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A modular convergence theorem for certain nonlinear integral operators with homogeneous kernel

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Abstract

Here we give modular estimates for nonlinear integral operators of the form $(Tf)(s) = \int_G K(s,t,f(t))dt$ in Orlicz spaces. Namely we obtain an estimate for the error of approximation $Tf - f$ and as a consequence we state the main result, i.e. $(T_w f - f)\xrightarrow{\mathcal{W}} 0$ with $w \in \mathcal{W}$.

1. Introduction

In previous papers ([10]-[14]), J.Musielak began the study of the modular convergence for the so-called "error of approximation" $T_w f - f$, where T_w is a family of nonlinear operators of convolution type of the form

$$(T_w f)(s) = \int_G K(s-t, f(t))dt$$

with respect to a filter of subsets of a set $\mathcal{W}$; here f belongs to a Orlicz space $L^\varphi(G)$, and G is a compact or a locally compact topological group provided with its Haar measure dt. Moreover in [2] these results has been extended to the case of a general modular function space; for literature on classical linear convolution operators in L^p spaces see [6].

Later, on [1, 4] extensions of the previous results has been obtained when the usual strongly-Lipschitz condition assumed on the kernel K is replaced by the weaker (L,ψ)-Lipschitz condition.

The aim of this paper is to obtain a modular convergence result for a filtered family of operators T_w of the form

$$(T_w f)(s) = \int_G K(s,t,f(t))dt,$$

where G is a locally compact topological group, dt is its Haar measure, $w \in \mathcal{W}$ where $\mathcal{W}$ is a set of indices and K satisfies a strongly-Lipschitz condition.

Here our operators are nonlinear and satisfying some general homogeneity assumptions.

Usually, on estimating the error of approximation for linear or nonlinear integral operators with homogeneous kernels using the approach of the previous papers [11]-[14], [1, 3, 4] an estimate and a convergence result for $T_w f - g$ is obtained, where $g = \eta f$, being η the function that appear in the definition of the subhomogeneity for $L(s,t)$ (see assumption (K.3)).

In this paper we are able to obtain a modular estimate just for $T_w f - f$, using the theory of the Young functions together with the condition $(\overline{\Delta_3})$; as a consequence of this result, we obtain a modular convergence theorem for $T_w f - f$.

A convergence result for the sequence of nonlinear operators with nonhomogeneous kernels is obtained in [5] using a density approach.

In section 2 we give first an estimate of the operator T in the modular sense for f belonging to the Orlicz space $L^\varphi(G)$ and taking the kernel in a certain class $\mathcal{K}$; moreover we give the main estimate for the error of approximation $Tf - f$, where f must be taken in the intersection of three different Orlicz spaces, where one of them is of weighted type.

Then, in section 3 using an extension of the Proposition 1 of [11], in the case of a weighted φ-modulus of smoothness and assuming the usual singularity assumptions, we obtain the requested convergence result, i.e.

$I_{\varphi_1}[\lambda(T_w f - f)] \xrightarrow{\mathcal{VW}} 0$, where $\xrightarrow{\mathcal{VW}}$ denotes the convergence with respect to the filter $\mathcal{VW}$ (see [10]).

2. Notations and definitions

Let G be a locally compact group and let dt be its Haar measure; for a sake of simplicity, we will assume G abelian.

Moreover let $|A|$ denote the Haar measure of a measurable set $A \subset G$ and let $\mathcal{U}$ be the neighborhoods base of the neutral element θ of G.

Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0, +\infty[$, be a function satisfying the following assumption:

i) φ is a convex function, with $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, and $\varphi(u)/u \to +\infty$ as $u \to +\infty$.

If φ satisfies assumption i), we will say that φ is a $\Phi-$function and we will write $\varphi \in \Phi$.

Now, for $\varphi \in \Phi$, on the space $X = \{f : G \to \mathbb{R} : f \text{ is Haar measurable}\}$ we define the modular

$$I_\varphi[f] = \int_G \varphi(|f(t)|)dt$$

and

$$L^\varphi(G) = \{f \in X : I_\varphi[\lambda f] < +\infty \text{ for some } \lambda > 0\}$$

will denote the corresponding Orlicz space (see [10, 9, 11, 3]).

Moreover, given a measurable function $g : G \to \mathbb{R}^+$, for $\varphi \in \Phi$ we define the functional

$$I_g^\varphi : X \to \overline{\mathbb{R}_0^+} = [0, +\infty]$$

by means of the formula

$$I_g^\varphi[f] = \int_G g(t)\varphi(|f(t)|)dt.$$

It is well known that I_g^φ is a modular and the corresponding modular space $L_g^\varphi(G) = \{f \in X : I_g^\varphi[\lambda f] < +\infty \text{ for some } \lambda > 0\}$ will be called the "weighted Orlicz space" generated by φ with weight g.

