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The density condition and the strong dual density condition by operator

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Abstract

The aim of the present article is to introduce and investigate topological properties by operator. We obtain good stability properties for the density condition and the strong dual density condition by taking injective tensor products. Further we analyze the connection to (DF)-properties by operator.

Introduction

Many function spaces are in fact injective tensor products or spaces of linear continuous mappings. In this paper we investigate the stability of the density condition (DC) and of the strong dual density condition (SDDC) under the formation of injective tensor products of Fréchet or (DF)-spaces with a Banach space and of spaces of linear continuous mappings from a Banach space into Fréchet or (DF)-spaces. The density condition was introduced by S. Heinrich [10] in the context of ultrapowers of locally convex spaces. The density condition plays an important role in the theory of Köthe echelon spaces [1] - [3], for extensions of linear operators [8] and in the theory of unbounded operator *-algebras [11].

This article is divided into two parts. In Section 1 we introduce and investigate the properties density condition by operator (DCO) and strong dual density condition by operator (SDDCO). By this properties we obtain good stability properties for density condition and strong dual density condition by taking tensor products and these properties are, in a certain sense, optimal. The method to define properties by operators was introduced by A. Peris for quasinormable spaces [15] and for (DF)-spaces [6].

Section 2 is devoted to investigate the relations between the density condition and strong dual density condition by operator and (DF)-spaces by operator (DFO). (DF)-spaces by operator and the density condition present a general frame in which the following problems of topologies of Grothendieck are solvable:

(1) Let E_1, E_2 be (DF)-spaces, is $E_1 \otimes_\varepsilon E_2$ a (DF)-space?

(2) Let (F_1, F_2) be Fréchet spaces. Can every bounded subset M of the projective tensor product $F_1 \tilde{\otimes}_\pi F_2$ be localized, i.e. there are bounded subsets $B_i \subset F_i,\ i = 1, 2$ such that $M \subset \overline{\Gamma}(B_1 \otimes B_2)$?

Due to J. Taskinen it is known that the answer to these problems is negative in general. The class of (DF)-spaces by operator is related to the class of (DF)-spaces satisfying strong dual density condition by operator and by dualization to the class of Fréchet spaces satisfying density condition by operator. For example the (DF) property by operator implies the equivalence of (SDDC) and (SDDCO). As against the situation for density condition and strong dual density condition there is no simple duality relation between density condition by operator and strong dual density condition by operator. There exists a duality theory in the frame of (DF)-spaces by operator, but it is impossible to construct the corresponding operators with the present methods. Further, we will show that equivalent definitions for the strong dual density condition give different properties by operator, see Example 2.5.(3). Therefore it is a non-trivial problem to find the right concepts by operator. We collect different definitions for (DF)-spaces by operator, (DFo) and (DFop) by A. Peris, and we prove the equivalence of the definitions for large classes of spaces.

There exist applications for the strong dual density condition by operator in the theory of unbounded operator *-algebras, too. This subject we are going to study in another article.

The notation for locally convex spaces is standard. If E is a locally convex space, $\mathcal{U}(E)$ stands for a basis of absolutely convex closed 0-neighborhoods. If E is a (DF)-space, then there exists a fundamental sequence $(M_k)_{k=1}^\infty$ of bounded sets, abbreviated fsb, such that each set M_k is absolutely convex and closed. $\mathcal{B}(E)$ stands for the system of all absolutely convex bounded sets in E. If V is an absolutely convex set, we denote by p_V the Minkowski functional of V, by $p_V^{-1}(0)$ the kernel of p_V and if V is in addition absorbent, we denote by E_V the quotient $E/p_V^{-1}(0)$. Further $\mathrm{FIN}(E)$ stands for the set of all finite-dimensional subspaces of E. If E and F are locally convex spaces, then $L_b(E,F)$ denotes the space of all continuous linear mappings from E into F endowed with the topology of uniform convergence on the bounded sets of E. We write $L_b(E)$ for $L_b(E,E)$. If $K \subset E$, $L \subset F$ and M is a linear subspace of $L_b(E,F)$, then we write $W(K,L) := \{T \in M : T(K) \subset L\}$. An operator $T \in L(E,F)$ is called bounded if there exists a 0-neighborhood $U \in \mathcal{U}(E)$ such that $T(U)$ is bounded. Let C_2 denote the space of Johnson. C_2 is the l_2-direct sum of the spaces $F_k,\ F_k'$ with $k \in \mathbb{N}$, where $(F_k)_{k=1}^{\infty}$ is a sequence of finite-dimensional Banach spaces which is dense in the set of all finite-dimensional Banach spaces endowed with the Banach-Mazur distance. If X is a Banach space, then U_X denotes its closed unit ball. The linear hull of a subset $M \subset E$ is denoted by $[M]$.

1. The density condition and the strong dual density condition by operator

S. Heinrich introduced the density condition in the context of ultrapowers of locally convex spaces in [10]. K. D. Bierstedt and J. Bonet intensively studied the density condition for Fréchet spaces in [1], [2], and [3].

Definition 1. Let F denote a metrizable space and $(U_k)_{k=1}^{\infty}$ a countable basis of closed absolutely convex 0-neighborhoods in F.

(1) F is said to satisfy the density condition (DC) if the following holds:

Given a positive sequence $(\lambda_k)_{k=1}^{\infty}$ and an $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $B \in \mathcal{B}(F)$ such that

$$\bigcap_{k=1}^{m} \lambda_k U_k \ \subset\ U_n \ +\ B\ . \tag{1}$$

(2) F is said to satisfy the density condition by operator (DCO) if the following holds:

Given a positive sequence $(\lambda_k)_{k=1}^{\infty}$ and an $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and bounded $P \in L(F)$ such that

$$(I\ -\ P)(\bigcap_{k=1}^{m} \lambda_k U_k)\ \subset\ U_n\ . \tag{2}$$

Quasinormable Fréchet spaces and Fréchet-Montel spaces are examples of spaces satisfying (DC), see [10]. The density condition for the Köthe echelon space was characterized in [1] and for Fréchet domains of unbounded operators *-algebras in [11]. It follows by definition, that every Fréchet space with (DCO) satisfies (DC). The purpose of Definition 1.(2) is to obtain good permanence properties by taking injective tensor products. Examples of spaces with (DCO) follow later. The next lemma will be very useful for the characterization of properties by operators. It is due to A. Peris, see [6], Lemma 3.

