Collect. Math. **48**, 3 (1997), 253–264

Quadratic stabilization of distributed parameter systems with norm-bounded time-varying uncertainty

Wanyi Chen

Department of Mathematics, Nankai University, Tianjin 300071, P. R. China

Fengsheng Tu

Department of Computer and System Sciences, Nankai University, Tianjin 300071, P. R. China

Received April 4, 1995. Revised June 10, 1996

Abstract

This note focuses on the study of robust H_∞ control design for a kind of distributed parameter systems in which time-varying norm-bounded uncertainty enters the state and input operators. Through a fixed Lyapunov function, we present a state feedback control which stabilizes the plant and guarantees an H_∞ norm bound on disturbance attenuation for all admissible uncertainties. In the process, we generalize some known results for finite dimensional linear systems.

1. Introduction and definitions

In the last decade, we have witnessed a significant research thrust in H_∞ *control theory*, a frequency domain methodology which is closely related with deep complex-function and operator techniques, see [3, 4, 8]. We also know that H_∞ control is greatly useful for robustness problem. To date, many papers have appeared on the robust control of finite dimensional linear systems with norm-bounded time-varying uncertainty. But only a few papers deal with the similar problem for distributed parameter systems, see [1, 6]. In this paper, via the use of some operator method, we characterize quadratic stabilizability with an H_∞ norm bound constraint for uncertain distributed parameter systems satisfying the so-called *matching condition*. In the process, we generalize the relevant results for finite dimensional systems to infinite dimensional ones, see Section 3.

In this paper, we discuss uncertain distributed parameter systems described by state-space models of the form:

$$\Sigma_0 \begin{cases} \dot{x}(t) = \big[A + \Delta A(t)\big]x(t) + B_1 w(t) + \big[B_2 + \Delta B_2(t)\big]u(t) \\ z(t) = C_1 x(t) + D_1 u(t) \\ x(0) = 0 \end{cases}$$

where A is the generator of a C_0-semigroup $\{T_t; t \geq 0\}$ of bounded operators in a Hilbert space $\mathcal{H}$, and $x(t) \in \mathcal{H}$ is the state, $u(t) \in H_i$ is the control input, $w(t) \in \mathcal{H}_d$ is the disturbance input which belongs to $L_2(0, \infty; \mathcal{H}_d)$, $z(t) \in \mathcal{H}_o$ is the controlled output, here $\mathcal{H}_d, \mathcal{H}_i, \mathcal{H}_o$ are Hilbert spaces, while B_1, B_2, C_1, D_1 are bounded operators on appropriate spaces. (A, B_1, B_2, C_1, D_1) describes the nominal system and $(\Delta A(\cdot), \Delta B_2(\cdot))$ are operator-valued functions representing time-varying uncertainty to the state and input operators, respectively. $(\Delta A(\cdot), \Delta B_2(\cdot))$ is in the following form:

$$\big(\Delta A(\cdot), \Delta B_2(\cdot)\big) = DF(t)\big(E_1, E_2\big)$$

Here D, E_1, E_2 are known bounded operators, from $\mathcal{H}_2$ to $\mathcal{H}$, from $\mathcal{H}$ to $\mathcal{H}_1$, and from $\mathcal{H}_i$ to $\mathcal{H}_1$, respectively. Also an admissible function $F(t)$ is any Lebesgue-Bochner measurable function from $[0, \infty)$ to $L(\mathcal{H}_1, \mathcal{H}_2)$, with $\|F(t)\| \leq 1$, $t \in [0, \infty)$. Similar to the finite dimensional case, we shall make the following assumption without loss of generality.

Assumption 1 $D_1^* [C_1, D_1] = [0, I]$.

The closed loop system with static state feedback $u(t) = Kx(t)$ is given by

$$\Sigma_g \begin{cases} \dot{x} = A_g(t) + B_1 w \\ z = C_g x \end{cases}$$

where

$$\begin{aligned} A_g(t) = {} & A + B_2 K + \Delta A(t) + \Delta B_2(t) K \\ = {} & A + B_2 K + DF(t)\big(E_1 + E_2 K\big) \\ C_g = {} & C_1 + D_1 K\,. \end{aligned}$$

DEFINITION 2 [8]. Let the constant $r > 0$ is given, the uncertain system Σ_0 is said to be quadratically stabilizable with an H_∞ norm bound r if there exist a fixed static state feedback $u(t) = Kx(t)$ and a self-adjoint, nonnegative operator $P \in L(\mathcal{H})$ such that for any $x \in \mathcal{D}(A)$,

$$\big\langle A_g(t)x, Px\big\rangle + \big\langle Px, A_g(t)x\big\rangle + r^{-2}\big\langle PB_1B_1^*Px, x\big\rangle + \|C_g x\|^2 \le -\alpha\big\langle x, x\big\rangle$$

holds for any admissible $F(\cdot)$, where α is a positive constant independent of x and $F(\cdot)$.

