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Necessary and sufficient conditions for oscillations of delay partial difference equations*

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Abstract

In this paper we study two classes of delay partial difference equations with constants coefficients. Explicit necessary and sufficient conditions for the oscillation of the solutions of these equations are obtained.

Recently, oscillation theory for partial difference equation have been studied by some authors [1–4]. In this paper we will concern with two classes of delay partial difference equations of the form

$$\begin{aligned} u(i+1,j) + bu(i,j+1) + cu(i,j) + du(i-r,j) + eu(i,j-s) \\ + fu(i-m,j-n) = 0, i,j = 0,1,2,\dots, \end{aligned} \tag{1}$$

and

$$\begin{aligned} u(i+1,j+1) + au(i+1,j) + bu(i,j+1) + cu(i,j) + du(i-r,j) \\ + eu(i,j-s) + fu(i-m,j-n) = 0, i,j = 0,1,2,\dots \end{aligned} \tag{2}$$

where a, b, c, d, e, and f are real numbers, r, s, m and n are positive integers.

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A solution of (1) or (2) is a real double sequence $u = \{u(i,j)\}_{i,j=0}^{\infty}$ which satisfies (1) or (2) respectively. Since (1) can be written in the form

$$\begin{aligned} u(i+1,j) = &-bu(i,j+1) - cu(i,j) - du(i-r,j) \\ &- eu(i,j-s) - fu(i-m,j-n), \quad i,j = 0,1,2,\ldots, \end{aligned}$$

it is clear that if we impose conditions such as

$$\begin{aligned} u(0,j) &= \phi(j), \quad j = 0,1,2,\ldots, \\ u(i,j) &= \alpha(i,j), \ -r \le i \le 0, \ \ j \ge 0, \\ u(i,j) &= \beta(i,j), \ -s \le j \le 0, \ \ i \ge 0, \\ u(i,j) &= \varphi(i,j), \ -m \le i \le 0, \ -n \le j \le 0, \end{aligned}$$

with

$$\phi(0) = \alpha(0,0) = \beta(0,0) = \alpha(0,0),$$

we can calculate $u(1,0); u(1,1), u(2,0); u(1,2), u(2,1), u(3,0); \ldots$ successively in a unique manner. An existence and uniqueness theorem for solutions of (1) is thus easily formulated and proved.

Similarly, since (2) can be written as

$$\begin{aligned} u(i+1,j+1) = &-au(i+1,j) - bu(i,j+1) - cu(i,j) - du(i-r,j) \\ &- eu(i,j-s) - fu(i-m,j-n), \quad i,j = 0,1,2,\ldots, \end{aligned}$$

it is clear that if the conditions

$$\begin{aligned} u(0,j) &= \phi(j), \quad j = 0,1,2,\ldots, \\ u(i,0) &= \psi(i), \quad i = 0,1,2,\ldots, \\ u(i,j) &= \alpha(i,j), \ -r \le i \le 0, \ \ j \ge 0, \\ u(i,j) &= \beta(i,j), \ -s \le j \le 0, \ \ i \ge 0, \\ u(i,j) &= \varphi(i,j), \ -m \le i \le 0, \ -n \le j \le 0, \end{aligned}$$

with

$$\phi(0) = \psi(0) = \alpha(0,0) = \beta(0,0) = \alpha(0,0)$$

are imposed, we can calculate $u(1,1); u(1,2), u(2,1); u(1,3), u(2,2), u(3,1); \ldots$ successively in a unique manner.

A double sequence $u = \{u(i,j)\}_{i,j}^{\infty} = 0$ is said to be eventually positive if $u(i,j) > 0$ for all sufficiently large i and j. An eventually negative sequence is similarly defined. The sequence u is said to be oscillatory if it is neither eventually positive nor eventually negative. We are interested in explicit condition imposed on the numbers a, b, c, d, e and f such that all solutions of (1) and (2) are oscillatory.