Lemma 2

Let E be a locally convex space which is complemented in $(E'_b)'_e$, H a space of linear mappings from the Johnson space C_2 to E which contains all linear operators with finite-dimensional range and $A, C, B_1, \dots, B_n$ absolutely convex subsets of E such that C is closed, B_k is bounded and closed for $k = 1, \dots, n$ and the following property is satisfied in H:

$$W(U_{C_2}, A) \subset W(U_{C_2}, C) + \sum_{k=1}^{n} W(U_{C_2}, B_k) . \tag{3}$$

Then there are linear operators $(P_k)_{k=1}^{n}$ in E such that

$$P_k(A) \subset 2B_k \quad \text{for} \quad k = 1, \dots, n \quad \text{and} \quad (I_E - \sum_{k=1}^{n} P_k)(A) \subset 2C . \tag{4}$$

The next theorem characterizes the density condition by operator for a large class of Fréchet spaces.

Theorem 3

Let F be a Fréchet space with a basis $(U_k)_{k=1}^{\infty}$ of 0-neighborhoods complemented in the strong bidual F''. The following assertions are equivalent:

1. *F satisfies the density condition by operator (DCO).*
2. *There exists a bounded set $B \subset F$ such that for all $n \in \mathbb{N}$ and all bounded sets $M \subset F$, we can find $\lambda > 0$ and $Q \in L(F)$ such that*

$$Q(M) \subset U_n \quad \text{and} \quad (I - Q)(M) \subset \lambda B$$

   (this is the property (DCo) in [16]).
3. *$X \otimes_\varepsilon F$ ($X \tilde{\otimes}_\varepsilon F$, $X \varepsilon F$ and $L_b(X, F)$, resp.) satisfies the density condition (DC) for each Banach space X.*

Proof. (1) ⇒ (2): We define countable families

$$\mathcal{A}_p := \Big\{ \bigcap_{k=1}^{m} \lambda_k U_k \;:\; m, \lambda_1, \ldots, \lambda_m \in \mathbb{N} \quad \text{and} \quad \exists P \in L(F) \quad \text{with}$$
$$C := P\Big(\bigcap_{k=1}^{m} \lambda_k U_k\Big) \in \mathcal{B}(F) \quad \text{and} \quad (I-P)\Big(\bigcap_{k=1}^{m} \lambda_k U_k\Big) \subset U_p \Big\}$$

for all $p \in \mathbb{N}$. There exist countable families $\mathcal{B}_p := \{C_{p,i} \;:\; i \in \mathbb{N}\}$ of bounded subsets in F and countable families $\mathcal{P}_p := \{P_{p,i} : i \in \mathbb{N}\}$ of operators in $L(F)$ for the characterization of $\mathcal{A}_p$ and we find a positive double sequence $(\varrho_{p,i})_{p,i=1}^{\infty}$ such that $B := \cup_{p,i=1}^{\infty} \varrho_{p,i} C_{p,i}$ is bounded.

Given a bounded set $M \subset F$ and an $n \in \mathbb{N}$, there exists a positive sequence $(\lambda_k)_{k=1}^{\infty}$ such that $M \subset \cap_{k=1}^{\infty} \lambda_k U_k$. By assumption F satisfies (DCO). Thus there exists an $m \in \mathbb{N}$ such that $\cap_{k=1}^{m} \lambda_k U_k \in \mathcal{A}_n$. Now we choose a fit $C_{n,i} \in \mathcal{B}_n$ and a fit $P \in \mathcal{P}_n$. We define $Q := I - P$ and $\lambda := \varrho_{n,i}^{-1}$. It follows $Q(M) \subset (I-P)(\cap_{k=1}^{m} \lambda_k U_k) \subset U_n$ and $(I-Q)(M) \subset P(\cap_{k=1}^{m} \lambda_k U_k) = C_{n,i} \subset \lambda B$.
(2) ⇒ (3): This follows by [16], Proposition 11, but we include the proof for the sake of completeness. We are going to prove the result for the injective tensor product $X \otimes_\varepsilon F \;\big(\subset L_b(X', F)\big)$, the proof will be similar for $X \tilde{\otimes}_\varepsilon F$, $X \varepsilon E$ and $L_b(X, F)$.

Given a Banach space X, a bounded set $M \subset F$ and an $n \in \mathbb{N}$, there exist $\lambda > 0$ and $Q \in L(F)$ such that $Q(M) \subset U_n$ and $(I-Q)(M) \subset \lambda B$. Define $\tilde{B} := W(U_{X'}, B)$ and $\tilde{Q} := I_X \otimes Q \in L(X \otimes_\varepsilon F)$, i.e. $\tilde{Q}(T) = Q \circ T \circ I_{X'}$. We conclude $\tilde{Q}\big(W(U_{X'}, M)\big) = W\big(U_{X'}, Q(M)\big) \subset W(U_{X'}, U_n)$ and $(I_{X \otimes F} - \tilde{Q})\big(W(U_{X'}, M)\big) = W\big(U_{X'}, (I_F - Q)(M)\big) \subset W(U_{X'}, \lambda B) = \lambda \tilde{B}$. It follows that

$$\exists \tilde{B} \in \mathcal{B}(X \otimes_\varepsilon F) \quad \forall n \in \mathbb{N} \quad \forall \tilde{M} \in \mathcal{B}(X \otimes_\varepsilon F) \quad \exists \lambda > 0 \;:\; \tilde{M} \subset \tilde{U}_n + \lambda \tilde{B}.$$

