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On the class of some projective varieties

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Abstract

Some inequalities between the class and the degree of a smooth complex projective manifold are given. Application to the case of low sectional genus are supplied.

1. Introduction

Let X be a complex projective algebraic variety of dimension r and L a very ample line bundle with $h^0(L) = N + 1$. Consider the polarized pair (X, L) embedded in $\mathbb{P}^N$ as a variety of degree $d = L^r$. Denote by μ the class of X, i.e. the number of hyperplanes in a general pencil which are tangent to it. Many authors [3, 4, 6, 7] have studied such X with low μ, with the aim of classifying them, following the idea that varieties with very small projective invariants are far from being of general type. Moreover in some of these papers relationship between the class and the degree of the varieties have been investigated in order to classify those varieties which do not satisfy them. E.g., as to the case of surfaces, it is known that, except for a finite number of known examples, it is $\mu \geq 2d$ ([7]).

In contrast with this kind of inequalities, in this paper we obtain an upper bound for the class in terms of the degree, namely

$$\mu = \mu(X) \leq \frac{d\big(2d+c-2\big)^{2r}}{\big(4cd\big)^{r}}\,, \tag{1}$$

where c denotes the codimension of X. The idea is that of applying the semipositivity of Schur polynomials of a spanned vector bundle to the first Jet-bundle of L and combining it with the generalized Hodge Index Theorem and Castelnuovo inequality. The inequality (1), when $d >> 0$, is an improvement of the one in [1]. A similar technique is then applied to get an inequality for the class of the varieties of a family C_a introduced in [5], whose sectional genus g is bounded by a linear function of the degree: $2g-2 \leq ad$. For such varieties we get the following bound

$$\mu \leq \big(a+(r+1)\big)^{r} d\,.$$

Finally, we apply the last inequality to some varieties which are proved to be of class C_a for suitable a. For instance we consider $\mathbb{P}^{r-1}$-bundles on a curve of genus g for which we obtain the following bound

$$\mu \leq \big(2g+r-1\big)^{r} d\,,$$

and $\mathbb{P}$-bundles $\mathbb{P}(E)$ on surfaces whose class we compare with the class of the base embedded via $\det E$.

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2. Notations

The notation used in this work is mostly standard from Algebraic Geometry. The ground field is always the field $\mathbb{C}$ of complex numbers. Unless otherwise stated all varieties are supposed to be projective. $\mathbb{P}^n$ denotes the n-dimensional complex projective space. Given a projective n-dimensional variety X, $Pic(X)$ denotes the group of line bundles over X. For $L \in Pic(X)$, $|L|$ denotes the complete linear system of effective divisors associated with L. Let E be a vector bundle of rank r on X; we denote by $c_k(E)$ the k-th Chern class of E and by $\mathbb{P}(E)$ the projectivized bundle over X.

3. An upper bound for the class

Let X be a smooth complex projective algebraic variety of dimension r and L a very ample line bundle with $h^0(L) = N+1$. Consider the polarized pair (X, L) and define $d = L^r, c = N - r$ and $\mu(X)$ the class of X, i.e. the degree of the discriminancy locus of L.

Proposition 3.1

With the above hypothesis one has

$$\mu = \mu(X) \leq \frac{d(2d + c - 2)^{2r}}{(4cd)^r} .$$

Proof. Since if L is very ample then its first jet bundle $J_1(L)$ is spanned, we have the inequality (see for example [5]):

$$\mu = c_r\big(J_1(L)\big) \leq \big(c_1(J_1(L))\big)^r$$

which follows from the semipositivity of suitable Schur polynomials.
Hence we get

$$\mu \leq \big((r+1)L + K_X\big)^r \leq \frac{\big(L^{r-1}(K_X + (r+1)L)\big)^r}{(L^r)^{r-1}} ,$$

the second inequality following from generalized Hodge index theorem, as in [5], Proposition 1, applied to L and $D = K_X + (r+1)L$ (which is nef indeed). Moreover, from the Castelnuovo inequality for a non hyperplane curve in $\mathbb{P}^n$,

$$(n-1)(2g-2) \leq d(d-n-1) + (n-3)^2/4 ,$$

applied to the hyperplane section $C \subset \mathbb{P}^n$, $n = N - r + 1$, of X one gets

$$\frac{\big[L^{r-1}(K_X + (r+1)L)\big]^r}{(L^r)^{r-1}} \leq \frac{d\big(2d + n - 3\big)^{2r}}{\big(4d(n-1)\big)^r}$$

hence

$$\mu \leq \frac{d(2d+c-2)^{2r}}{(4cd)^r} \,. \square$$

Remark 3.1. Notice that the inequality of Proposition 3.1 is similar, but stronger than the one appearing in [1], Theorem 3.1, for $d >> 0$.

Now, following [5], we will say that X is in class C_a, for a a real number, if

$$K_X L^{r-1} \leq ad\,;$$

obviously $a \geq -(r+1)$, since $K_X + (r+1)L$ is nef.

Applying the definition of class C_a one gets

$$\mu \leq \frac{\big(L^{r-1}(K_X + (r+1)L)\big)^r}{(L^r)^{r-1}} \leq \big(a + (r+1)\big)^r d\,.$$

So we have the following.

