Collect. Math. **48**, 3 (1997), 303–314

On the dimension of ordered spaces

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Received December 20, 1995. Revised March 28, 1996

Abstract

In this paper, we prove that, for every ordered space (ordered set with its order topology i.e. with the topology generated by the family of all intervals $]\leftarrow, a[$ and $]a, \rightarrow[$) and more generally for every line (space homeomorphic to a subspace of an ordered space and called in (7) generalized ordered space), the small inductive dimension (*ind*), the large inductive dimension *(Ind),* the covering dimension *(dim)* and the nonstandard definition or thickness *(ep)* coincide. More precisely, we prove, that for every line $X \neq \emptyset$, we have:

1) *ep X = ind X = Ind X = dim X = 0* if and only if *X* is totally disconnected,
2) *ep X = ind X = Ind X = dim X = 1* if and only if *X* is not totally disconnected.

Introduction

We will consider the following four definitions of the topological dimension of a Hausdorff space: three classical - the small inductive dimension *(ind),* the large inductive dimension *(Ind)* and the covering dimension *(dim)* - and a nonstandard definition or thickness (2) (denoted by *ep* for épaisseur in French). We recall that, if we except the case of separable metric spaces, these four definitions do not necessarily coincide: thus, there exists (8) a non separable metric space X such that $indX = 0$ (and therefore (2) such that $ep\ X = 0$) and $IndX = dimX = 1$, and (2) a (non metric) compact space X such that $indX = Ind\ X < epX$. We recall lastly that every space X such that $indX = 0$ is totally disconnected (i.e. such that each connected component is a one-point set) but there exists (9) a totally disconnected space (which is not compact Hausdorff of course) such that $indX > 0$.

About ordered spaces, if we except a statement of H. Herrlich (see Remark 5.1.4), it seems there is no general result. Here, we prove that, for every non empty ordered space and more generally for every non empty line X, we have:

1) $epX = indX = IndX = dimX = 0$ if and only if X is totally disconnected,
2) $epX = indX = IndX = dimX = 1$ if and only if X is not totally disconnected.

Terminology: Let X be an ordered set.

We will call on the one hand *interval* of X any non empty subset I of X such that:

$$\forall x \in I \quad \forall y \in I \quad \forall z \in X, \quad x \leq z \leq y \Longrightarrow z \in I.$$

We will say on the other hand that an interval I of X is *unlimited* on the left (resp. on the right) if and only if:

$$\forall x \in I \quad \forall y \in X, \quad y \leq x \text{ (resp. } y \geq x) \Longrightarrow y \in I.$$

1. Notion of line

We recall that:

- an ordered space is a hereditarily normal space (see for example (10)) but is not necessarily a paracompact space and therefore not necessarily a Lindelöf space as shown by the example of the ordinal space $\Omega = [0, \omega_1[$.
- If A is a subset of an ordered space X, the induced topology on A is finer than the order topology on A.

Let us note that, if A is not an interval, it might happen that these two topologies are different as shown by the example where $X = \mathbb{R}$ and $A = [0, 1[\cup\{2\}$: A is compact Hausdorff for the order topology (because it is homeomorphic to $[0, 1]$) but A is not a compact Hausdorff subspace of $\mathbb{R}$, whence the following definition.

DEFINITION 1.1. We will say that a topology on an ordered set X is *linear* if it is finer than the order topology on X and moreover generated by a set of left or right unlimited intervals, and we will call *line* every ordered set with a linear topology.

It follows from this definition that every ordered space and every subspace of an ordered space are lines. Conversely, one can prove (see (3) or (7)) that every line is homeomorphic to a subspace of an ordered space. Let us note this result implies that every line is a normal space.

DEFINITION 1.2. We will say that an interval of a line X *is open* (resp. *closed*) if it is an open (resp. closed) subset of X.

Proposition 1.3

Every connected subset of a line is an interval.

Proof. Let C be a subset of a line X and let us suppose C is not an interval. There exists then (x, y, z) such that $x < z < y$, $(x, y) \in C \times C$ and $z \notin C$. Since C is the union of the two non empty disjoint open subsets $C \cap\,]\leftarrow, z[$ and $C \cap\,]z, \rightarrow[$, C is not a connected subset of X. $\square$

2. Constituents of a subset

Let A be a subset of an ordered set X. Let us denote by $\mathcal{I}(A)$ the non empty set ($\phi \in \mathcal{I}(A)$) consisting of all intervals of X contained in A. Thus, $\mathcal{I}(A)$ is a partially ordered set such that every chain in $\mathcal{I}(A)$ has an upper bound, whence the definition: We will call *constituents* (or convex components) of A the maximal non empty elements of $\mathcal{I}(A)$.