This implies that bounded sets in $(X \otimes_\varepsilon F)'_b$ are metrizable and $X \otimes_\varepsilon F$ has (DC), see [1], Theorem 1.4.
(3) ⇒ (1): We are also going to prove the result for the injective tensor product. Setting $X := C_2$, by assumption $X \otimes_\varepsilon E\big(\subset L_b(X', E)\big)$ satisfies (DC). We have: $\forall (\lambda_k)_{k=1}^{\infty}$ positive sequence, $\forall n \in \mathbb{N}$, $\exists m \in \mathbb{N}$, $\exists B \in \mathcal{B}(F)$ such that

$$2W\Big(U_{X'}, \bigcap_{k=1}^{m} \lambda_k U_k\Big) \subset 2 \bigcap_{k=1}^{m} \lambda_k W(U_{X'}, U_k) \subset W(U_{X'}, U_n) \;+\; W(U_{X'}, B)\;.$$

We set $A := \cap_{k=1}^{m} \lambda_k U_k$, $B_1 := B$ and $C := U_n$ in Lemma 2. It follows that F satisfies (DCO). □

EXAMPLE 4: (1) (DC) is equivalent to (DCO) for Köthe echelon spaces $\lambda_p(A)$ of order p, $1 \leq p \leq \infty$. For a proof see Example 2.5. (1). We recall that $\lambda_p(A)$ satisfies (DC) if and only if the Köthe matrix A satisfies condition (D), see [1].

(2) (DC) is equivalent to (DCO) for Fréchet domains D of unbounded operator *-algebras. This equivalence was proved in [11], Theorem 3.2. by implicate (see the last lines of the proof).

(3) A Fréchet space has (DCO) whenever it is quasinormable by operator (QNo) in the sense of Peris, see [15]. For instance the space $C_c(\mathbb{R})$ of all continuous functions endowed with the compact-open topology is (QNo).

(4) Let X, Y be Banach spaces such that Y is a topological subspace of X, let $q : X \longrightarrow X/Y$ be the quotient map and let λ be a normal Banach sequence space. The standard quojection of Moscatelli type is defined as:

$$\lambda(X, X/Y) := \left\{ (x_n)_{n=1}^{\infty} \subset X \; : \; (\|q(x_n)\|)_{n=1}^{\infty} \in \lambda \right\}$$

and a basis of 0-neighborhoods is given by $(\frac{1}{k} W_k)_{k=1}^{\infty}$, where

$$W_k := \left\{ (x_n)_{n=1}^{\infty} \subset X \; : \left\| \left((\|x_n\|)_{n<k}, (\, \|q(x_n)\| \,)_{n \geq k} \right) \right\|_{\lambda} \leq 1 \right\} .$$

By A. Peris, see [15], Proposition 3.16, the standard quojection of Moscatelli type $F := \lambda(X, X/Y)$ is quasinormable by operator if and only if Y is complemented in X. The same method works for (DCO), too. It follows that F satisfies (DCO) if and only if Y is complemented in X. Setting $X := l_\infty$ and $Y := c_0$, then F does not satisfy (DCO).

Now, we introduce the strong dual density condition by operator.

DEFINITION 5. Let E denote a locally convex space with an increasing fsb $(M_k)_{k=1}^{\infty}$.

(1) E is said to satisfy the strong dual density condition (SDDC), resp. dual density condition (DDC), if the following holds:
Given any positive sequence $(\lambda_k)_{k=1}^{\infty}$ and an $n \in \mathbb{N}$, there always exist $m \in \mathbb{N}$ and $U \in \mathcal{U}(E)$ such that

$$M_n \cap U \subset \Gamma \bigcup_{k=1}^{m} \lambda_k M_k \; , \quad \text{resp.} \quad M_n \cap U \subset \overline{\Gamma} \bigcup_{k=1}^{m} \lambda_k M_k \; . \tag{5}$$

(2) E is said to satisfy the strong dual density condition by operator (SDDCO) if the following holds:

Given any positive sequence $(\lambda_k)_{k=1}^{\infty}$ and an $n \in \mathbb{N}$, there always exist $m \in \mathbb{N}$, $U \in \mathcal{U}(E)$ and linear operators $(Q_k)_{k=1}^{m}$ in E such that

$$\sum_{k=1}^{m} Q_k = I_E \qquad \text{and} \qquad Q_k(M_n \cap U) \subset \lambda_k M_k \ , \qquad k = 1, \ldots, m \ . \tag{6}$$

It is not hard to see, that (SDDCO) implies (SDDC). By taking polars, it follows that the Fréchet space E satisfies the (DC) if and only if the strong dual E'_b satisfies the (DDC). Since polars of 0-neighborhoods are $\sigma(E', E)$-compact, (DDC) and (SDDC) are equivalent for E'_b. A (DF)-space has (DDC) if and only if its bounded sets are metrizable. We refer to [2] and [3] for details and examples. Clearly, for (DCO) and (SDDCO) there is no simple duality by taking polars and adjoint operators.

Theorem 6

Let E be a (DF)-space complemented in the strong bidual E''. The following assertions are equivalent:

1. *E satisfies the strong dual density condition by operator (SDDCO).*
2. *$X \otimes_\varepsilon E$ ($X \tilde{\otimes}_\varepsilon E$, $X \varepsilon E$ and $L_b(X, E)$, resp.) satisfies the strong dual density condition (SDDC) for each Banach space X.*

Proof. We are going to prove the result for the injective tensor product $X \otimes_\varepsilon E$ ($\subset L_b(X', E)$), the proof will be similar for $X \tilde{\otimes}_\varepsilon E$, $X \varepsilon E$ and $L_b(X, E)$.

