lambda$, norm convexity and monotony of ρ, and the fact that the triple $\{\rho, \psi, \eta\}$ is properly directed. Then we obtain

$$J_1 \le \big\|p(\cdot)\eta\big[\lambda(f(:\cdot) - f(:))\big]\big\| \le J_1^1 + J_1^2,$$

where

$$J_1^1 = \big\|p(\cdot)\chi_U(\cdot)\ \eta[\lambda(f(:\cdot) - f(:))]\big\|, \quad J_1^2 = \big\|p(\cdot)\chi_{G\setminus U}(\cdot)\ \eta[\lambda(f(:\cdot) - f(:))]\big\|\,,$$

with an arbitrary $U \in \mathcal{U} \cap \Sigma$. But

$$\eta\big[\lambda(f(:t) - f(:))\big] \le \omega_\eta^r\big(\lambda f, U\big), \quad t \in U.$$

Hence, by monotony of the $\|\ \|$ we get $J_1^1 \le \omega_\eta^r(\lambda f, U)$.

In order to estimate J_1^2, we apply right τ-boundedness of η, with $h(t) \le h_o$, and the monotony of η, obtaining easily the following inequality:

$$\eta\big[\lambda(f(:t) - f(:))\big] \le 2\eta\big(2c\lambda f\big) + h_o.$$

Hence, by monotony of $\|\ \|$, we have $J_1^2 \le \big[2\eta(2c\lambda f) + h_o\big]\big\|p\chi_{G\setminus U}\big\|$. Consequently

$$J_1 \le \omega_\eta^r\big(\lambda f, U\big) + \big[2\eta(2c\lambda f) + h_o\big]\ \big\|p\chi_{G\setminus U}\big\|\,.$$

In order to finish the proof, we have to show that $J_2 \le R_k$ for $k = 0, 1, 2, \ldots$ First we prove this for $k = 0$.

It is easily seen that $|\mathcal{S}K(\cdot, f(t)) - f(t)| \le r_o\ |f(t)|$. Hence we get, by monotony of ρ, the inequality $J_2 \le \rho(2ar_of) = R_o$. Now, let $k \ge 1$. Taking $A \in \Sigma$ arbitrarily, we have

$$G = D_1 \cup D_2 \cup D_3 \cup D_4 \ \text{ with }\ D_1 = G \setminus A,\ D_2 = A \cap A_k,\ D_3 = A \cap B_k,\ D_4 = A \cap C_k,$$

and the sets $D_1, D_2, D_3, D_4 \in \Sigma$ are pairwise disjoint. Hence $J_2 \le J_2^1 + J_2^2 + J_2^3 + J_2^4$, where

$$J_2^i = \rho\big[8a|\mathcal{S}K(\cdot, f(:)) - f(:)|\chi_{D_i}(:)\big], \quad i = 1, 2, 3, 4.$$

Let $P \in \Sigma$ be arbitrary. By (L, ψ)-Lipschitz condition, $|K(t,u)| \leq L(t)\ \psi(t, |u|)$ for $t \in G,\ u \in \mathbb{R}$. Hence $|K(\cdot, f(t)\chi_P(t))| \leq L(\cdot)\ \psi(\cdot, |f(t)|\ \chi_P(t))$ for $t \in G$. Applying monotony of $\mathcal{S}$, we thus easily obtain

$$\begin{aligned}&\left|\,\mathcal{S}K(\cdot, f(t)) - f(t)\right|\ \chi_P(t) \leq \mathcal{S}\left|K(\cdot, f(t))\ \chi_P(t)\right| + |f(t)|\ \chi_P(t)\\ &\leq \left\|p(\cdot)\ \|L\|\ \|\mathcal{S}\|\ \psi(\cdot, |f(t)|\ \chi_P(t))\right\| + |f(t)|\ \chi_P(t)\end{aligned}$$

for all $t \in G$. Hence, applying monotony of ρ, inequality $16a\|L\|\ \|\mathcal{S}\| \leq C_\lambda$, again monotony of ρ, norm convexity of ρ, monotony of $\|\ \|$ and the fact that $\{\rho, \psi, \eta\}$ is properly directed, we obtain

$$\rho\big[8a|\mathcal{S}K(\cdot, f(:)) - f(:)|\chi_P(:)\big] \leq \eta(\lambda f\chi_P) + \rho(16af\chi_P)\,.$$

Applying this inequality for $P = D_1,\ P = D_2$ and $P = D_3$, we get the inequalities

$$J_2^i \leq \eta\big(\lambda f\chi_{D_i}\big) + \rho\big(16af\chi_{D_i}\big), \quad \text{for} \quad i = 1, 2, 3. \tag{7}$$