We recall that a function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a Young function if it satisfies the following assumption:

i$'$) φ is convex with $\varphi(0) = 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$ (see [10, 15]).

Now it is easy to verify that every function $\varphi \in \Phi$ is a Young function. For $\varphi \in \Phi$, we will denote by ψ the complementary function to φ, defined as

$$\psi(y) = \sup\{xy - \varphi(x) : x \in \mathbb{R}_0^+\}, \quad y \in \mathbb{R}_0^+. \tag{1}$$

It results that ψ is a Young function and from (1) we may deduce the Young's inequality for the pair (φ, ψ), i.e.

$$xy \le \varphi(x) + \psi(y), \quad x, y \in \mathbb{R}_0^+. \tag{2}$$

We will need the following condition (Δ_g').

Let $\varphi_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+,\ i = 1,2,3$ *be three functions with* $\varphi_i \in \Phi,\ i = 1,2,3$ *(or Young functions) satisfying the following condition (see [15]):*
there exists $a > 0$ *such that*

$$(\Delta_g^{'}) \qquad \varphi_1(auv) \leq \varphi_2(u)\ \varphi_3(v), \quad \forall u, v \in \mathbb{R}_0^+.$$

Moreover if $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is a function belonging to the class* Φ *(or is a Young function) and if* ψ *denotes the complementary function to* φ*, we say that* φ *is a* $\overline{\Delta_3}$ *function if:*

$$(\overline{\Delta_3}) \qquad (\psi \circ \varphi)(u) \leq \varphi(cu),$$

for some $c > 0$ *and* $u \geq u_o \geq 0$ *(see [15]).*

Remark 1. We observe that if in the condition $\Delta_g^{'}$ we put $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$ and φ is a N-function, i.e. φ is a Young function with $\varphi(x) = 0$ iff $x = 0$ and $\lim_{x\to 0^+} \varphi(x)/x = 0,\ \lim_{x\to+\infty} \varphi(x)/x = +\infty$, then our condition $\Delta_g^{'}$ becomes equivalent to the original $\Delta^{'}$ condition with $x_o = 0$, i.e. $\Delta_g^{'}$ becomes a global $\Delta^{'}$ condition (see [15], pp. 28–29).

Moreover we observe that in the case when (φ, ψ) is a complementary pair of N-functions, our condition $\overline{\Delta_3}$ becomes equivalent to the original Δ_3-condition, i.e. there exists a $b > 0$ such that

$$(\Delta_3) \qquad x\varphi(x) \leq \varphi(bx), \quad x \geq x_o \geq 0.$$

An example of function φ satisfying Δ_3-condition is given by $\varphi(x) = e^{x^2} - 1$. For other examples see [15].

Let now denote by $\mathcal{K}$ the class of all functions $K : G \times G \times \mathbb{R} \to \mathbb{R}$ satisfying the following assumptions:

K.1) $K(\cdot,\cdot,u)$ is measurable on $G \times G$ for every $u \in \mathbb{R}$ and $K(s,t,0) = 0$ for every $(s,t) \in G \times G$.

K.2) $K(s,t,\cdot)$ is strongly Lipschitz, i.e. there exists a globally measurable function $L : G \times G \to \mathbb{R}_0^+$ such that

$$|K(s,t,u) - K(s,t,u+h)| \leq L(s,t)|h|,$$

for every $s,t \in G, u,h \in \mathbb{R}$.

K.3) $L(s,t)$ is $\eta-$subhomogeneous (see [3]), i.e. there exists a measurable function $\eta : G \to \mathbb{R}^+$ such that the following inequality holds:

$$L(s+v, t+v) \leq \eta^{-1}(t)\ \eta(t+v)\ L(s,t)$$

for every $t, s, v \in G$.

As example of property K.3), we may consider the case of $G = (\mathbb{R}^+, \cdot)$ and L homogeneous of degree $\alpha \in \mathbb{R}$; then L is $\eta-$homogeneous with respect to $\eta(t) = t^\alpha$ and the inequality in K.3) becomes an equality.

We now consider the following integral operator T :

$$(Tf)(s) = \int_G K(s,t,f(t))dt, \quad f \in \operatorname{Dom} T$$

where $\operatorname{Dom} T$ is the subset of X on which Tf is well defined as an Haar integral for almost all $s \in G$ and Tf is measurable on G.

3. Modular estimates for Tf and for the error of approximation $Tf - f$

First of all we establish an estimate for Tf which is an extension to classical Orlicz spaces of Theorem 1 of [3] for nonlinear operators.