Lemma 1

Suppose $c \geq 0,\ d > 0,\ e > 0$ *and* $f > 0$. *If* $b \geq 0$, *then there cannot be any pair of positive numbers* α *and* β *such that*

$$\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n} = 0. \tag{3}$$

The converse also holds.

Proof. Suppose $b \geq 0$. Since $c \geq 0,\ d > 0$, $e > 0$ and $f > 0$, for any pair of positive numbers α and β, we have

$$\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n} > 0.$$

This shows that (3) cannot hold.

Conversely, if $b < 0$, let $\alpha = 1$ then

$$\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n} = 1 + b\beta + c + d + e\beta^{-s} + f\beta^{-n}.$$

Set $F(\beta) = 1 + b\beta + c + d + e\beta^{-s} + f\beta^{-n}$. Since $\lim_{\beta\to 0+} F(\beta) = +\infty$, $\lim_{\beta\to +\infty} F(\beta) = -\infty$ and F is continuous on $(0, +\infty)$, there exists $\beta_0 \in (0, +\infty)$ such that $F(\beta_0) = 0$. So that (3) has a solution pair $\alpha = 1,\ \beta = \beta_0$, and the solution pair are positive. The proof of Lemma 1 is complete. $\square$

Theorem 1

Suppose $c \geq 0$, $d > 0$, $e > 0$ *and* $f > 0$. *Then every solution of (1) is oscillatory if and only if* $b \geq 0$.

Proof. If $b < 0$, then by means of Lemma 1, we can find a pair of positive numbers α and β such that

$$\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n} = 0.$$

Then, as can easily be verified, the double sequence $\{u(i,j)\}$ defined by

$$u(i,j) = \alpha^i\beta^j, \quad i, j = 0, 1, 2, \ldots$$

is a positive solution of (1).

Conversely, suppose $b \geq 0$ and that (1) has a nonoscillatory solution $u = \{u(i,j)\}_{i,j}^{\infty} = 0$. We may assume without loss of generality that $u(i,j) > 0$ for $i, j \geq 0$. But since $c \geq 0$, $d > 0$, $e > 0$ and $f > 0$, then left side of (1) is strictly greater than 0. This contradiction establishes our proof of Theorem 1. □

We now deal with the question of oscillation of (2).

Lemma 2

Suppose $b \neq 0$, $c \geq 0$, $d > 0$, $e > 0$ and $r < m$, $s \neq n$. If $b > 0$, $a \geq 0$ and $f \geq 0$, then there cannot be any pair of positive numbers α and β such that

$$\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n} = 0. \tag{4}$$

The converse also holds.

Proof. If $a \geq 0$, $b > 0$, $f \geq 0$ as well as $\alpha > 0$ and $\beta > 0$, since $c \geq 0$, $d > 0$, $e > 0$ and $f > 0$, it is obvious that (4) cannot hold:

Conversely, we need to consider seven cases:

(a) Assume $a \geq 0$, $b < 0$ and $f \geq 0$, let $\alpha = -b/2$, then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad = \frac{b}{2}\beta - \frac{ab}{2} + c + d\left(-\frac{b}{2}\right)^{-r} + e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}.\end{aligned}$$

Set

$$H(\beta) = \frac{b}{2}\beta - \frac{ab}{2} + c + d\left(-\frac{b}{2}\right)^{-r} + e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}.$$

Since $\lim_{\beta\to 0+} H(\beta) = +\infty$, $\lim_{\beta\to+\infty} F(\beta) = -\infty$ and H is continuous on $(0, +\infty)$, there exists a $\beta_0 > 0$ such that $H(\beta_0) = 0$. So that (4) has a solution pair $\alpha = -b/2$, $\beta = \beta_0$.

(b) Assume $a \geq 0$, $b < 0$ and $f < 0$.