Now, J_2^4 is estimated analogously as J_2 in case $k = 0$, since supposing $t \in D_4 = A \cap C_k$ we have

$$\frac{1}{k} \leq |f(t)| \leq k \quad \text{and so} \quad \left|\mathcal{S}K(\cdot, f(t)) - f(t)\right| \leq r_k\left|f(t)\right|.$$

Thus, $J_2^4 \leq \rho\ (8ar_kf)$ for $k \geq 1$. This, together with the inequality (7), shows that $J_2 \leq R_k$ for $k \geq 1$. □

5. Convergence theorem

Taking a family $\mathbb{K} = (K_w)_{w\in\mathcal{W}}$, where $\mathcal{W}$ is an abstract set of indices, we shall ask now the question, under what conditions the functions $T_w^\circ f$ approximate f in the sense of the modular ρ, i.e. we shall look for conditions under which $\rho[a(T_w^\circ f - f)]$ tends to zero. In order to make sense to statement of this kind we suppose that there is given a filter $\mathcal{WW}$ of subsets of $\mathcal{W}$ and the above convergence will be understood in the sense of this filter, i.e. $\rho[a(T_w^\circ f - f)]\xrightarrow{\mathcal{WW}}0$.

Here we will assume that the filter $\mathcal{U}$ of subsets of G has a basis $\mathcal{U}_o$ with $\mathcal{U}_o \subset \Sigma$.

A family $\mathbb{K} = (K_w)_{w\in\mathcal{W}}$ of kernel functions is called a **kernel.** Let

$$(T_w^l f)(t) = \mathcal{S}K_w(\cdot, f(t\cdot)),\ (T_w^r f)\ (t) = \mathcal{S}K_w(\cdot, f(\cdot t))\,,\ w \in \mathcal{W}.$$

We denote $\mathbb{T}^l = (T^l_w)_{w\in\mathcal{W}}$ and $\operatorname{Dom}\mathbb{T}^l = \bigcap_{w\in\mathcal{W}} \operatorname{Dom}\mathbb{T}^l$ and similarly for $\mathbb{T}^r$, see also [1]. Let $\mathbb{L} = (L_w)_{w\in\mathcal{W}}$, where $L_w : G \to \mathbb{R}^+_0$, $L \in L^0(G)$ and $L = \sup_{w\in\mathcal{W}}\|L_w\| < +\infty$; we write $p_w = L_w/\|L_w\|$. Moreover let $\psi \in \Psi$. The kernel $\mathbb{K}$ is $(\mathbb{L},\psi)$-**Lipschitz**, if the kernel functions K_w are (L_w,ψ)-Lipschitz for all $w \in \mathcal{W}$. An $(\mathbb{L},\psi)$-Lipschitz kernel $\mathbb{K}$ is called a **singular kernel**, if for every $U \in \mathcal{U}\cap\Sigma$ there hold the following conditions:

$$\left\|p_w \chi_{G\setminus U}\right\| \xrightarrow{\mathcal{W}} 0, \quad r_k(w) = \sup_{\frac{1}{k}\le|u|\le k}\left|\frac{1}{u}\mathcal{S}K_w\,(\cdot,u) - 1\right| \xrightarrow{\mathcal{W}} 0\,,$$

for $k = 1, 2, \ldots$.

A singular kernel is called **strongly singular** if additionally $r_o(w) = \sup_k r_k(w) \xrightarrow{\mathcal{W}} 0$ (for this terminology see [1]). Now we are able to prove the following convergence theorem.

Theorem 3

Let the system $\mathcal{G} = \{G, \mathcal{U}, \Sigma, \mu\}$ be right-correctly filtered [resp. left-correctly filtered] and let the space $L^0(G)$ be translation invariant. Let $\|\;\|$ be a monotone, extended seminorm on $L^0(G)$. Let $\psi \in \Psi$ and let η, ρ be two monotone modulars on $L^0(G)$ such that η is right τ-bounded [resp. left τ-bounded], absolutely finite and absolutely continuous, ρ is norm convex and the triple $\{\rho,\psi,\eta\}$ is properly directed. Let $\mathbb{K} = (K_w)_{w\in\mathcal{W}}$ be a singular $(\mathbb{L},\psi)$-Lipschitz kernel and let at least one of the following two conditions be satisfied:

(a) $\mathbb{K}$ is strongly singular;

(b) ρ is finite and absolutely continuous. Then, for every function $f \in \operatorname{Dom}\mathbb{T}^l \cap (L^0(G))_{\eta+\rho}$ [resp. $f \in \operatorname{Dom}\mathbb{T}^r \cap (L^0(G))_{\eta+\rho}$] there is a constant $a_o > 0$ such that

$$\rho\big[a(T^l_w f - f)\big] \xrightarrow{\mathcal{W}} 0, \quad 0 < a \le a_o \quad [resp.\, \rho\,[a(T^r_w f - f)] \xrightarrow{\mathcal{W}} 0, \quad 0 < a \le a_o]\,.$$