Theorem 1

Let $K \in \mathcal{K},\ \varphi \in \Phi,\ \lambda > 0$ and let us suppose that

$$0 < D := \int_G \eta^{-1}(z)\, L(\theta, z) dz < +\infty.$$

Then for every $f \in L^\varphi(G) \cap \operatorname{Dom} T$ such that $g := \eta f \in L^\varphi(G)$, it results $Tf \in L^\varphi(G)$ and

$$I_\varphi[\lambda(Tf)] \le I_\varphi[\lambda D\eta\, f]. \tag{3}$$

Proof. By the properties of $K \in \mathcal{K}$, we may write

$$\begin{aligned} I_\varphi[\lambda(Tf)] &= \int_G \varphi\Big(\lambda\Big|\int_G K(s,t,f(t))dt\Big|\Big) ds \\ &\le \int_G \varphi\Big(\lambda \int_G L(s,t)|f(t)|dt\Big) ds = \int_G \varphi\Big(\lambda \int_G L(s,z+s)|f(z+s)|dz\Big) ds \\ &\le \int_G \varphi\Big(\lambda \int_G \eta^{-1}(z)\eta(z+s)L(\theta,z)|f(z+s)|dz\Big) ds. \end{aligned}$$

Now, by Jensen inequality and Fubini-Tonelli's theorem, we have

$$\begin{aligned} I_\varphi[\lambda(Tf)] &\le \frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)\Big\{\int_G \varphi(\lambda D\eta(z+s)|f(z+s)|)ds\Big\} dz \\ &= \frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z) I_\varphi[\lambda D g] dz = I_\varphi[\lambda D\eta f]. \ \square \end{aligned}$$

Remark 2. We remark that the inequality of Theorem 1 states that $Tf \in L^{\varphi}(G)$ whenever $\eta f \in L^{\varphi}(G)$ and in general it is not possible to replace the assumption $\eta f \in L^{\varphi}(G)$ with $f \in L^{\varphi}(G)$ (see [3] for some particular cases). Moreover if $\eta(t) \leq C$, for some $C > 0$, a.e. $t \in G$, the inequality of Theorem 1 implies that $Tf \in L^{\varphi}(G)$ whenever $f \in L^{\varphi}(G)$. This happens for example for homogeneous kernel of degree zero, as in the convolution type operators.

In order to state a modular estimate for the error of approximation $Tf - f$, we will use the following notations:

a) In a weighted Orlicz space $L^{\varphi}_g(G)$ with weight g, we will denote by $\omega_{\varphi}(g, f, U)$ the weighted φ-modulus of smoothness of the function f, for the neighborhood $U \in \mathcal{U}$ of θ, that is we put

$$\omega_{\varphi}(g, f, U) = \sup_{z \in U} \int_G g(t)\varphi[|f(t) - f(t-z)|]dt.$$

b) for $h > 0$, $A^h_z(\mu)$, $A_n, A^h(\mu)$ will denote respectively the sets

$$A^h_z(\mu) = \{t \in G : \mu\ |f(t) - f(t-z)| > h\},$$
$$A_n = \{s \in G : |f(s)| > n\}, \quad n = 1, 2, \ldots,$$
$$A^h(\mu) = \{s \in G : \mu |f(s)| > h\},$$

for a suitable constant $\mu > 0$.

c) $r_n(s) = \sup\limits_{0<|u|\leq n} \left| \frac{1}{u} \int_G K(s, t, u)dt - 1 \right|, \quad s \in G, \quad n = 1, 2, \ldots.$

Now we are ready to formulate the following

Theorem 2

Let $K \in \mathcal{K}$, and let $\varphi_i \in \Phi$, $i = 1, 2, 3$ be the functions satisfying condition Δ'_g and let (φ_3, ψ_3) be a complementary pair of Young functions satisfying condition $\overline{\Delta_3}$. Let moreover $f \in L^{\varphi_1+\varphi_3}(G) \cap L^{\varphi_3}_{\varphi_2 \circ \eta}(G) \cap \operatorname{Dom} T$, $\eta \in L^{(\varphi_3 \circ \varphi_2)+\varphi_2}(G)$ and take $\lambda > 0$ sufficiently small such that $\sqrt{\lambda} < \min\left\{\frac{a}{4D}, 1\right\}$ where a is the constant of Δ'_g. Then for every $U \in \mathcal{U}$, $h \geq u_o$, u_o being the constant from the condition $\overline{\Delta_3}$, and $n = 1, 2, 3, \ldots$, the operator Tf satisfies the inequality