(1) $\Rightarrow$ (2): Let X be a Banach space, let $(\lambda_k)_{k=1}^{\infty}$ be a positive sequence and let $n \in \mathbb{N}$. By assumption there exist $m \in \mathbb{N}$, $U \in \mathcal{U}(E)$ and linear operators $(Q_i)_{i=1}^{m}$ in E such that

$$\sum_{i=1}^{m} Q_i \ = \ I_E \qquad \text{and} \qquad Q_i(M_n \cap U) \subset \lambda_i M_i \ , \qquad i = 1, \ldots, m \ .$$

If we define $\tilde{Q}_i := I_X \otimes Q_i \in L_b(X \otimes_\varepsilon E)$ for $i = 1, \ldots, m$, then we can conclude that $\sum_{i=1}^{m} \tilde{Q}_i \ = \ I_X \otimes \sum_{i=1}^{m} Q_i \ = \ I_{X \otimes E}$ and

$$\begin{aligned} \tilde{Q}_i \left(W(U_{X'}, M_n) \cap W(U_{X'}, U) \right) &= \tilde{Q}_i\big(W(U_{X'}, M_n \cap U)\big) \\ &= W\big(U_{X'}, Q_i(M_n \cap U)\big) \\ &\subset \lambda_i W(U_{X'}, M_i) \ , \end{aligned}$$

for all $i = 1, \ldots, m$. Thus $X \otimes_\varepsilon E$ satisfies (SDDCO) and (SDDC).

(2) $\Rightarrow$ (1): E is a complemented subspace of $X \otimes_\varepsilon E$, this implies that E has (SDDC). Then by [2] E is quasibarrelled and the strong bidual E'' is $(E'_b)'_e$. Let

$X := C_2$, by assumption $X \otimes_\varepsilon E \big(\subset L_b(X', E)\big)$ satisfies (SDDC). We have: $\forall (\lambda_k)_{k=1}^\infty$ positive sequence, $\forall n \in \mathbb{N}$, $\exists m \in \mathbb{N}$, $\exists U \in \mathcal{U}(E)$ such that

$$W(U_{X'}, M_n \cap U) \subset \Gamma \bigcup_{i=1}^m 2^{-1} \lambda_i W(U_{X'}, M_i) \ ,$$
$$W(U_{X'}, M_n \cap U) \subset \sum_{i=1}^m W(U_{X'}, 2^{-1} \lambda_i M_i) \ .$$

Now we set $A := M_n \cap U$, $B_i := 2^{-1}\lambda_i M_i$ for $i = 1, \ldots, m$ and $C := \{0\}$ in Lemma 2. Then we obtain linear operators $(\tilde{Q}_i)_{i=1}^m$ in E such that

$$\tilde{Q}_i(M_n \cap U) \subset \lambda_i M_i, \quad i = 1, \ldots, m \quad \text{and} \quad \sum_{i=1}^m \tilde{Q}_i|_{[M_n \cap U]} \ = \ I_{[M_n \cap U]} \ .$$

We define $Q_i := \tilde{Q}_i$ for $i = 1, \ldots, m-1$ and $Q_m := I_E - \sum_{i=1}^{m-1} \tilde{Q}_i$. For $x \in M_n \cap U$ immediately follows that $Q_m(x) = \tilde{Q}_m(x)$. This implies

$$Q_i(M_n \cap U) \subset \lambda_i M_i, \quad i = 1, \ldots, m \quad \text{and} \quad \sum_{i=1}^m Q_i \ = \ I_E$$

and E satisfies (SDDCO). □

Remark. By Corollary 1.6. in [2] a (DF)-space E with fsb $(M_k)_{k=1}^\infty$ has (SDDC) if and only if for each positive sequence $(\lambda_k)_{k=1}^\infty$, there exists $U \in \mathcal{U}(E)$ such that for every $n \in \mathbb{N}$ we can find $m \in \mathbb{N}$ with

$$M_n \cap U \subset \Gamma \bigcup_{k=1}^m \lambda_k M_k \ . \tag{7}$$

If $X \otimes_\varepsilon E$ is a (DF)-space satisfying (SDDC) for each Banach space X, then we can use (7) in the implication (2) $\Rightarrow$ (1). In this case it follows that E has (SDDCO) if and only if E satisfies the property:

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos.} \quad \exists U \in \mathcal{U}(E) \quad \forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} \quad \exists (Q_k)_{k=1}^m \text{ lin. op.}:$$
$$\sum_{k=1}^m Q_k = I_E \qquad \text{and} \quad Q_k(M_n \cap U) \subset \lambda_k M_k \ , \quad k = 1, \ldots, m \ . \tag{8}$$

EXAMPLE 7: (1) A (DF)-space E with fsb $(M_k)_{k=1}^\infty$ is said to satisfy the strict Mackey condition (s.M.c.) if

$$\forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} \quad \forall \varepsilon > 0 \quad \exists U \in \mathcal{U}(E): \quad U \cap M_n \subset \varepsilon M_m$$

and this is equivalent to

$$\forall(\lambda_k)_{k=1}^{\infty} \text{ pos.} \quad \forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} \quad \exists U \in \mathcal{U}(E): \quad U \cap M_n \subset \lambda_m M_m \ .$$

We define linear operators by $Q_i := 0$ for $i = 1, \dots, m-1$ and $Q_m := I_E$. This demonstrates that E has (SDDCO) if E satisfies (s.M.c.). For instance $l_{p^+} := \cap_{k=1}^{\infty} l_{p+\frac{1}{k}}$ is a quasinormable Fréchet space for $1 \leq p < \infty$, see [15], and the strong dual l_{q^-} with $\frac{1}{p} + \frac{1}{q} = 1$ has (s.M.c.) and (SDDCO).

(2) Let X, Y be Banach spaces such that Y is a topological subspace of X and Y has the approximation property. Further let λ be a normal Banach sequence space with $\lambda' = \lambda^X$. J. C. Díaz and G. Metafune proved in [9], Theorem 2.8. the following equivalent assertions:

(a) Y'' is complemented in X''.

(b) The quojection $\lambda(X, X/Y)$ is an (FBa)-space (for the definition see Section 2).

(c) The strong dual $\lambda(X, X/Y)'_b$ is a countable direct sum of Banach spaces.

