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One-parameter family of cubic Kolmogorov system with an isochronous center

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Abstract

We show the existence of an one-parameter family of cubic Kolmogorov system with an isochronous center in the realistic quadrant.

1. Introduction

A center is said to be isochronous if all periodic orbits in a neighborhood of it have the same period. Recently, several papers have been devoted to the study of conditions on polynomial systems so that its centers are isochronous [4], [3], [5].

The autonomous differential system on the plane given by

$$\begin{cases} \dot{x} = x\ F(x,y) \\ \dot{y} = y\ G(x,y) \end{cases} \tag{1}$$

known as Kolmogorov system, is frequently used to model the interaction of two species occupying the same ecological niche.

If F and G are linear (Lotka-Volterra-Gause model), then it is well known that there is at most one critical point in the interior of the realistic quadrant ($x \geq 0$,

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$y \geq 0$), and that there are no limit cycles. In particular, in this situation periodic solutions can only occur nested around a center.

For the classical Lotka-Volterra system, J. Waldvogel [7] proved the monotonicity of the period function and, in particular, the non-existence of isochronous centers. For the general case of quadratic Kolmogorov System we show (by use of the classification given in [2]) the non-existence of isochronous centers (Theorem A).

In the case that F and G are both quadratic, that is, for cubic Kolmogorov system, one might think by analogy that the behavior within the first quadrant is similar to that of a quadratic Kolmogorov system. In this paper we show that this is not the case (Theorem B).

A technique to prove the isochronicity of some centers is the Urabe's Criteria [6]. To apply these criteria we must find two functions $\phi, \psi \in C^2$ such that the system (1) can be transformed, with the change of variables

$$x = \phi(u) \quad \text{and} \quad u = \psi(t)\,,$$

into an equivalent equation of the form

$$\ddot{u} + g(u) = 0\,. \tag{2}$$

If $g'(0) = 1$, then all the solutions of (2) are periodic of period 2π if and only if

$$\frac{1}{2}\,X^2 \;=\; \int_0^u g(\xi)\,d\xi \Longleftrightarrow u \;=\; X + F(X)\,,$$

where F(X) is an even function of X and $sgn(X) = sgn(u)$.

2. Main results

Theorem A

If F and G are linear, then the system (1) has no isochronous centers.

Proof. If the point (x_0, y_0) is a center of (1), then there exists a non-singular linear transformation $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Psi^{-1}(x_0, y_0) = (0, 0)$, such that (1) is conjugate with a system of the form:

$$\begin{cases} \dot{u} = -v \;+\; a_{20}u^2 + \;a_{11}uv \;+ a_{02}v^2 \\ \dot{v} = u \;+\; b_{20}u^2 + \;b_{11}uv \;+ b_{02}v^2\,. \end{cases} \tag{3}$$

In [2] Loud proved that the system of the form (3) has an isochronous center at the origin only in four cases. Moreover, in [4] the first rational integrals are shown. It follows that for each case there exists at most one invariant straight line as level curve. As Ψ preserve the invariant straight lines of (1), we conclude that the point (x_0, y_0) cannot be an isochronous center. □

Theorem B

Let us consider the one real parameter family of cubic Kolmogorov system

$$\begin{cases} \dot{x} = x\left(1-a-2\,x\,(1-a)-a\,(x^2-y^2)\right) \\ \dot{y} = y\left(-(1-a)+2\,y\,(1-a)-a\,(x^2-y^2)\right). \end{cases} \tag{4}$$

Then, for $a > 1$, the point (1/2,1/2) is an isochronous center of period $\frac{2\pi}{\sqrt{a-1}}$

Proof. The system (4) has a first rational integral

$$H_a(x,y) \;=\; \frac{(a-1)\,(1-2(x+y))+a\,(-x+y)^2}{a\,(a-1)\,(-1+2(x+y)+a\,(-x-y+1)^2)}\,. \tag{5}$$

For $c \in]-1/a, -1/a+\epsilon[\ ,\ 0 < \epsilon \ll 1$, the level curves $H_a(x,y) = c$ are ellipse and for $c = -1/a$ we have $H_a^{-1}(-1/a) = \{(1/2,1/2)\}$. It follows then that the singularity $(1/2,1/2)$ is a center.