Let us note it follows from this definition that the set of all constituents of A is a partition of A and that every non empty interval contained in A is contained in one and only one constituent of A.

Proposition 2.1

Let A be a subset of a line X. Then:

a) A is an open subset of X if and only if each constituent of A is an open subset of X.

b) If A is a closed subset of X, each constituent of A is also a closed subset of A.

Proof. a) The condition is obviously sufficient since every union of open subsets is an open subset. Let us prove now it is also necessary. Let I be a constituent of A and x be an element of I. Since A is open, there exists an open interval J such that $x \in J$ and $J \subset A$. Since $I \cup J$ is an interval such that $I \subset I \cup J \subset A$ and since I is maximal, we have $I \cup J = I$ and therefore $J \subset I$. Consequently, I is an open subset of X.

b) Let I be a constituent of A and let $x \in \overline{I}$. Since A is a closed subset, x belongs to A. Consequently $I \cup \{x\}$ is an interval of X contained in A and therefore such that $I \cup \{x\} = I$, which implies I is a closed subset of X.

Let us note that it might happen that all constituents of A are closed but A is not closed as shown by the following example where $X = \mathbb{R}$ and $A = \mathbb{Q}$: $\mathbb{Q}$ is not a closed subset of $\mathbb{R}$ but all constituents of $\mathbb{Q}$ are closed since these are one-point sets. □

Proposition 2.2

Let C be a non empty connected subset of a line X.

If C is contained in an open subset O of X, then C is contained in one and only one constituent of O.

It is an immediate consequence of the definition of constituents and of 1.3.

Corollary 2.3

Every connected subset of a line X has a neighborhood base consisting of open intervals.

3. Borders of an interval

Definition 3.1. Let I be an interval of a line X. We will call *borders* of I the elements of $\overline{I} \setminus I$, when they exist.

Let us note that an interval is without border if and only if it is closed.

Proposition 3.2

Every interval I of a line X has at most two borders, one on the left, the other on the right.

Proof. Let us suppose $x < y < I$. Then $]\leftarrow, y[\, \cap I = \emptyset$ and therefore x is not a left border of I. In the same way, if $I < y < x, x$ is not a right border of I. □

Examples. Let us suppose that $X = \mathbb{Q}$, $I =]0, 2[$, $J = \{r \in \mathbb{Q} : r^2 < 2\}$.

Then I has two borders 0 and 2, J has no border, while $I \cap J$ has a left border but no right border.

Proposition 3.3 ("Shortening" of an interval.)

Let X be a line, I be an interval of X, y be a left (resp. right) border of I, and J be an open-closed neighborhood of y. Then $I \setminus J$ is an interval without left (resp. right) border.

Proof. Let us note it follows from the hypothesis that $I \setminus J$ is indeed an interval. Let us suppose that $I \setminus J$ has a left border z. Then $z \in (\overline{I \setminus J}) \setminus (I \setminus J)$. Since I is open, we have $\overline{I \setminus J} \subset \overline{I} \setminus J$, which implies that $z \in (\overline{I} \setminus J) \setminus (I \setminus J)$ and consequently $z \in (\overline{I} \setminus I) \setminus J$. It follows then from 3.2 that necessarily $y = z$, which is impossible because $y \in J$. □

4. Meatus of a line

Definition 4.1. We will call *meatus* of an ordered set X any pair (I, J) of complementary intervals such that $I < J$ (i.e. such that : $\forall x \in I \quad \forall y \in J,\ x < y$).

Among the meatus, we will distinguish:

– the improper meatus: these are the meatus (ϕ, X) and (X, ϕ),
– the proper meatus and among these:
  a) the *gaps*: these are the meatus (I, J) where I and J are not empty, I having no last element and J no first element.
  b) the *holes*: these are the meatus (I, J) where I has a last element and J a first element.
  c) the *left faults*: these are the meatus (I, J) where I is not empty but without last element and J has a first element.
  d) the *right faults*: these are the meatus (I, J) where I has a last element and J is not empty but without first element.