$$\begin{aligned} I_{\varphi_1}[\lambda(Tf - f)] &\leq \frac{1}{2}\,\omega_{\varphi_3}(\varphi_2 \circ \eta, \sqrt[4]{\lambda}\, f, U) \\ &+ \left\{ I_{\varphi_3 \circ \varphi_2}[\sqrt[4]{\lambda}\ \eta] + I_{\varphi_3}[2c\sqrt[4]{\lambda}\, f] + H\ I_{\varphi_2}[\sqrt[4]{\lambda}\,\eta] \right\} \frac{1}{2\,D} \int_{G \setminus U} \eta^{-1}(z)\, L(\theta, z)dz \\ &+ \frac{1}{4\,D} \int_G \eta^{-1}(z)\, L(\theta, z) I_{\varphi_3 \circ \varphi_2}[\sqrt[4]{\lambda}\ \eta(z + \cdot)\chi_{A_n}]dz + \frac{1}{4}\, I_{\varphi_3}[c\sqrt[4]{\lambda}\ f\chi_{A_n}] \qquad (4) \\ &+ \frac{H}{4\,D} \int_G \eta^{-1}(z)\, L(\theta, z) I_{\varphi_2}[\sqrt[4]{\lambda}\ \eta(z + \cdot)\chi_{A_n}]dz + \frac{1}{4}\, I_{\varphi_1}[4\lambda f\chi_{A_n}] + \frac{1}{2}\, I_{\varphi_1}[2\lambda r_n f] \end{aligned}$$

where $D > 0$ *is the constant of Theorem 1 that we suppose to be finite and* $H = \varphi_3[h]$.

Proof. We may suppose that $\lambda > 0$ is so small that $I_{\varphi_1}[4\lambda f] < +\infty$, $I_{\varphi_3}[2c\sqrt[4]{\lambda}\, f] < +\infty$, $I_{\varphi_3\circ\varphi_2}[\sqrt[4]{\lambda}\,\eta] < +\infty$ and $I_{\varphi_2}[\sqrt[4]{\lambda}\,\eta] < +\infty$. Moreover we can assume $r_n(s) < +\infty$ a.e. $s \in G$. Since

$$\begin{aligned} |(Tf)(s) - f(s)| &= \Big|\int_G K(s,t,f(t))dt - f(s)\Big| \\ &\leq \Big|\int_G \{K(s,t,f(t)) - K(s,t,f(s))\}dt\Big| \\ &\quad + \Big|\int_G K(s,t,f(s))dt - f(s)\Big|, \end{aligned}$$

then, by convexity of $\varphi_1 \in \Phi$, we have

$$\begin{aligned} I_{\varphi_1}[\lambda(Tf - f)] &\leq \frac{1}{2}\int_G \varphi_1\Big[2\lambda\int_G L(s,t)|f(t) - f(s)|dt\Big]ds \\ &\quad + \frac{1}{2}\int_G \varphi_1\Big[2\lambda\Big|\int_G K(s,t,f(s))dt - f(s)\Big|\Big]ds \\ &=: J_1 + J_2. \end{aligned}$$

Now we evaluate J_1. Since $K \in \mathcal{K}$ and putting $t = z + s$, it results

$$\begin{aligned} J_1 &= \frac{1}{2}\int_G \varphi_1\Big[2\lambda\int_G L(s,t)|f(t) - f(s)|dt\Big]ds \\ &= \frac{1}{2}\int_G \varphi_1\Big[2\lambda\int_G L(s,z+s)|f(z+s) - f(s)|dz\Big]ds \\ &\leq \frac{1}{2}\int_G \varphi_1\Big[2\lambda\int_G \eta^{-1}(z)\eta(z+s)L(\theta,z)|f(z+s) - f(s)|dz\Big]ds \\ &\leq \frac{1}{2}\int_G \left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)\varphi_1[2\lambda D|f(z+s) - f(s)|\ \eta(z+s)]dz\right\} ds. \end{aligned}$$

Now, since $2\sqrt{\lambda}D \leq a$, by condition $\Delta_g^{'}$, putting in the inner integral $z = t - s$ and applying Fubini-Tonelli's theorem, we may write

$$\begin{aligned} J_1 &\leq \frac{1}{2}\int_G \left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)\varphi_2[\sqrt[4]{\lambda}\,\eta(z+s)]\varphi_3[\sqrt[4]{\lambda}\,|f(z+s) - f(s)|]dz\right\} ds \\ &= \frac{1}{2}\int_G \left\{\frac{1}{D}\int_G \eta^{-1}(t-s)L(\theta,t-s)\varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(s)|]dt\right\} ds \\ &= \frac{1}{2\,D}\int_G \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \left\{\int_G \eta^{-1}(t-s)L(\theta,t-s)\varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(s)|]ds\right\} dt. \end{aligned}$$

Now, putting in the inner integral $s = t - z$ and applying Fubini-Tonelli's theorem, we have

$$\begin{aligned} J_1 &\leq \frac{1}{2\,D} \int_G \varphi_2[\sqrt[4]{\lambda}\,\eta(t)] \left\{ \int_G \eta^{-1}(z)\, L(\theta, z)\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dz \right\} dt \\ &= \frac{1}{2\,D} \int_U \eta^{-1}(z)\, L(\theta, z) \left\{ \int_G \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \right\} dz \\ &\quad + \frac{1}{2\,D} \int_{G\setminus U} \eta^{-1}(z) L(\theta, z) \left\{ \int_G \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \right\} dz \\ &=: J_1^1 + J_1^2. \end{aligned}$$