We can easily see the following fact from the definition.

Lemma 3

If the uncertain system Σ_0 is quadratically stabilizable with H_∞ an norm bound r, then there exists a $\delta_0 > 0$ such that for any $\delta \in [0,\ \delta_0]$, the uncertain system

$$\Sigma(\delta) : \begin{cases} \dot{x}(t) = \big[A + \Delta A(t) + \delta I\big]x(t) + B_1 w(t) + \big[B_2 + \Delta B_2(t)\big]u(t) \\ z(t) = C_1 x(t) + D_1 u(t) \\ x(0) = 0 \end{cases}$$

is also quadratically stabilizable with the H_∞ norm bound r.

Setting $u(t) \equiv 0$, we obtain the unforced system of Σ_0 of the following form:

$$\Sigma_1 : \begin{cases} \dot{x}(t) = \big[A + \Delta A(t)\big]x(t) + B_1 w(t) \\ z(t) = C_1 x(t), \quad x(0) = 0\,. \end{cases}$$

In order to guarantee an H_∞ performance for all admissible $\Delta A(\cdot)$, and like Definition 2, we use a fixed Lyapunov function in the following notion of quadratic stability with disturbance attenuation, providing a practical way of handling both parameter uncertainty and disturbance input.

DEFINITION 2' [8]. Given a real number $r > 0$, the system Σ_1 is said to be quadratically stable with disturbance attenuation r if there exist $P_1 \in L(\mathcal{H}), P_1 \ge 0$, and a positive number α_1 such that for all $x \in \mathcal{D}(A)$ and all admissible $\Delta A(\cdot)$,

$$\begin{aligned} &\big\langle [A + \Delta A(t)]x, P_1 x\big\rangle + \big\langle P_1 x, [A + \Delta A(t)]x\big\rangle \\ &\qquad + r^{-2}\big\langle P_1 B_1 B_1^* P_1 x, x\big\rangle + \big\langle C_1 x, C_1 x\big\rangle \le -\alpha_1\big\langle x, x\big\rangle\,. \end{aligned}$$

Remark. The notion of quadratic stability with disturbance attenuation is a direct extension of quadratic stability to give an H_∞ performance description in the face of time-varying state parameter uncertainty, see [1, 5]. Also for system Σ_1, under the above notion, $||z||_2 < r||w||_2$ for all admissible uncertainty $\Delta A(\cdot)$ and all nonzero $w \in L_2(0, \infty; \mathcal{H})$, see [1, 5].

DEFINITION 4 [1]. Suppose that X, Y, Z are bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that the triple (X, Y, Z) has property P if there exists a $\omega > 0$ such that for all $x \in \mathcal{H}$,

$$\langle Yx, x\rangle^2 - 4|\langle Xx, x\rangle\langle Zx, x\rangle| \geq \omega||x||^4$$

Lemma 5 [1]

Assume that the triple (X, Y, Z) has the property P, and $X \geq 0$, $Y \leq 0$ and $Z \geq 0$, then there exists a $\lambda > 0$ such that

$$\lambda^2 X + \lambda Y + Z$$

is negative and invertible on $\mathcal{H}$ (see [7]).

Like [1], we also make the following assumption on the semigroup $\{T_t; t \geq 0\}$.

Assumption 6 For $\{T_t; t \geq 0\}$, there are $\tau, m_0 > 0$ such that

$$\int_0^\tau ||T_t x||^2 \, dt \geq m_0 ||x||^2$$

for any $x \in \mathcal{H}$.

Lemma 7

If A_0 is a bounded operator on $\mathcal{H}$, and $\{T_t\}$ satisfies Assumption 6, then the semigroup generated by $(A + A_0)$ still satisfies Assumption 6.