Let us note it follows from these definitions that:

– In an ordered set, every element x defines a left fault $(]\leftarrow, x[, [x, \rightarrow[)$ and a right fault $(]\leftrightarrow, x],]x, \rightarrow[)$.
– In $\mathbb{R}$, there are no holes and no gaps.

Definition 4.2. We will say that a meatus (I, J) of a line X is open if I and J are both open (and therefore both closed) for the linear topology on X.

Remark 4.3. It follows from these definitions that improper meatus, gaps and holes are always open meatus but left and right faults are not necessarily open: thus, no fault is open for the order topology. Moreover, let us note that, if (I, J) is a left fault (resp. a right fault) of a line X, the interval I (resp. J) is always open.

Lemma 4.4.1

Let $X = [a, b]$ be a line with a first element a and a last element b. If X is not a connected space, there exists a partition of X in two closed subsets A and B such that $a \in A$ and $b \in B$.

Proof. Since X is not connected, there exists a partition of X in two closed subsets F and G. Let us suppose $a \in F$.

First case: $b \in G$. We set then $A = F$ and $B = G$.
Second case: $b \in F$. Let then c be an element of G (such an element exists since $G \neq \emptyset$.) We have $a < c < b$ and we set $A = F \cap [a, c]$ and $B = G \cup [a, b]$. A and B are then two non empty disjoint closed subsets of X such that $A \cup B = X, a \in A$ and $b \in B$. □

Lemma 4.4.2

Let $X = [a, b]$ be a line with a first element a and a last element b. If X is not a connected space, there exists a partition of X in two complementary open-closed intervals $I < J$, i.e. an open proper meatus (I, J) of X.

Proof. It follows from 4.4.1 there exists a partition of X in two closed subsets A and B such that $a \in A$ and $b \in B$. Let us consider then the following intervals $I = \bigcup_{x \in A}[a, x]$ and $J = X \setminus I$. These two intervals are complementary and such that $I < J$, $A \subset I$ and $J \subset B$. Let us prove they are open-closed subsets of X. This result is obvious if the meatus (I, J) is a gap or an hole. Let us suppose now (I, J) is a fault.

i) If (I, J) is a right fault, I has a last element s and therefore $I = [a, s]$. Consequently $s \in A$ and therefore I is closed. Since $s \notin B$, we have $J = B \cap [s, b]$, which implies that J is also closed.

ii) If (I, J) is a left fault, I is, from 4.3, an open subset of X. Let us prove J is also an open subset of X. Let t be the first element of J. Since $t \notin A$ (because $J \subset B$) and since A is a closed subset of X, there exists an open neighborhood V of t such that $V \cap A = \emptyset$. Consequently, we have $V \subset J$ and therefore J is open. □

DEFINITION 4.4.3. We will say that a line X is *disconnected* between two subsets A and B if there exists an interval $[x, y]$ which is not connected and has an extremity in A and the other in B.

Lemma 4.4.4

If a line X is disconnected between to subsets A and B, there exists an open meatus (U, V) of X such that one of the two intervals U and V meets A, and the other meets B.

Proof. Since X is disconnected between A and B, there exists (exchanging if necessary A and B) an interval $[x, y]$ which is not connected and such $x \in A$ and $y \in B$. It follows from 4.4.2, there exist two complementary intervals I and J of $[x, y]$ such that $I < J$, $x \in I$ and $y \in J$. Let us put then $U =]\rightarrow, x] \cup I$ and $V = J \cup [y, \rightarrow[$. These two intervals are closed and complementary in X and such that $U < V$, $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. □

Proposition 4.4

A line X is a connected space if and only if its only open meatus are its improper meatus.

Proof. The condition is obviously necessary. It is also sufficient. Indeed, if X is not a connected space, X is disconnected between two points x and y. Consequently, from 4.4.4, there exists an open meatus (U, V) such that $x \in U$ and $y \in V$ and therefore a proper meatus. □

Proposition 4.5

A non empty ordered space X is compact Hausdorff if and only if X has a first and a last element and has no gaps.