Now, being $\lambda < 1$, we have $J_1^1 \leq \dfrac{1}{2}\,\omega_{\varphi_3}(\varphi_2 \circ \eta, \sqrt[4]{\lambda}\, f, U)$, while in order to evaluate J_1^2 we first consider the inner integral; let us fix $h \geq u_o$. We have

$$\begin{aligned} &\int_G \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \\ &= \int_{A_z^h(\sqrt[4]{\lambda})} \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \\ &\quad + \int_{G\setminus A_z^h(\sqrt[4]{\lambda})} \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \\ &=: Q_1 + Q_2. \end{aligned}$$

By Young's inequality and condition $\overline{\Delta_3}$ applied to the complementary pair (φ_3, ψ_3), it results

$$\begin{aligned} Q_1 &= \int_{A_z^h(\sqrt[4]{\lambda})} \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \\ &\leq \int_{A_z^h(\sqrt[4]{\lambda})} (\varphi_3 \circ \varphi_2)[\sqrt[4]{\lambda}\,\eta(t)] dt + \int_{A_z^h(\sqrt[4]{\lambda})} (\psi_3 \circ \varphi_3)[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \\ &\leq \int_G (\varphi_3 \circ \varphi_2)[\sqrt[4]{\lambda}\,\eta(t)] dt + \int_G \varphi_3[c\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt. \end{aligned}$$

Moreover

$$\begin{aligned} Q_2 &= \int_{G\setminus A_z^h(\sqrt[4]{\lambda})} \varphi_2[\sqrt[4]{\lambda}\,\eta(t)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(t) - f(t-z)|] dt \\ &\leq H \int_G \varphi_2[\sqrt[4]{\lambda}\,\eta(t)] dt. \end{aligned}$$

Hence

$$\begin{aligned}
J_1^2 &\le \frac{1}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)\left\{\int_G(\varphi_3\circ\varphi_2)[\sqrt[4]{\lambda}\,\eta(t)]dt\right\}dz\\
&\quad+\frac{1}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)\left\{\int_G\varphi_3[c\sqrt[4]{\lambda}\,|f(t)-f(t-z)|]dt\right\}dz\\
&\quad+\frac{H}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)\left\{\int_G\varphi_2[\sqrt[4]{\lambda}\,\eta(t)]dt\right\}dz\\
&=: R_1+R_2+R_3.
\end{aligned}$$

Now,

$$\begin{aligned}
R_1 &= I_{\varphi_3\circ\varphi_2}[\sqrt[4]{\lambda}\,\eta]\;\frac{1}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)dz,\\
R_2 &\le \frac{1}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)\left\{\frac{1}{2}\int_G\varphi_3[2c\sqrt[4]{\lambda}\,|f(t)|]dt\right\}dz\\
&\quad+\frac{1}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)\left\{\frac{1}{2}\int_G\varphi_3[2c\sqrt[4]{\lambda}\,|f(t-z)|]dt\right\}dz\\
&\le I_{\varphi_3}[2c\sqrt[4]{\lambda}\,f]\;\frac{1}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)dz,
\end{aligned}$$

and

$$R_3 = I_{\varphi_2}[\sqrt[4]{\lambda}\,\eta]\;\frac{H}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)dz.$$

Hence

$$J_1^2 \le \left\{I_{\varphi_3\circ\varphi_2}[\sqrt[4]{\lambda}\,\eta]+I_{\varphi_3}[2c\sqrt[4]{\lambda}\,f]+HI_{\varphi_2}[\sqrt[4]{\lambda}\,\eta]\right\}\;\frac{1}{2\,D}\int_{G\setminus U}\eta^{-1}(z)\,L(\theta,z)dz.$$

We now estimate J_2.

$$\begin{aligned}
J_2 &= \frac{1}{2}\int_G\varphi_1\Big[2\lambda\Big|\int_G K(s,t,f(s))dt-f(s)\Big|\Big]ds\\
&\le \frac{1}{2}\int_{A_n}\varphi_1\Big[\frac{1}{2}\,4\lambda\int_G\big|K(s,t,f(s))|dt+\frac{1}{2}\,4\lambda|f(s)\big|\Big]ds\\
&\quad+\frac{1}{2}\int_{G\setminus A_n}\varphi_1\Big[2\lambda\Big|\int_G K(s,t,f(s))dt-f(s)\Big|\Big]ds\\
&=: J_2^1+J_2^2.
\end{aligned}$$