If we consider the conjugation

$$\Psi(X,Y) = \frac{1}{2}\left(-X-\sqrt{a-1}\,Y+1, -X+\sqrt{a-1}\,Y+1\right),$$

then (4) is conjugate to the system

$$\begin{cases} \dot{X} = \sqrt{a-1}\,Y\,(-1+2\,X-a\,X^2) \\ \dot{Y} = \sqrt{a-1}\,(X-X^2+Y^2-a\,X\,Y^2)\,. \end{cases} \tag{6}$$

By rescaling the time

$$t = \frac{1}{\sqrt{a-1}}\,T, \tag{7}$$

the system (6) is transformed into the below system (in which we have replaced the capital letters T, X and Y by t, x and y, respectively).

$$\begin{cases} \dot{x} = y\,(-1+2\,x-a\,x^2) \\ \dot{y} = \ x-x^2+y^2-a\,x\,y^2\,. \end{cases} \tag{8}$$

Firstly, we try to find an appropriate change of coordinates, to transform the system into a second order differential equation of the form (2). For this, consider $x = \phi(u)$ with $u = u(t)$, both of class C^2. Replacing at the first equation of (8) we obtain

$$\phi'(u)\,\dot u = -y\,\big(1 - 2\,\phi(u) + a\phi^2(u)\big)$$

where $\phi'(u)$ is the derivative of ϕ with respect to u and $\dot u$ is the derivative of u with respect to the time t.

By the analyticity of the period function (see [4]), is sufficient to consider a small neighborhood $\mathcal{V}$ of the origin such that $1 - 2\,\phi(u) + a\phi^2(u) > 0$.

Taken

$$y = \frac{\phi'(u)\,\dot u}{-1 + 2\,\phi(u) - a\,\phi^2(u)}$$

in the second equation of (8), we get

$$\begin{aligned}&\big(-1 + 2\,\phi(u) - a\,\phi^2(u)\big)\,\phi'(u)\,\ddot u + \big[\big(-1 + 2\,\phi(u) - a\,\phi^2(u)\big)\,\phi''(u) -\\ &3\,\big(1 - a\,\phi(u)\big)\,\phi'(u)^2\big]\,\dot u^2 - \phi(u)\,\big(1 - a\,\phi(u)\big)\,\big(-1 + 2\,\phi(u) - a\,\phi(u)^2\big)^2 = 0\,.\end{aligned} \tag{9}$$

Therefore, to obtain an equation of the form (2) we must consider the following initial value problem

$$\begin{aligned}\big(-1 + 2\,\phi(u) - a\,\phi^2(u)\big)\,\phi''(u) - 3\,\big(1 - a\,\phi(u)\big)\,\phi'(u)^2 = 0\\ \phi(0) = 0, \phi'(0) = 1\,.\end{aligned}$$

Integrating the above equation over the neighborhood $\mathcal{V}$, we obtain

$$\begin{aligned}\phi'(u) &= \sqrt{(1 - 2\,\phi(u) + a\,\phi^2(u))^3}\\ \phi(0) &= 0\,.\end{aligned}$$

This initial value problem has the solution

$$\phi(u) = \frac{1}{a}\left(1 + \frac{a\,u - u - 1}{\sqrt{1 + 2\,u + (1 - a)\,u^2}}\right). \tag{10}$$

With this ϕ, equation (9) becomes in the form (2) with

$$g(u) = \frac{\phi(u)\,(1 - \phi(u))}{\sqrt{1 - 2\,\phi(u) + a\,\phi^2(u)}}. \tag{11}$$

Using (10) and (11), the equation

$$\frac{1}{2}\,X^2 = \int_0^u g(\xi)\,d\xi$$

transforms into $s \cdot z = 0$, where

$$s = (-2u - u^2 + au^2 + 2X + 2uX - 2auX + aX^2)$$
$$z = (-2u - u^2 + au^2 - 2X - 2uX + 2auX + aX^2)\,.$$

From $s = 0$, we obtain u in terms of X

$$u = X + F(X)$$

where

$$F(X) = \frac{1 - \sqrt{1 + (1-a)\,X^2}}{a-1}\,.$$

As $F(X)$ is even function, the Urabe's Criteria is satisfied and consequently for $a > 1$ the origin of the system (8) is an isochronous center. By the linear part of system (8) we conclude that the period is 2π. By (7) the origin of (6) is an isochronous center of period $2\pi/\sqrt{a-1}$ and, by the conjugation Ψ, the singularity $(1/2, 1/2)$ of the system (4) is an isochronous center of the same period. □

Remark. By (5) it is easy to see that for $c \in]-1/a, 0[$, the level curves of $H_a(x,y) = c$ are the periodic orbits (ellipse) in the basin of the centre and, for $c = 0$, is the parabola $1 - 2(x+y) + a(x-y)^2 = 0$, the boundary of the basin.

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