We set $X := l_\infty$ and $Y := c_0$. By Example 4.(4) the quojection $\lambda(X, X/Y)$ does not satisfy (DCO), but the strong dual $\lambda(X, X/Y)'_b$ is a countable direct sum of Banach spaces and it is not difficult to check that it has (SDDCO). See Proposition 2.2, too.

More examples will follow in the next section.

2. Relations to the (DF)-property by operator

In this section we will study the permanence of the (DF)-property in injective tensor products. Similarly to the method in the first section, it is possible to define (DF)-properties by operator. Further, we describe the relations of the (DF)-properties by operator with (DCO) and (SDDCO).

Definition 1. A locally convex space E with an increasing fsb $(M_k)_{k=1}^{\infty}$ is said to be a (DFO)-space if for every positive sequence $(\varepsilon_k)_{k=1}^{\infty}$ and for every sequence $(U_n)_{n=1}^{\infty}$ of 0-neighborhoods in E, there are $U \in \mathcal{U}(E)$ and $(S_k)_{k=1}^{\infty} \subset L(E)$ such that

$$S_k(U) \subset \varepsilon_k M_k \ , \quad (I_E - \sum_{k=1}^{n} S_k)(U) \subset U_n \quad k, n \in \mathbb{N} \ . \tag{9}$$

Remark. From the characterization of (DF)-spaces given by Bierstedt and Bonet in [3], Lemma 5.A., we deduce that a (DFO)-space is always a (DF)-space:

A locally convex space is a (DF)-space if and only if it has an increasing fsb $(M_k)_{k=1}^{\infty}$ such that for every positive sequence $(\varepsilon_k)_{k=1}^{\infty}$ and for every sequence $(U_n)_{n=1}^{\infty}$ of 0-neighborhoods in E, there is $U \in \mathcal{U}(E)$ such that

$$U = \bigcap_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \varepsilon_k M_k + U_n \Big) . \tag{10}$$

We get the following simple hereditary properties for (DFO)-spaces and spaces satisfying (SDDCO). Recall that a subspace F of a locally convex space E is called large if for every bounded set B in E there is a bounded set M in F such that $B \subset \bar{M}$.

Proposition 2

1. *Suppose E is a (DFO)-space (resp., has (SDDCO)) and F is a complemented subspace of E. Then F is a (DFO)-space (resp., has (SDDCO)).*
2. *Every normed space is a (DFO)-space satisfying (SDDCO).*
3. *The countable direct sum of (DFO)-spaces (resp., of spaces satisfying (SDDCO)) is a (DFO)-space (resp., has (SDDCO)).*
4. *Let F be a (DFO)-space (resp., has (SDDCO)) and a large subspace of a complete locally convex space E. Then E is a (DFO)-space (resp., has (SDDCO)).*

Proof. We are going to prove (3) for the (DFO) property. Let $E = \oplus_{i=1}^{\infty} E_i$. Given a positive sequence $(\varepsilon_k)_{k=1}^{\infty}$ and a sequence $(\mathcal{V}_k)_{k=1}^{\infty} = (\oplus_{i=1}^{\infty} U_{i,k})_{k=1}^{\infty} \subset \mathcal{U}(E)$, we obtain an increasing fsb for E by $\mathcal{M}_k := \oplus_{i=1}^{k} M_{i,k-i+1}$, where $(M_{i,k})_{k=1}^{\infty}$ is an increasing fsb for E_i. We define $\lambda_{i,k} := \varepsilon_{k+i-1}$ and we choose $U_i \in \mathcal{U}(E_i)$, $U_i \subset U_{i,i}$ and $S_{i,k} \in L(E_i)$ with

$$S_{i,k}(U_i) \subset \lambda_{i,k} M_{i,k-i+1} \quad \text{for } i \le k \quad \text{and} \quad \Big(I_{E_i} - \sum_{k=i}^{n} S_{i,k} \Big)(U_i) \subset U_{i,n} .$$

Setting $\mathcal{V} := \oplus_{i=1}^{\infty} U_i$ and $\mathcal{S}_k := \oplus_{i=1}^{k} S_{i,k}$, it is easy to check that $\mathcal{S}_k \in L(E)$ are the desired mappings. □

Theorem 3

Let E be a (DF)-space with an increasing fsb $(M_k)_{k=1}^{\infty}$ complemented in $(E_b')_e'$. The following assertions are equivalent:

1. *E is a (DFO)-space.*
2. *E satisfies the following condition:*

Given any positive sequence $(\varepsilon_k)_{k=1}^{\infty}$ and any sequence $(U_k)_{k=1}^{\infty} \subset \mathcal{U}(E)$, there always exists $U \in \mathcal{U}(E)$ such that for all $n \in \mathbb{N}$ there exists $(S_k^n)_{k=1}^{n} \subset L(E)$ such that

$$S_k^n(U) \subset \varepsilon_k M_k \ , \quad \Big(I_E - \sum_{k=1}^{n} S_k^n\Big)(U) \subset U_n \qquad n \in \mathbb{N},\ k = 1, \dots, n$$

(this is the property (DFop) in [6]).

3. *$X \otimes_\varepsilon E$ ($X \tilde{\otimes}_\varepsilon E$, $X \varepsilon E$ and $L_b(X, E)$, resp.) is a (DF)-space for each Banach space X.*

Remark. The equivalence (2) ⇔ (3) was proved in [6] for the case $L_b(X, E)$. It is a consequence of the characterization of (DF)-spaces by (10) and of Lemma 1.2. The direction (1) ⇒ (3) was proved in [16], Proposition 7 and Theorem 13.