Now, since $K(s,t,0)=0$, we have

$$\begin{aligned} J_2^1 &\le \frac{1}{4}\int_{A_n}\varphi_1\Big[4\lambda\int_G L(s,t)|f(s)|dt\Big]ds + \frac{1}{4}\int_{A_n}\varphi_1[4\lambda|f(s)|]ds \\ &= P_1 + \frac{1}{4}I_{\varphi_1}[4\lambda f\chi_{A_n}]. \end{aligned}$$

Since $4\sqrt{\lambda}D < a$, using condition $\Delta_g^{'}$ and Jensen inequality, we have

$$\begin{aligned} P_1 &\le \frac{1}{4}\int_{A_n}\varphi_1\Big[4\lambda\int_G \eta^{-1}(z)\, L(\theta,z)\eta(z+s)|f(s)|dz\Big]ds \\ &\le \frac{1}{4}\int_{A_n}\left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)\varphi_2[\sqrt[4]{\lambda}\,\eta(z+s)]\ \varphi_3[\sqrt[4]{\lambda}\,|f(s)|]dz\right\}ds. \end{aligned}$$

Now, using Young's inequality with $\varphi=\varphi_3$ and $\psi=\psi_3$ in (2), and applying condition $\overline{\Delta_3}$ and Fubini-Tonelli's theorem, we obtain for $h\ge u_o$,

$$\begin{aligned} P_1 &\le \frac{1}{4}\int_{A_n\cap A^h(\sqrt[4]{\lambda})}\left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)(\varphi_3\circ\varphi_2)[\sqrt[4]{\lambda}\,\eta(z+s)]dz\right\}ds \\ &\quad + \frac{1}{4}\int_{A_n\cap A^h(\sqrt[4]{\lambda})}\left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)\varphi_3[c\sqrt[4]{\lambda}\,|f(s)|]dz\right\}ds \\ &\quad + \frac{1}{4}\int_{A_n\setminus A^h(\sqrt[4]{\lambda})}\left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)\varphi_2[\sqrt[4]{\lambda}\,\eta(z+s)]\ \varphi_3[h]dz\right\}ds \\ &\le \frac{1}{4}\int_{A_n}\left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)(\varphi_3\circ\varphi_2)[\sqrt[4]{\lambda}\,\eta(z+s)]dz\right\}ds \\ &\quad + \frac{1}{4}\int_{A_n}\left\{\frac{1}{D}\int_G \eta^{-1}(z)\, L(\theta,z)\varphi_3[c\sqrt[4]{\lambda}\,|f(s)|]dz\right\}ds \\ &\quad + \frac{H}{4\,D}\int_{A_n}\left\{\int_G \eta^{-1}(z)\, L(\theta,z)\varphi_2[\sqrt[4]{\lambda}\,\eta(z+s)]dz\right\}ds \\ &\le \frac{1}{4\,D}\int_G \eta^{-1}(z)\, L(\theta,z)I_{\varphi_3\circ\varphi_2}[\sqrt[4]{\lambda}\,\eta(z+\cdot)\chi_{A_n}]dz \\ &\quad + \frac{1}{4}I_{\varphi_3}[c\sqrt[4]{\lambda}\,f\chi_{A_n}] + \frac{H}{4\,D}\int_G \eta^{-1}(z)\, L(\theta,z)I_{\varphi_2}[\sqrt[4]{\lambda}\,\eta(z+\cdot)\chi_{A_n}]dz. \end{aligned}$$

Hence, we can conclude that

$$\begin{aligned} J_2^1 &\le \frac{1}{4\,D}\int_G \eta^{-1}(z)\, L(\theta,z)I_{\varphi_3\circ\varphi_2}[\sqrt[4]{\lambda}\,\eta(z+\cdot)\chi_{A_n}]dz \\ &\quad + \frac{1}{4}I_{\varphi_3}[c\sqrt[4]{\lambda}\,f\chi_{A_n}] + \frac{H}{4\,D}\int_G \eta^{-1}(z)\, L(\theta,z)I_{\varphi_2}[\sqrt[4]{\lambda}\,\eta(z+\cdot)\chi_{A_n}]dz \\ &\quad + \frac{1}{4}I_{\varphi_1}[4\lambda f\chi_{A_n}]. \end{aligned}$$

Now, we evaluate J_2^2.

$$J_2^2 = \frac{1}{2}\int_{G\setminus A_n} \varphi_1[2\lambda|\int_G K(s,t,f(s))dt - f(s)|]ds$$
$$\leq \frac{1}{2} I_{\varphi_1}[2\lambda r_n f].$$

Hence the assertion follows □

4. A convergence theorem

Let G be an abelian locally compact topological group. First of all we recall (see [4]) that for every measurable subset $A\subset G$ of finite measure $|A|$, there holds the condition

$$(o) \qquad \lim_{s\to\theta} |A\Delta(A-s)| = 0,$$

where Δ denotes the symmetric difference of sets.