Proof. The proof is similar to that in [1] and therefore we give only its outline here. Due to the assumptions on the system $\mathcal{G}$ and on $L^0(G)$ and η, we have $\omega^r_\eta(\lambda_o f, U) \xrightarrow{\mathcal{U}} 0$ for sufficiently small $\lambda_o > 0$ ([2], Theorem 2). Let us choose an arbitrary $\varepsilon > 0$. Fixing λ_o and $U \in \mathcal{U}_o$ we get $\omega^r_\eta(\lambda_o f, U) < \varepsilon/4$. By singularity of $\mathbb{K}$, we may find a $W_1 \in \mathcal{W}$ such that $J_1 < \varepsilon/2$ for $w \in W_1$, where J_1 is defined by (6).

Supposing (a) we get our result from (5) with $k = 0$ immediately. If we suppose (b) then ρ and η are both absolutely continuous. Since $f \in (L^0(G))_{\eta+\rho}$ we have $\mu(A\cap A_k)\to 0$ as $k\to+\infty$, for any set $A \in \Sigma$ such that $\mu(A) < +\infty$; moreover taken such a set A, we have also $\eta(\lambda f\chi_{A\cap B_k})\to 0$ and $\rho(16af\chi_{A\cap B_k})\to 0$ as $k\to+\infty$. Thus we infer that taking k sufficiently large, the terms in the formula for R_k with the exception of the last one are smaller than $\varepsilon/24$ each of them, if only $\lambda > 0$ and $a > 0$ are sufficiently small. Now keeping k fixed we derive from singularity of $\mathbb{K}$ that $\rho\,(8ar_k(w)f) < \varepsilon/4$ for all $w \in W_2$, where W_2 is a set from $\mathcal{VW}$. This finish the proof. For further details see [1]. $\square$

6. Examples

1. We give first examples of functionals $\mathcal{S}$ of integral type on a seminormed subspace $\{\mathcal{F}, \|\ \|\}$ of $L^0(G)$, where $\mathcal{F}$ is correct and translation invariant. Let $A \in \Sigma$ and let $\|f\|_p = \|f\| = (\int_A\ |f(t)|^p d\mu)^{1/p}$, $1 \le p < +\infty$. Moreover let $g \in L^q(G)$, where $1/p + 1/q = 1$ if $1 < p < +\infty$, and $q = +\infty$ if $p = 1$. Let $A \in \Sigma$ be arbitrary. We take

$$\mathcal{S}f = \left|\int_A\ f(t)g(t)d\mu\right|$$

or we may also take

$$\mathcal{S}f = \int_A\ f(t)g(t)d\mu, \quad \text{supposing} \quad g \ge 0\,.$$

It is easily seen that $\mathcal{S}$ are continuous functionals of integral type over $\mathcal{F} = L^p(G)$.

2. We provide now some examples of (L,ψ)-Lipschitz, singular and strongly singular kernels $\mathbb{K}$ which are not Lipschitz in the usual sense, i.e. with $\psi(t,u) = |u|$, taking $\mathcal{S}f = \int_G\ f(t)d\mu$ and $\|f\| = \|f\|_1$. For a sake of simplicity we put $\mathcal{W} = \mathbb{N} = \{1,2,3,\ldots\}$, $\mathcal{VW} =$ the filter of neighborhoods of $+\infty$ in $\overline{\mathbb{N}} = \mathbb{N}\cup\{+\infty\}$. Let $L_n \in L^1(G)$, for $n \in \mathbb{N}$, $\|L_n\|_1 \to 1$ as $n\to+\infty$ and $\int_{G\setminus U} L_n(t)d\mu(t)\to 0$ as $n\to+\infty$, for every $U \in \mathcal{U}_o$.

Next we define $H_n(u)$ putting for $u \ge 0$,

$$H_n(u) = \left\{\frac{1}{n}\Big(u - \frac{k}{n}\Big)\right\}^{1/2} + \frac{k}{n}, \quad u \in \left[\frac{k}{n}, \frac{k+1}{n}\right], \quad k = 0,1,2,\ldots, \quad n = 1,2\ldots.$$

Then we extend the definition for $u < 0$ on putting $H_n(u) = -H_n(-u)$. We define $K_n(t,u) = L_n(t)H_n(u)$ for $n \in \mathbb{N}$. It is easily to show that $\mathbb{K} = (K_n)_{n\in\mathbb{N}}$ is a kernel satisfying an $(\mathbb{L},\psi)$-Lipschitz condition with $\mathbb{L} = (L_n)_{n\in\mathbb{N}}$ and $\psi(t,u) = \sqrt{|u|}$, but does not satisfy a Lipschitz condition $\psi(t,u) = |u|$. Moreover $\int_G\ K_n(t,u)d\mu(t)$ converges uniformly to u on every interval $[a,b[\subset \mathbb{R}^+$, where $0 < a < b \in \overline{\mathbb{R}}$, as $n\to+\infty$. Hence we deduce the singularity of $\mathbb{K}$ immediately.