Proof. (2) ⇒ (1): Given a positive sequence $(\varepsilon_k)_{k=1}^{\infty}$ and a sequence $(U_k)_{k=1}^{\infty} \subset \mathcal{U}(E)$, we choose a new sequence $(\lambda_k)_{k=1}^{\infty}$ with $0 < \lambda_k \leq \varepsilon_k$ for all $k \in \mathbb{N}$ and $\sum_{k=n}^{m} \lambda_k M_k \subset U_n$ for all $m, n \in \mathbb{N}$ with $n < m$. By assumption there exist $U \in \mathcal{U}(E)$ and $(S_k^n)_{k=1}^{n} \subset L(E)$ such that

$$S_k^n(U) \subset \lambda_k M_k \ , \quad \Big(I_E - \sum_{k=1}^{n} S_k^n\Big)(U) \subset U_n \quad n \in \mathbb{N},\ k = 1, \dots, n \ . \tag{11}$$

We define $S_k^n := 0$ for $k > n$. Let $\mathcal{D}$ be any free ultrafilter on $\mathbb{N}$, i.e. it contains all sets $\{n \in \mathbb{N} : n_0 \leq n\}$ for each $n_0 \in \mathbb{N}$. We define $\tilde{S}_k \in L\big(E, (E_b')_e'\big)$ by setting

$$\tilde{S}_k(x) := \sigma(E'', E)\text{-}\lim_{\mathcal{D}} S_k^n(x) \qquad x \in E,\ n \in \mathbb{N} \ .$$

Since M_k is $\sigma(E'', E)$-relatively compact in E'', the linear operator $\tilde{S}_k$ is well-defined. The first relation in (11) implies $\tilde{S}_k(U) \subset \lambda_k M_k^{\circ\circ} \subset \varepsilon_k M_k^{\circ\circ}$ for all $k \in \mathbb{N}$. Let x be an element in U. It follows

$$\begin{aligned}\Big(I_E - \sum_{k=1}^{m} \tilde{S}_k\Big)(x) &= \sigma\text{-}\lim_{\mathcal{D}} \Big(I_E - \sum_{k=1}^{m} S_k^n\Big)(x) \\ &= \sigma\text{-}\lim_{\mathcal{D}} \Big(I_E - \sum_{k=1}^{n} S_k^n\Big)(x) + \sigma\text{-}\lim_{\mathcal{D}} \Big(\sum_{k=m+1}^{n} S_k^n\Big)(x)\end{aligned}$$

where the limit is taken over n and we can assume $n > m$. Since $(I_E - \sum_{k=1}^{n} S_k^n)(x) \in U_n \subset U_m$ for all $n > m$ and $(\sum_{k=m+1}^{n} S_k^n)(x) \in \sum_{k=m+1}^{n} \lambda_k M_k \subset U_m$ for all $n > m$ we get $(I_E - \sum_{k=1}^{m} \tilde{S}_k)(U) \subset 2U_m^{\circ\circ}$ for all $m \in \mathbb{N}$. By assumption E is a complemented subspace of $(E_b')_e'$ and let P be the projection onto E. We set $S_k := P\tilde{S}_k$ for all $k \in \mathbb{N}$. Then E is a (DFO)-space. □

Following Taskinen we say that a pair of Fréchet spaces (F_1, F_2) satisfies property (BB) if any bounded subset M of the complete projective tensor product $F_1\tilde{\otimes}_\pi F_2$ is localizable, i.e. there are bounded subsets $B_i \subset F_i$, $i = 1, 2$ such that $M \subset \overline{\Gamma}(B_1 \otimes B_2)$. It is well-known that, as a consequence of property (BB), $(F_1\tilde{\otimes}_\pi F_2)'_b \cong L_b\big(F_1, (F_2)'_b\big)$ topological. We say that a Fréchet space F is an (FBa)-space if (X, F) satisfies property (BB) for all Banach spaces X. The next theorem uncovers the relations between the properties (FBa), (DCO), (SDDCO) and (DFO). It is remarkable that in the context of the strong dual density condition the (DFO) property is characterizable by a finite sum of operators and we obtain the characterization by exchanging quantors in the definition of (SDDCO).

Theorem 4

Let F be a Fréchet space and let $E := F'_b$ be the strong dual of F. Then the following are equivalent:

1. *F is an (FBa)-space satisfying the density condition (DC).*
2. *F is an (FBa)-space and F'' satisfies the density condition by operator (DCO).*
3. *E is a (DFO)-space satisfying the strong dual density condition (SDDC).*
4. *E is a (DFO)-space satisfying the strong dual density condition by operator (SDDCO).*
5. *$L_b(X, E)$ is a bornological (DF)-space for each Banach space X.*
6. *E satisfies the following condition:*
   Given a positive sequence $(\lambda_k)_{k=1}^\infty$, there always exists $U \in \mathcal{U}(E)$ such that for all $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and linear operators $(Q_k)_{k=1}^m$ in E such that

$$\sum_{k=1}^{m} Q_k = I_E \qquad \text{and} \quad Q_k(M_n \cap U) \subset \lambda_k M_k\ , \qquad k = 1, \dots, m$$

   (this is property (DFo1,2) in [15]).

Proof. (3) $\Rightarrow$ (4), (5): By Theorem 3 we know that $L_b(X, E)$ is a (DF)-space for each Banach space X. Since $E = F'_b$ satisfies (SDDC), it follows that F satisfies (DC) and (X, F) has the property (BB) by Proposition 4.2.(i) in [5]. This implies $(X\tilde{\otimes}_\pi F)'_b \cong L_b(X, F'_b)$ holds topologically and $L_b(X, E)$ is the strong dual of a Fréchet space. Since the bounded sets of F'_b are metrizable, it follows that $L_b(X, F'_b)$ has a fundamental sequence of bounded subsets which are metrizable. That gives that $L_b(X, E)$ satisfies (DDC). Since $L_b(X, E)$ is the strong dual of a Fréchet space, it follows that $L_b(X, E)$ satisfies (SDDC) and is bornological for each Banach space X. By Theorem 1.6 the space E satisfies (SDDCO).

(5) $\Rightarrow$ (1): Since $L_b(X, E)$ is bornological for each Banach space X, it follows that $L_b(l_1, E) \cong l_\infty(E)$ is bornological and Theorem 1.4. in [1] gives that E satisfies (SDDC). Hence F satisfies (DC). Proposition 4.2.(i) in [5] gives again that the pair (X, F) has the property (BB) for an arbitrary Banach space X and F is an (FBa)-space.