This property follows by using an approximation of the characteristic function of A by means of continuous functions with compact support (see [8]).

Now we may state the following

Proposition 1

Let $\varphi \in \Phi$. Then, for every function $f \in L_g^{\varphi}(G)$ there exists a $\lambda_o \in \mathbb{R}^+$ such that for every $\varepsilon > 0$ there exists a neighborhood $U \in \mathcal{U}$ of θ such that

$$\omega_\varphi(g, \lambda f, U) < \varepsilon, \quad \text{for } 0 < \lambda \leq \lambda_o,$$

where $g : G\to\mathbb{R}^+$ is a weight function.

Proof. Let $\mu : \mathcal{B}(G)\to\mathbb{R}^+$ be defined by $\mu(E) = \int_E g(t)dt$; we may write

$$\omega_\varphi(g, \lambda f, U) = \sup_{z\in U}\int_G \varphi(\lambda|f(t) - f(t-z)|)d\mu(t).$$

Now, the proof follows using (o) and taking into account that when φ does not depend on the parameter t, Theorem 5 of [11] holds also for an abstract measure μ. □

Now, let us take an abstract set of indices and let $\mathcal{WW}$ be a filter of subsets of a set $\mathcal{W}$.

We take a family $\mathbb{K} = (K_w)_{w\in\mathcal{W}}$ of functions $K_w \in \mathcal{K}$ such that

$$|K_w(s,t,u+h) - K(s,t,u)| \leq L_w(s,t)|h|$$

for every $s,t \in G,\ u,h \in \mathbb{R}$ and where $\mathbb{L} = (L_w)_{w\in\mathcal{W}}$ is a family of functions $L_w : G \times G \to \mathbb{R}_0^+,\ \eta-$subhomogeneous. We shall denote this by $\boldsymbol{\mathcal{K}}$. The corresponding family of operators $\mathbb{T} = (T_w)_{w\in\mathcal{W}}$ is defined by

$$(T_w f)(s) = \int_G K_w(s,t,f(t))dt$$

where $f \in \mathrm{Dom}\mathbb{T} = \cap_{w\in\mathcal{W}} \mathrm{Dom}\, T_w$ (see [4]).

We will say that the family of kernels $\mathbb{K} \in \boldsymbol{\mathcal{K}}$ is $\mathbb{L}$-singular if

a) $D_w = \displaystyle\int_G \eta^{-1}(z)L_w(\theta,z)dz \leq S < +\infty$, for all $w \in \mathcal{W}$.

b) for every $U \in \mathcal{U}$, $\displaystyle\frac{1}{D_w}\int_{G\setminus U} \eta^{-1}(z)L_w(\theta,z)dz \xrightarrow{\mathcal{WW}} 0.$

c) $r_n^w(s) = \displaystyle\sup_{0<|u|\leq n}\left|\frac{1}{u}\int_G K_w(s,t,u)dt - 1\right| \xrightarrow{\mathcal{WW}} 0$ uniformly with respect to $s \in G$ (see [11]-[14], [1, 3, 2]).

Now, we may prove the following convergence theorem.

Theorem 3

Let $\mathbb{K} = (K_w)_{w\in\mathcal{W}} \in \boldsymbol{\mathcal{K}}$ be a family of $\mathbb{L}$-singular kernels and let be $\varphi_i \in \Phi,\ i = 1,2,3$ satisfying the conditions of Theorem 2. Then for any $f \in L^{\varphi_1+\varphi_3}(G) \cap L^{\varphi_3}_{\varphi_2\circ\eta}(G) \cap \mathrm{Dom}\mathbb{T}$ and $\eta \in L^{(\varphi_3\circ\varphi_2)+\varphi_2}(G)$, we have that $I_{\varphi_1}[\lambda(T_w f - f)] \xrightarrow{\mathcal{WW}} 0$, for sufficiently small $\lambda > 0$.

Proof. We replace in inequality (4), D, L, r_n with D_w, L_w, r_n^w. For a sake of simplicity we write the second member of (4) in the form

$$I_{\varphi_1}[\lambda(T_w f - f)] \leq F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7.$$

Let $\lambda_o > 0$ be the number of Proposition 1 and let $\varepsilon > 0$ be fixed. Since $f \in L^{\varphi_3}_{\varphi_2\circ\eta}(G)$ for sufficiently small $\lambda > 0$, there is an $U \in \mathcal{U}$ such that $F_1 < \varepsilon$. Moreover by the assumptions and property b) of $\mathbb{L}$-singularity, there is a set $W_1 \in \mathcal{W}$ such that $F_2 < \varepsilon$.