Let us remark that in a similar manner one can define also a strongly singular kernel. For example we modify the definition of $H_n(u)$ near the point $u = 0$ in such a way that

$$\left|\frac{H_n(u)}{u} - 1\right| \le \frac{1}{n}, \quad 0 < u \le \frac{1}{n}\,.$$

3. Next, we consider some particular cases of our theory.

(a) Let $G = ([0,1], \cdot)$ with Lebesgue measure m; the operation "$\cdot$" is here the usual multiplication. Then G is a semigroup and m is not invariant. Here, we can consider a net of linear operators of the form

$$(T_r f)(t) = \int_0^1 w_r(s) f(st) ds, \quad r \in \mathbb{R}^+\,,$$

where $(w_r)_{r\in\mathbb{R}^+}$ satisfies the assumptions i) $\int_0^1 w_r(t)dt = 1$ for every $r \in \mathbb{R}^+$, ii) $\lim_{r\to+\infty} \int_0^{1-\delta} w_r(t)dt = 0$, for every $\delta \in]0,1/2[$; we take the family $\mathcal{U}_o = \{[1-\delta, 1] : \delta \in]0,1/2[\}$ as a basis of the filter $\mathcal{U}$. Operators of this form are called **average**, or **moment operators** (see [3]).

Moreover, a general filtered family of nonlinear integral operators becomes now of the form

$$(T_w f)\ (t) = \int_0^1 K_w\big(s, f(st)\big) ds\,.$$

(b) Let G be the semigroup $G = ([0,+\infty[, +)$ be provided with a measure μ defined on the σ-algebra Σ of all Lebesgue measurable subset of $[0,+\infty[$ by means of $\mu(A) = \int_A g(t)dm(t)$, $A \in \Sigma$ where m is the Lebesgue measure and g is a non negative function, locally integrable with respect to m. It is easy to show that, denoting by $\mathcal{U}$ the filter of right neighborhoods of zero, $\mathcal{G} = \{G, \mathcal{U}, \Sigma, \mu\}$ becomes correctly filtered. Now, we may take a filtered family of integral operators

$$(T_w f)\ (t) = \int_0^{+\infty} K_w\big(s, f(t+s)\big) d\mu(s)\,.$$

(c) Let G be the semigroup $G = (\mathbb{N}_o, +)$, $\mathbb{N}_o = \{0,1,2,3,\ldots\}$ with a counting measure.

Here we can consider a filtered family of integral operators of the form

$$(T_w f)_j = \sum_{i=0}^{+\infty} K_{w,i}\ (t_{i+j}), \quad T_w f = \big((T_w f)_j\big)_{j=0}^{+\infty}\,,$$

where f is now the sequence $(t_i)_{i=0}^{+\infty}$. Let $\|(t_i)_{i=0}^{+\infty}\| = \sum_{i=0}^{+\infty}|t_i|$. Take as $\mathcal{U}$ the family of neighborhoods of $+\infty$ in $\overline{\mathbb{N}_o} = \{\mathbb{N}_o, +\infty\}$, i.e. $U \in \mathcal{U}$ when the complement $\mathbb{N}_o \setminus U$ is finite or empty. The sets $U_n = \{n, n+1, n+2, \ldots\}$ where $n = 0, 1, 2, \ldots$ form a basis $\mathcal{U}_o$ of this filter. Then singularity of the kernel $\mathbb{K}$ means that $p_{w,i} \xrightarrow{\mathcal{W}} 0$ for every $i = 0, 1, 2, \ldots$ separately and

$$\sup_{\frac{1}{k} \le |u| \le k} \left| \frac{1}{u} \sum_{i=0}^{+\infty} K_{w,i}(u) - 1 \right| \xrightarrow{\mathcal{W}} 0$$

for every $k = 1, 2, \ldots$. Moreover we have the η-modulus of continuity

$$\omega_\eta\big(f, U_n\big) = \sup_{j \ge n} \eta\big(t_{\cdot+j} - t_\cdot\big)\,.$$

(d) The last example may be generalized as follows. Let $a = (a_i)_{i=0}^{+\infty}$ be a sequence of non negative numbers and let μ be the measure on $\mathbb{N}_o$ defined by $\mu(A) = \sum_{i \in A} a_i$ for every $A \subset \mathbb{N}_o$. Then the family $\mathbb{T} = (T_w)_{w \in \mathcal{W}}$ consists of operators $T_w f = \big((T_w f)_j\big)_{j=0}^{+\infty}$, where $(T_w f)_j = \sum_{i=0}^{+\infty} a_i K_{w,i}(t_{i+j})$.

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