Case (i): if $s > n$, let $\alpha = -b/2$, then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad = \frac{b}{2}\beta - \frac{ab}{2} + c + d\left(-\frac{b}{2}\right)^{-r} + e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}.\end{aligned}$$

Set

$$H(\beta) = \frac{b}{2}\beta - \frac{b}{2}a + c + d\left(-\frac{b}{2}\right)^{-r} + e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}.$$

Then $\lim_{\beta\to+\infty} H(\beta) = -\infty$ and

$$\begin{aligned}&\lim_{\beta\to 0+}\left[e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}\right]\\ &= \lim_{\beta\to 0+}\beta^{-s}\left[e + f\left(-\frac{b}{2}\right)^{-m}\beta^{s-n}\right] = +\infty\,.\end{aligned}$$

Thus $\lim_{\beta\to 0+} H(\beta) = +\infty$. So there exists $\beta_0 > 0,$ such that $H(\beta_0) = 0$ and (4) has a solution pair $\alpha = -b/2$, $\beta = \beta_0$.

Case (ii): if $s < n,$ let $\alpha = -2b,$ then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad= -b\beta - 2ab + c + d(-2b)^{-r} + e\beta^{-s} + f(-2b)^{-m}\beta^{-n}\,.\end{aligned}$$

Set $H(\beta) = -b\beta - 2ab + c + d(-2b)^{-r} + e\beta^{-s} + f(-2b)^{-m}\beta^{-n},$ then $\lim_{\beta\to+\infty} H(\beta) = +\infty$ and

$$\lim_{\beta\to 0+}\left[e\beta^{-s} + f(-2b)^{-m}\beta^{-n}\right] = \lim_{\beta\to 0+}\beta^{-n}\left[e\beta^{n-s} + f(-2b)^{-m}\right] = -\infty\,.$$

So $\lim_{\beta\to 0+} H(\beta) = -\infty$ and there exists $\beta_0 > 0$ such that $H(\beta_0) = 0$. Hence (4) has a solution pair $\alpha = -2b$, $\beta = \beta_0$.

(c) Assume $a < 0$, $b < 0$ and $f \geq 0$. Let $\beta = -a/2$, then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad= \frac{a}{2}\alpha - \frac{ab}{2} + c + d\alpha^{-r} + e\left(-\frac{a}{2}\right)^{-s} + f\alpha^{-m}\left(-\frac{a}{2}\right)^{-n}.\end{aligned}$$

Set

$$H(\alpha) = \frac{a}{2}\alpha - \frac{ab}{2} + c + d\alpha^{-r} + e\left(-\frac{a}{2}\right)^{-s} + f\alpha^{-m}\left(-\frac{a}{2}\right)^{-n}.$$

Since $\lim_{\alpha\to 0+} H(\alpha) = +\infty$ and $\lim_{\alpha\to+\infty} H(\alpha) = -\infty$, there exists $\alpha_0 > 0$ such that $H(\alpha_0) = 0$, so that (4) has a solution pair $\alpha = \alpha_0$, $\beta = -a/2$.

(d) Assume $a < 0$, $b < 0$ and $f < 0$.

Case (i): if $s > n$, let $\alpha = -b/2$, then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad= \frac{b}{2}\beta - \frac{ab}{2} + c + d\left(-\frac{b}{2}\right)^{-r} + e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}.\end{aligned}$$

Set

$$H(\beta) = \frac{b}{2}\beta - \frac{ab}{2} + c + d\left(-\frac{b}{2}\right)^{-r} + e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}\,,$$

then $\lim_{\beta\to+\infty} H(\beta) = -\infty$ and

$$\begin{aligned}&\lim_{\beta\to 0+}\left[e\beta^{-s} + f\left(-\frac{b}{2}\right)^{-m}\beta^{-n}\right]\\ &= \lim_{\beta\to 0+}\beta^{-s}\left[e + f\left(-\frac{b}{2}\right)^{-m}\beta^{s-n}\right] = +\infty\,.\end{aligned}$$

Thus $\lim_{\beta\to 0+} H(\beta) = +\infty$. Hence there exists $\beta_0 > 0$ such that $H(\beta_0) = 0$, so that (4) has a solution pair $\alpha = -b/2$, $\beta = \beta_0$.