Now since $f \in L^{\varphi_1+\varphi_3}(G)$ and $\eta \in L^{(\varphi_3\circ\varphi_2)+\varphi_2}(G)$, there is $\overline{n}(\varepsilon) > 0 : \ \forall n \geq \overline{n}, \ F_4 + F_6 < \varepsilon$ and moreover, by absolute continuity of the integrals,

$$I_{\varphi_3\circ\varphi_2}[\sqrt[4]{\lambda}\,\eta(z+\cdot)\chi_{A_n}] + I_{\varphi_2}[\sqrt[4]{\lambda}\,\eta(z+\cdot)\chi_{A_n}] < \varepsilon/S,$$

uniformly with respect to $z \in G$. Hence by a) of $\mathbb{L}$-singularity, we have that $F_3+F_5 < \varepsilon$.

Now, keeping $\overline{n}$ fixed, by c) of $\mathbb{L}$-singularity, we have that $r^w_{\overline{n}}(s) \stackrel{\mathcal{W}}{\longrightarrow} 0$ uniformly with respect to $s \in G$ and so, from $f \in L^{\varphi_1}(G)$ we deduce that there is a set $W_2 \in \mathcal{W}$ such that $F_7 < \varepsilon$ for every $w \in W_2$. Thus we obtain that $I_{\varphi_1}[\lambda(T_w f - f)] < \varepsilon$ for $w \in W_1 \cap W_2 \in \mathcal{W}$ $\square$

Remarks 3.

3.a) We want to point out that if we assume the condition $\eta \in L^\infty(G)$, where $\eta : G \rightarrow \mathbb{R}^+$ is the function of the assumption (K.3), then following the same idea and techniques of Theorem 3 of [3] and taking into account the new assumptions of the nonlinear case (see also Theorem 3 of [4]), it is possible to obtain an easier estimate for the error of approximation $Tf - f$ and hence, as consequence, we may state a convergence theorem for $T_w f - f$. So, in this case (i.e. $\eta \in L^\infty(G)$), it is possible to obtain extensions of the results of [3] to the operator T of the form: $(Tf)(s) = \int_G K(s,t,f(t))dt$.

Of course the assumption $\eta \in L^\infty(G)$ is more meaningful for compact groups G. Indeed, when $G = (\mathbb{R}^+, \cdot)$, among the kernels homogeneous of degree $\alpha \in \mathbb{R}$, the only one which satisfies $\eta(t) = t^\alpha \in L^\infty(\mathbb{R}^+)$ is a homogeneous kernel of degree zero, and this happens for example for convolution type operators.

Moreover given $\eta \in L^\infty(\mathbb{R}^+)$, $\eta > 0$ and a function $L : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+_0$ homogeneous of degree zero, then $H(s,t) = \eta(t)L(s,t)$ is a non trivial example of $\eta-$homogeneous kernel with bounded η.

In the case of compact groups G, as for example $G = (S^1, \cdot)$, where $S^1 = \{z \in C : |z| = 1\}$ with Lebesgue measure, the assumption $\eta \in L^\infty(G)$ is always satisfied by $\eta(t) = t^\alpha$, for each $\alpha \in \mathbb{R}^+_0$; in this case the operator T takes the form

$$(Tf)(s) = \int_0^{2\pi} K(s,t,f(t))dt.$$

3.b) Other particular cases of our operator T of the form

$$(Tf)(s) = \int_G K(s,t,f(t))dt,$$

are the following:

α) $G = (\mathbb{R}, +)$ with Lebesgue measure; in this case the operator T is of the form

$$(Tf)(s) = \int_{-\infty}^{+\infty} K(s, t, f(t))dt.$$

β) $G = (Z, +)$ with the counting measure; in this case T takes the form

$$(Tf)_j = \sum_{i=-\infty}^{+\infty} K_{j,i}(a_i), \quad j \in Z$$

where $f = (a_i)_{i \in Z}$ and $K_{j,i}(a_i) = K(j, i, a_i), \quad i, j \in Z$.

γ) $G = (Z_n + (\bmod n)) = \{0, 1, \ldots, n-1\}$; in this case T becomes

$$(Tf)_j = \sum_{i=0}^{n-1} K_{j,i}(a_i)$$

where $f = (a_o, a_1, \ldots, a_{n-1})$ and $K_{j,i}(a_i) = K(j, i, a_i)$, $i, j = 0, 1, \ldots, n-1$.

δ) Other examples are $G = (\mathbb{R}^n, +)$, $G = (Z^n, +)$.

3.c) As a particular case of Theorem 3, we have that when $\mathcal{W} = \mathbb{N}$ and $\mathcal{VW}$ is the filter of all sets of the form $\mathbb{N} \setminus B$, where $B \subset \mathbb{N}$ is finite and if $\varphi(u) = u^p$, $p \geq 1$, we obtain $\|T_n f - f\|_p \to 0$ as $n \to +\infty$.

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