Proof. i) Let us suppose X is compact Hausdorff. Then $\bigcap_{x\in X}] \leftarrow, x]$ and $\bigcap_{x\in X}[x, \rightarrow [$ are one-point sets, so that X has a first and a last element. Let now $s = (I, J)$ be a proper open meatus of X. Then I and J are compact Hausdorff subsets of X which implies, in particular, that I has a last element and J has a first element, so that s in a hole.

ii) Conversely, let us suppose X has a first element a and a last element b, but is not compact. There exists then a set $\mathcal{E}$ consisting of open intervals covering X and such that no finite subset of $\mathcal{E}$ covers X. Let then A be the set of all points of X which are connected to a by a finite chain of intervals $U_1, \ldots, U_n$ belonging to $\mathcal{E}$ and such that $a \in U_1$, $U_i \cup U_{i+1}$ is an interval for every $i \in \{1, \ldots, n-1\}$, and $x \in U_n$. By construction $a \in A$, $b \notin A$ and A is an open-closed interval. Consequently, $s = (A, X \setminus A)$ is a proper open meatus of X. It is not a hole because, if not, $X \setminus A$ would have a first element c and this element c could be connected to a. Consequently, s is a gap. □

Remark 4.6. It follows from the definition of a linear topology that every compact Hausdorff line is necessarily a compact Hausdorff ordered space.

5. Characterization of zero-dimensional lines

Let X be a non empty topological space. We recall that:

(1) $IndX = 0$ if and only if every closed subset of X has a neighborhood base consisting of open-closed subsets of X,

(2) $indX = 0$ if and only if every point of X has a neighborhood base consisting of open-closed subsets of X,

(3) X is *totally disconnected* if and only if all connected components of X are one-pointsets,

(4) X is *punctiform* if and only if X does not contain any continuum (a connected and compact Hausdorff subset) of cardinality larger than one.

We recall also it follows from classical results in dimension theory (see for example (5)) that if:

a) X is a Hausdorff space, we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4)

b) X is a locally compact Hausdorff space, we have (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4),

c) X is a Lindelöf Hausdorff space, we have (1) $\Longleftrightarrow$ (2),

d) X is a compact Hausdorff space, we have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4).

Theorem 5.1

For every non empty line X, the following assertions are equivalent:

(1) $IndX = 0$, (2) $indX = 0$,

(3) X is totally disconnected, (4) *X is punctiform.*

Proof. Since every line is a Hausdorff space, it suffices to prove (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

Let us note that, since a line is neither necessarily a locally compact space, nor necessarily a Lindelöf space, we cannot use the previous results.

5.1.1 *Every non empty punctiform line is totally disconnected.*

Proof. Let C be a connected component of X and $x \leq y$ two points of C. It follows from 4.4 and 4.5 that $[x, y]$ is a continuum. Consequently, since X is punctiform, we have $x = y$ and therefore C is a one-point set.

5.1.2 *If X is a non empty totally disconnected line, then $indX = 0$.*

This assertion is an obvious consequence of the following lemma:

Lemma

In a line X, every connected component has a neighborhood base consisting of open-closed intervals.

Proof of the lemma: Let C be a connected component of X and O be an open neighborhood of C. It follows from 2.3 there exists an open interval I such that $C \subset I \subset O$.

If I is without border, I is closed and all is said.

If not, let us suppose I has a border y, on the left for example.

Since $y \notin I$ and therefore $y \notin C$, $C \cup \{y\}$ is not a connected subset of X so that X is disconnected between y and C. Consequently, from 4.4, there exist two open-closed complementary intervals U and V of X such that $U < V, y \in U$ and $V \cap C \neq \emptyset$. Since C is connected, we deduce from this that $U \cap C = \emptyset$ and therefore $C \subset V$. Consequently, $V \cap I$ is an open interval containing C and contained in O and without border. We shorten also, if necessary, $V \cap I$ on the right. We obtain, in this way, an open-closed (because without border) interval containing C and contained in O. □

5.1.3 *If X is a non empty line such that $indX = 0$, then $IndX = 0$.*

Proof. Let F be a closed subset of X and U be an open neighborhood of F. It follows from 2.1 that U is the union of all its constituents (which are open intervals). Let then I be a constituent of U such that $I \cap F \neq \emptyset$.

i) If I is without border, I is closed and we keep it without further modification.

ii) If not, let us suppose I has a border y, on the left for example. Since $y \notin U$ and therefore $y \notin F$, and since $indX = 0$, there exists an open-closed interval J_y containing y and such that $J_y \cap F = \emptyset$. We consider then the « shortened »interval $I \setminus J_y$. It follows from 3.3 that $I \setminus J_y$ is without left border. We shorten also, if necessary, on the right. In this way, we replace I by an open interval L, without border and therefore also closed, such that $L \subset I$ and $F \cap L = F \cap I$.