Case (ii): if $s < n$, let $\alpha = -2b$, then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad = -b\beta - 2ab + c + d(-2b)^{-r} + e\beta^{-s} + f(-2b)^{-m}\beta^{-n}\,.\end{aligned}$$

Set $H(\beta) = -b\beta - 2ab + c + d(-2b)^{-r} + e\beta^{-s} + f(-2b)^{-m}\beta^{-n}$, then $\lim_{\beta\to+\infty} H(\beta) = +\infty$ and

$$\lim_{\beta\to 0+}\left[e\beta^{-s} + f(-2b)^{-m}\beta^{-n}\right] = \lim_{\beta\to 0+}\beta^{-n}\left[e\beta^{n-s} + f(-2b)^{-m}\right] = -\infty\,.$$

So $\lim_{\beta\to 0+} H(\beta) = -\infty$ and there exists $\beta_0 > 0$ such that $H(\beta_0) = 0$. Hence (4) has a solution pair $\alpha = -2b$, $\beta = \beta_0$.

(e) Assume $a < 0$, $b > 0$ and $f \geq 0$. Let $\beta = -a/2$, then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad = \frac{a}{2}\alpha - \frac{ab}{2} + c + d\alpha^{-r} + e\left(-\frac{a}{2}\right)^{-s} + f\alpha^{-m}\left(-\frac{a}{2}\right)^{-n}\,.\end{aligned}$$

Set

$$H(\alpha) = \frac{a}{2}\alpha - \frac{ab}{2} + c + d\alpha^{-r} + e\left(-\frac{a}{2}\right)^{-s} + f\alpha^{-m}\left(-\frac{a}{2}\right)^{-n}\,.$$

Since $\lim_{\alpha\to 0+} H(\alpha) = +\infty$ and $\lim_{\alpha\to+\infty} H(\alpha) = -\infty$, there exists $\alpha_0 > 0$ such that $H(\alpha_0) = 0$, so that (4) has a solution pair $\alpha = \alpha_0$, $\beta = -a/2$.

(f) Assume $a < 0$, $b > 0$ and $f < 0$. Let $\beta = -2a$, then

$$\begin{aligned}&\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\\ &\quad = -a\alpha - 2ab + c + d\alpha^{-r} + r(-2a)^{-s} + f\alpha^{-m}(-2a)^{-n}\,.\end{aligned}$$

Set $H(\alpha) = -a\alpha - 2ab_c + d\alpha^{-r} + r(-2a)^{-s} + f\alpha^{-m}(-2a)^{-n}$, then $\lim_{\alpha\to+\infty} H(\alpha) = +\infty$ and

$$\lim_{\alpha\to 0+} \left[d\alpha^{-r} + f\alpha^{-m}(-2a)^{-n}\right] = \lim_{\alpha\to 0+} \alpha^{-m}\left[d\alpha^{m-r} + f\right] = -\infty\,.$$

Thus $\lim_{\alpha\to 0+} H(\alpha) = -\infty$ and there exists a $\alpha_0 > 0$ such that $H(\alpha_0) = 0$. Hence (4) has a solution pair $\alpha = \alpha_0$, $\beta = -2a$.

(g) Assume $a \geq 0$, $b > 0$ and $f < 0$, let $\beta = 1$ then

$$\begin{aligned} &\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n} \\ &\quad = (1+a)\alpha + b + c + d\alpha^{-r} + e + f\alpha^{-m}. \end{aligned}$$

Set $H(\alpha) = (1+a) + b + c + d\alpha^{-r} + e + f\alpha^{-m}$, then $\lim_{\alpha\to+\infty} H(\alpha) = +\infty$ and

$$\lim_{\alpha\to 0+} \left[d\alpha^{-r} + f\alpha^{-m}\right] = \lim_{\alpha\to 0+} \alpha^{-m}\left[d\alpha^{m-r} + f\right] = -\infty\,.$$

Thus $\lim_{\alpha\to 0+} H(\alpha) = -\infty$ and there exists a $\alpha_0 > 0$ such that $H(\alpha_0) = 0$. Hence (4) has a solution pair $\alpha = \alpha_0$, $\beta = 1$.