Proposition 3.2

If X is of class C_a, then

$$\mu \leq (a + (r+1))^r d\,.$$

4. Applications

We want now to investigate some special families of varieties which are of class C_a, for some a.

Notice first that trivial examples are given by abelian or pluricanonical varieties. Indeed any abelian variety is of class C_0, so that the class of its embeddings satisfy

$$\mu \leq (r+1)^r d\,;$$

as to pluricanonical varieties, i.e. varieties embedded by a very ample divisor $L = mK$, with $m \geq 1$, they are of class C_1, so that

$$\mu \leq (r+2)^r d\,.$$

4.1 $\mathbb{P}$-bundles on a curve

Let E be a rank r vector bundle over a smooth curve C of genus g and consider the $\mathbb{P}$-bundle $\pi : X = \mathbb{P}(E) \to C$. Denoting by t the tautological line bundle on X, one has

$$Pic(X) \simeq t\mathbb{Z} \oplus \pi^* Pic(C)\,.$$

Now let $L = \alpha t + \pi^* D$, for some $D \in Pic(C)$, be a very ample line bundle on X. Denoting by β the degree of D and putting $e = -c_1(E)$, one has

$$\alpha > 0, r\beta > e\alpha\,.$$

Proposition 4.1

With the notations above, X is of class C_a where $a = \frac{2g-2}{r\beta-\alpha e} - \frac{r-1}{\alpha}$, in particular, if $g \geq 1$, X is of class C_{2g-2}, if $g = 0$ then X is of class C_0.

Proof. The canonical divisor of X is numerically equivalent to

$$-rt + (2g - 2 - e)F\,,$$

F denoting a general fiber of π, so that one has

$$K_X L^{r-1} = r\alpha^{r-1} e + \alpha^{r-1}(2g - 2 - e) - r(r - 1)\alpha^{r-2}\beta$$

and

$$d = L^r = -\alpha^r e + r\alpha^{r-1}\beta\,.$$

We are looking for an a satisfying

$$r\alpha^{r-1} e + \alpha^{r-1}(2g - 2 - e) - r(r - 1)\alpha^{r-2}\beta \leq a\alpha^{r-1}(-\alpha e + r\beta)\,.$$

Since it is $-\alpha e + r\beta > 0$, the last inequality obviously implies the thesis. □

Corollary 4.1

For $X = \mathbb{P}(E)$ as above and $g \geq 1$, one has

$$\mu \leq \left[\frac{2g-2}{r} + r + 1\right]^r d\,.$$

Proof. Twisting E by a suitable line bundle, it is possible to get $e \geq 0$. Under this hypothesis the inequality above is obtained from Proposition 4.1 and Proposition 3.2, observing that

$$\frac{2g-2}{r\beta - \alpha e} - \frac{r-1}{\alpha} \leq \frac{2g-2}{r}. \ \square$$

Remark 4.1. As it is clear, if $r \geq 3$, Corollary 4.1 applies only when $\alpha \geq 2$, since otherwise it is $\mu = 0$ ([3]).

Proposition 4.2

Let E be a very ample rank r vector bundle over a smooth curve C, and consider $\pi : X = \mathbb{P}(E) \to C$. Then (X, t) is in class C_a, if and only if $(C, \det E)$ is in class C_{a+r-1}.

Proof. Recalling the expression of K_X and the Wu-Chern relation we get that the inequality

$$K_X t^{r-1} - at^r \leq 0$$

is equivalent to

$$K_C + \det E - (a+r) \det E \leq 0$$

which implies the thesis. $\square$

4.2. $\mathbb{P}$-bundles on a surface

The following Lemma is an immediate consequence of Wu-Chern relation.

Lemma 4.1

Let E be a rank r vector bundle over a smooth surface B, and consider $\pi : X = \mathbb{P}(E) \to B$. Then for all $D \in Pic(B)$,

$$\pi^*(D)t^r = D \det(E).$$

Proposition 4.3

Let E be a very ample rank 2 vector bundle over a smooth surface B, and consider $\pi : X = \mathbb{P}(E) \to B$. If (X, t) is in class C_a $(a \geq -2)$, then $(B, \det E)$ is in class C_{a+1}.

Proof. By assumption we have

$$K_X t^2 - at^3 \leq 0$$

which implies

$$K_X t^2 - at^3 - (a+2)\pi^* c_2(E) t \leq 0\,.$$

Recalling the expression of K_X and the Wu-Chern relation we get

$$K_B \det E + (\det E)^2 - (a+2)\pi^* c_1(E) t^2 \leq 0\,.$$

By Lemma 4.1 we get

$$K_B \det E - (a+1)(\det E)^2 \leq 0$$

which implies the thesis. □

By the definition of class C_a and by adjunction, we easily get the following.

Lemma 4.2

Let (X, L) be a smooth projective variety of dimension n, and consider the generic hyperplane section $(H, L_{|H})$ for $H \in |L|$. Then (X, L) is in class C_a if and only if $(H, L_{|H})$ is in class C_{a+1}.

By Proposition 4.3 and Lemma 4.2 we immediately get.

Corollary 4.2

Let E be a very ample rank r vector bundle over a smooth surface B, and consider $\pi : X = \mathbb{P}(E) \to B$. Then if (X, t) is in class C_a, (with $a \geq -r$) then $(B, detE)$ is in class C_{a+r-1}.

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