Let V be the union of all those intervals L thus obtained. Then V is an open subset of X such that $F \subset V \subset U$. Let us prove V is also a closed subset of X. Let $x \in X \setminus V$. Since $x \notin F$, there exists an open interval J such that $x \in J$ and $J \cap F = \emptyset$. This interval J meets at most two intervals L (indeed, if not, it would contain one of these and consequently would meet F.) Since the union A of these two intervals L is a closed subset which does not contain x, there exists then a neighborhood W

of x such that $W \subset J$ and $W \cap A = \emptyset$ and therefore such that $W \cap V = \emptyset$, which implies $W \subset X \setminus V$. Consequently, $X \setminus V$ is an open subset of X and therefore V is a closed subset of X. □

Remark 5.1.4. In (6), H. Herrlich gives a similar result for ordered space but his proof is not very convincing. Indeed, he claims, without any justification, that, if an open interval I contains a closed subset F, there exists an open-closed interval J such that $F \cap I \subset J \subset I$. Now, the existence of such an interval J is not quite obvious. Indeed, his J is our L.

6. The coincidence theorem for lines

Lemma 6.1

In a line, every connected subset of cardinality larger that one has a non empty interior.

Proof. Let $a < b$ be two points of a connected subset C of a line X. Since $[a, b]$ is connected Hausdorff, there exists a point c of X such that $c \in]a, b[$. Consequently the interior of C is not empty. □

Lemma 6.2

For every open subset U of a line X, we have $Ind(FrU) = 0$ (where FrU denotes the boundary of U in X).

Proof. Since the interior of $F = FrU$ is empty, it follows from 6.1, that every connected subset of F is of cardinality at most one. Consequently, F is totally disconnected, which implies $IndF = 0$ from 5.1. □

Proposition 6.3

For every line X, we have $ep\ X \leq 1$ and $IndX \leq 1$ and therefore $indX \leq 1$ and $dimX \leq 1$.

Proof. i) Let $\mathcal{B}$ be the base of X consisting of all open intervals. Since $ep\ \mathcal{B} \leq 1$, we have $ep\ X \leq 1$ (see (2)).

ii) It follows immediately from 6.2 that $IndX \leq 1$, which implies $indX \leq 1$. Moreover, since X is a normal space, we have (see for example (5)) $dimX \leq IndX$ and therefore $dimX \leq 1$. □

Theorem 6.4 The coincidence theorem

For every line X, we have ep $X = indX = IndX = dimX \in \{-1, 0, 1\}$. More precisely, if $t(X)$ denotes the common value of these four dimensions, we have:
1) $t(X) = -1$ if and only if X is empty,
2) $t(X) = 0$ if and only if X is totally disconnected,
3) $t(X) = 1$ if and only if X is not totally disconnected.

Proof. 1) because each for these four dimensions is equal to -1 if and only if X is empty.

2) On one hand, it follows from 5.1 that the assertions « X is totally disconnected », « $indX = 0$ » , and « $IndX = 0$ » are equivalent. On the other hand, we know that, for every topological space Y, the assertions « $indY = 0$ » and « $ep\ Y = 0$ » are equivalent (see (2)) and that, for every normal space Z, the assertions « $IndZ = 0$ » and « $dimZ = 0$ » are equivalent (see for example(5)), whence the result.

3) This assertion is an obvious consequence of 2) and 6.3. □

Remark 6.5. In another paper (3), we have proved, that for every normal space X, if we denote by $\mathcal{U}$ the uniform structure on X induced by the only uniform structure on the Stone-$\check{C}$ech compactification $\beta(X)$ of X and by $\mu dim(X, \mathcal{U})$ the Alexandroff dimension of the uniform space $(X, \mathcal{U})$,we have $\mu dim(X, \mathcal{U}) = dimX$. Consequently, since every line X is a normal space, we have also, with the previous notations $t(X) = \mu dim(X, \mathcal{U})$.

Acknowledgment. I express my deep gratitude to Professor Labib Haddad for many valuable suggestions.

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