In all the cases above, the solution pairs are positive. The proof of Lemma 2 is complete. □

Similarly we have.

Lemma 3

Suppose $b \neq 0$, $c \geq 0$, $\alpha > 0$, $e > 0$, $s < n$ and $r \neq m$. If $b > 0$, $a \geq 0$ and $f \geq 0$, then there cannot be any pair of positive numbers α and β such that (4) holds. The converse also holds.

Theorem 2

Suppose $b \neq 0$, $c \geq 0$, $d > 0$, $e > 0$, $r < m$ and $s \neq n$. Then every solution of (2) is oscillatory if and only if $b > 0$, $a \geq 0$ and $f \geq 0$.

Proof. If one of the numbers a, b and f is negative, by Lemma 2, we can find a pair of positive numbers α and β such that

$$\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\alpha^{-s} + f\alpha^{-m}\beta^{-n} = 0\,.$$

Then, as can easily be verified, the double sequence $u = \{u(i,j)\}$ defined by $u(i,j) = \alpha^i\beta^j$ is a positive solution of (2).

Conversely, suppose $b > 0$, $a \geq 0$, $f \geq 0$ and that (2) has a nonoscillatory solution $u = \{u(i,j)\}_{i,j=0}^{\infty}$. We may assume without loss of generality that $u(i,j) > 0$ for $i,j \geq 0$. But as $c \geq 0$, $d > 0$, and $e > 0$, the left side of (2) is strictly positive, which is a contradiction. The proof of Theorem 2 is complete. □

Similarly, from Lemma 3 we have.

Theorem 3

Suppose $b \neq 0$, $c \geq 0$, $d > 0$, $e > 0$, $s < n$ and $r \neq m$, then every solution of (2) is oscillatory if and only if $b > 0$, $a \geq 0$ and $f \geq 0$.

We remark that equation (3) will occur in a natural manner if we seek solution of the form $u = \{u(i,j)\}$ defined by $u(i,j) = \alpha^i\beta^j$. Indeed, if we substitute $u(i,j) = \alpha^i\beta^j$ into (1), we obtain

$$\alpha^i\beta^j\left\{\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\right\} = 0\,.$$

Similarly, if we substitute $u(i,j) = \alpha^i\beta^j$ into (2) we obtain

$$\alpha^i\beta^j\left\{\alpha\beta + a\alpha + b\beta + c + d\alpha^{-r} + e\beta^{-s} + f\alpha^{-m}\beta^{-n}\right\} = 0.$$

EXAMPLE 1: Consider the equation

$$\begin{aligned} &u(i+1,j) + 2u(i,j+1) + 5u(i,j) + 5u(i-2,j) \\ &\qquad + 3u(i,j-3) + 4u(i-4,\,j-5) = 0\,, \quad i,j \geq 5\,. \end{aligned} \tag{5}$$

According Theorem 1, every solution of equation (5) is oscillatory. In fact, this equation has an oscillatory solution $u(i,j) = (-1)^{i+j}$.

EXAMPLE 2: Consider the equation

$$\begin{aligned} &u(i+1,j+1) + 3u(i+1,\,j) + 3u(i,j+1) + 2u(i,j) + 3u(i-1,j) \\ &\qquad + 3u(i,j-2) + u(i-3,\,j-4) = 0, \quad i,j \geq 4. \end{aligned} \tag{6}$$

According Theorem 2, every solution of (6) is oscillatory. In fact, (6) has an oscillatory solution $u(i,j) = (-1)^i j$.

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