(1) $\Rightarrow$ (3): Clearly, E satisfies the strong dual density condition (SDDC). If (X, F) has the property (BB), then we have $(X\tilde{\otimes}_\pi F)'_b \cong L_b(X, F'_b)$ holds topologically and $L_b(X, E)$ is a (DF)-space for each Banach space X. Hence E is a (DFO)-space.

Since the direction (4) $\Rightarrow$ (3) is trivial, we get the equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

(1) $\Leftrightarrow$ (2): Since the (DC) of F'' implies the (DC) of F, we have only to prove the direction $\Rightarrow$. Let F be an (FBa)-space satisfying (DC). By (1) $\Rightarrow$ (3) E is a (DFO)-space, i.e.

$$\forall(\varepsilon_k)_{k=1}^\infty \text{ pos.} \quad \forall(U_n)_{n=1}^\infty \subset \mathcal{U}(E) \quad \exists U \in \mathcal{U}(E) \quad \exists(S_k)_{k=1}^\infty \subset L(E):$$
$$S_k(U) \subset \varepsilon_k M_k \quad \text{and} \quad \Big(I_E - \sum_{k=1}^m S_k\Big)(U) \subset U_m \quad k, m \in \mathbb{N}\ . \tag{12}$$

By assumption E has (SDDC), it follows that E is quasibarrelled. We get for $F'' = E'_b$ by dualization of (12):

$$\forall(\varepsilon_k)_{k=1}^\infty \text{ pos.} \quad \forall(B_n)_{n=1}^\infty \subset \mathcal{B}(F'') \quad \exists B \in \mathcal{B}(F'') \quad \exists(Q_k)_{k=1}^\infty \subset L(F''):$$
$$Q_k(V_k) \subset \varepsilon_k B \quad \text{and} \quad \Big(I_{F''} - \sum_{k=1}^m Q_k\Big)(B_m) \subset B \quad k, m \in \mathbb{N} \tag{13}$$

where $(V_k)_{k=1}^\infty = (M_k^\circ)_{k=1}^\infty$ is a basis of 0-neighborhoods in F'' and $Q_k := S'_k$ are the adjoint operators.

Now, let $(\lambda_k)_{k=1}^\infty$ be a positive sequence and let $n_0 \in \mathbb{N}$. Since F'' also has (DC) and the bounded set in the Definition 1.1.(1) can be chosen not depending on n, see [10], 1.4., we obtain

$$\exists B_0 \in \mathcal{B}(F'')\ \forall n \in \mathbb{N}\ \exists m \in \mathbb{N}: \bigcap_{k=1}^m \lambda_k V_k \subset V_n + B_0\ . \tag{14}$$

We set $B_k := kB_0$. By (13) we find $B \in \mathcal{B}(F'')$ and $(Q_k)_{k=1}^\infty \subset L(F'')$ such that

$$Q_k(V_k) \subset 2^{-k}\lambda_k^{-1} B\ , \quad \Big(I_{F''} - \sum_{k=1}^{m'} Q_k\Big)(B_0) \subset \frac{1}{m'} B \quad k, m' \in \mathbb{N}\ . \tag{15}$$

We choose m' with $\frac{1}{m'}B \subset \frac{1}{2}V_{n_0}$. There exists an $n' \in \mathbb{N}$ with $(I_{F''} - \sum_{k=1}^{m'} Q_k)(V_{n'}) \subset \frac{1}{2}V_{n_0}$. Now, we find an $m'' \geq m'$ such that (14) is satisfied for n', i.e.

$$\bigcap_{k=1}^{m''} \lambda_k V_k \subset V_{n'} + B_0 \ . \tag{16}$$

We set $P := \sum_{k=1}^{m'} Q_k \ \in \ L(F'')$. By (15) it follows that

$$P\Big(\bigcap_{k=1}^{m''} \lambda_k V_k\Big) \subset \Big(\sum_{k=1}^{m'} Q_k\Big)\Big(\bigcap_{k=1}^{m'} \lambda_k V_k\Big) \subset \sum_{k=1}^{m'} \lambda_k Q_k(V_k) \subset B$$

and P is a bounded operator. Further, (15) and (16) imply

$$(I-P)\Big(\bigcap_{k=1}^{m''} \lambda_k V_k\Big) \subset (I-P)(V_{n'}) + (I-P)(B_0) \subset \frac{1}{2}V_{n_0} + \frac{1}{m'}B \subset V_{n_0}$$

and as desired F'' has (DCO).

(4) $\Rightarrow$ (6): By assumption $X \otimes_\varepsilon E$ is a (DF)-space satisfying (SDDC). The assertion follows by the remark after the Theorem 1.6.

(6) $\Rightarrow$ (3): An idea for the proof of this can be found in [15], Theorem 4.2. There is another argument. Let us be given $(\lambda_k)_{k=1}^\infty$ positive sequence and $(U_k)_{k=1}^\infty \subset \mathcal{U}(E)$. Define a new sequence $(\varepsilon_k)_{k=1}^\infty$ with $0 < \varepsilon_k < \lambda_k$ such that $\sum_{k=p}^{q} \varepsilon_k M_k \subset \ U_p$ for all $p, q \in \mathbb{N}$ with $p < q$, where $(M_k)_{k=1}^\infty$ is an increasing fsb of E such that each set M_k is absolutely convex and closed. By assumption

$$\exists U \in \mathcal{U}(E) \quad \forall n \in \mathbb{N} \quad \exists m(n) \in \mathbb{N} \quad \exists (Q_k^n)_{k=1}^{m(n)} \text{ lin. op.}:$$
$$\sum_{k=1}^{m(n)} Q_k^n = I_E \quad \text{and} \quad Q_k^n(M_n \cap U) \subset \varepsilon_k M_k \ , \quad k = 1, \ldots, m(n) \ .$$

We set $Q_k^n := 0$ for $k > m(n)$. Let $\mathcal{D}$ be a free ultrafilter on $\mathbb{N}$. Now we define a linear continuous operator by $Q_k(x) := \sigma(E,F)\text{-}\lim_{\mathcal{D}} Q_k^n(x)$ and it follows $Q_k(U) \subset \varepsilon_k M_k \subset \lambda_k M_k$ for all $k \in \mathbb{N}$, remark that M_k is $\sigma(E,F)$-compact. Fix $x \in U$. Then

$$\Big(I_E - \sum_{k=1}^{p} Q_k\Big)(x) = \sigma\text{-}\lim_{\mathcal{D}} \Big(I_E - \sum_{k=1}^{p} Q_k^n\Big)(x) = \sigma\text{-}\lim_{\mathcal{D}} \Big(\sum_{k=p+1} Q_k^n\Big)(x)$$
$$\subset \sum_{k=p+1} \varepsilon_k M_k \subset U_p$$

and this shows the required condition for E to be a (DFO)-space. □

Remarks. (1) The equivalences (1) $\Leftrightarrow$ (3) (with (DFop)) $\Leftrightarrow$ (5) can be found in [6], too.

(2) In the above proof we have shown, that an (FBa)-space satisfying (13) has always (DCO). A. Peris and M. J. Rivera proved that an (FG)-space satisfying (DC) has (DCo), see [16], Proposition 9. If the space is complemented in its strong bidual, then (DCo) is equivalent to (DCO) by Proposition 1.3. Moreover, it is unknown if the property (13) is equivalent to the (FG) property.

Example 5: (1) Let $F := \lambda_p(A)$ be a Köthe echelon space of order p with $1 \leq p \leq \infty$. By [9], Theorem 3.1 F is an (FBa)-space. Using Theorem 4 F satisfies (DCO) if and only if F has (DC). Recall that $\lambda_1(A)$ is a complemented subspace of $\lambda_1(A)''$ if $\lambda_1(A)$ has (DC) and that $\lambda_\infty(A)$ is the strong bidual of the (FBa)-space $\lambda_0(A)$. Let $E := \lambda_p(A)'_b$. Then E is a (DFO)-space and we conclude that (SDDC) and (SDDCO) are equivalent conditions for E.

(2) A locally convex space E is called a (gH)-space if its topology can be defined by a family of Hilbertian seminorms. Then we can choose an fsb $(M_k)_{k=1}^\infty$ such that each M_k is a Hilbert disc, see [13], Corollary 3.8. Now, let E be a reflexive (gH)-(DF)-space. By R. Hollstein [12], Proposition 3.3 we have for a Banach space X

$$X \otimes_\varepsilon E \; = \; \mathrm{ind}_k (X \otimes_\varepsilon E_{M_k}) \; ,$$

where E_{M_k} is the normed space associated to the Hilbert disc M_k. It follows that $X \otimes_\varepsilon E$ is a bornological (DF)-space for each Banach space X and by Theorem 4 E is a (DFO)-space.

(3) Let $F := l_{p^+} = \cap_{k=1}^\infty l_{p+\frac{1}{k}}$ with $1 \leq p < \infty$. By A. Peris F is not an (FBa)-space but $E := F'_b$ satisfies (SDDCO), see Example 1.7.(1). Using Theorem 4 it turns out that we cannot exchange the quantors "$\forall n$" and "$\exists U$" in the definition of (SDDCO) contrary to the situation for (SDDC).

(4) Let F be the standard quojection of Moscatelli type as in Example 1.7.(2). We set again $X := l_\infty$ and $Y := c_0$. Then F is an (FBa)-space satisfying (DC). By Example 1.4.(4) F does not satisfy (DCO), but F'' is a countable product of Banach spaces and F'' satisfies (DCO). This is contrary to the situation for (DC). For a Fréchet space G we have always the implication: G'' satisfies (DC) $\Rightarrow$ G satisfies (DC).

(5) Let $E := L_b(\lambda_1(A), l_{q^-})$ where $\lambda_1(A)$ is not quasinormable and $1 < q < \infty$. Since $l_{q^-} = (l_{p^+})'_b$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $(\lambda_1(A), l_{p^+})$ has property (BB), see [2], it follows $E \cong \left(\lambda_1(A) \tilde{\otimes}_\pi l_{p^+}\right)'_b$. Thus E is a dual of a Fréchet space, but not (DFO) and not satisfying (s.M.c.). Problem: Is (SDDC) equivalent to (SDDCO) for E?

Proposition 6

Let E be a complete (DF)-space complemented in the strong bidual E'' with a fsb $(M_k)_{k=1}^{\infty}$ such that $E_k := E_{M_k}$ are Banach spaces for all $k \in \mathbb{N}$. Then are equivalent:

1. *E is (DFO)-space satisfying the strong dual density condition (SDDC).*
2. *The canonical mapping $L_b(X, \bigoplus_{k=1}^{\infty} E_k) \longrightarrow L_b(X, E)$ is a topological surjection for each Banach space X.*

Proof. (1) $\Rightarrow$ (2): Since E is a (DFO)-space satisfying (SDDC), it follows that $X \otimes_{\varepsilon} E$ is a (DF)-space satisfying (SDDC) for all Banach spaces X. By the remark after Theorem 1.6 the property (8) is valid. The rest is due to [15], the proof of Theorem 4.2.

(2) $\Rightarrow$ (1): By assumption $L_b(X, E)$ is a quotient of the bornological (DF)-space $L_b(X, \bigoplus_{k=1}^{\infty} E_k) = \bigoplus_{k=1}^{\infty} L_b(X, E_k)$. Then $L_b(X, E)$ is a bornological (DF)-space, too. By Theorem 3 it follows that E is a (DFO)-space. Since $l_{\infty}(E) = L_b(l_1, E)$ is bornological, Theorem 1.5. in [1] gives that E satisfies (SDDC). □

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