Proof. See Section 3. $\square$

2. Main results

Theorem 8

Under Assumption 1, 6, uncertain system Σ_0 *is quadratically stabilizable with an* H_∞ *norm bound* r *if and only if there exist constant* $\varepsilon, \mu > 0$ *and* $P \in L(\mathcal{H}), P \geq 0$ *such that following Riccati inequality holds for all* $x \in \mathcal{D}(A)$,

$$\begin{aligned}\langle Ax, Px\rangle + \langle Px, Ax\rangle + r^{-2}\langle PB_1B_1^*Px, x\rangle + ||C_1x||^2 \\ - \Big\langle R_\varepsilon^{-1}\big(B_2^*P + \frac{1}{\varepsilon}E_2^*E_1\big)x, \big(B_2^*P + \frac{1}{\varepsilon}E_2^*E_1\big)x\Big\rangle \\ + \varepsilon\langle PDD^*Px, x\rangle + \frac{1}{\varepsilon}||E_1x||^2 \leq -\mu||x||^2 \end{aligned} \tag{2.1}$$

where $R_\varepsilon = I + \frac{1}{\varepsilon}E_2^*E_2$. *Moreover, a suitable feedback control law is given by* $u(t) = K_\varepsilon x(t)$, *and*

$$K_\varepsilon = -R_\varepsilon^{-1}\Big(B_2^*P + \frac{1}{\varepsilon}E_2^*E_1\Big) \tag{2.2}$$

Corollary 9

Under the condition of Theorem 8, the following uncertain control system

$$\begin{cases} \dot{x}(t) = [A + \Delta A(t)]x(t) + [B_2 + \Delta B_2(t)]u(t) \\ x(0) = 0 \end{cases}$$

is quadratically stabilizable, i.e., there exists a static feedback $u(t) = Kx(t)$ *such that the closed loop system is quadratically stable, see [1, 5] for the definition of quadratic stability.*

Corollary 10 [1, 2]

Under Assumption 6, system Σ_1 *is quadratically stable with disturbance attenuation* r *if and only if one of following conditions holds:*

(1) There exist $P \in L(\mathcal{H}), P \geq 0$ *and* $\mu, \varepsilon > 0$ *such that that for all* $x \in \mathcal{D}(A)$,

$$\begin{aligned}\langle Ax, Px\rangle + \langle Px, Ax\rangle + r^{-2}\langle PB_1B_1^*Px, x\rangle + \langle C_1x, C_1x\rangle \\ + \varepsilon\langle PDD^*Px, x\rangle + \frac{1}{\varepsilon}\langle E_1x, E_1x\rangle \leq -\mu\langle x, x\rangle\end{aligned}$$

(2) There exist $P \in L(\mathcal{H}), P \geq 0$ *and* $\varepsilon > 0$ *such that* $(A + \frac{1}{r^2}B_1B_1^*P + \varepsilon DD^*P)$ *generates an exponentially stable* C_0 *-semigroup on* $\mathcal{H}$, *and the following algebraic Riccati equation holds:*

$$\begin{aligned}\langle Ax, Px\rangle + \langle Px, Ax\rangle + r^{-2}\langle PB_1B_1^*Px, x\rangle \\ + \varepsilon\langle PDD^*Px, x\rangle + \frac{1}{\varepsilon}\langle E_1^*E_1x, x\rangle + \langle C_1x, C_1x\rangle = 0\end{aligned}$$

for all $x \in \mathcal{D}(A)$.

(3) $\{T_t\}$ *is exponentially stable and there exists an* $\varepsilon > 0$ *such that*

$$\left\| \Big(\frac{1}{\varepsilon}E_1^*E_1 + C_1^*C_1\Big)^{1/2}(sI-A)^{-1}\Big(\frac{1}{r^2}B_1B_1^* + \varepsilon DD^*\Big)^{1/2} \right\|_\infty < 1$$

or

$$\left\| \begin{bmatrix} C_1 \\ \frac{1}{\sqrt{\varepsilon}}E_1 \end{bmatrix}(sI-A)^{-1}\begin{bmatrix} \frac{1}{r}B_1 & \sqrt{\varepsilon}D \end{bmatrix} \right\|_\infty < 1\,.$$

3. Proofs

Proof of Lemma 7. Since the semigroup $\{T_t\}$ has the property P, there exist $\tau, m_0 > 0$ such that

$$\int_0^\tau ||T_tx||^2 dt \geq m_0||x||^2\,.$$

We assume that the semigroup generated by $(A + A_0)$ is $\{S_t\}$, then for all $x \in \mathcal{D}(A)$

$$S_tx = T_tx + \int_0^t T_{t-s}A_0S_sx\,ds, \quad t > 0\,.$$

Hence

$$T_tx = S_tx - \int_0^t T_{t-s}A_0S_sx\,ds$$

$$\Big(\int_0^\tau ||T_tx||^2\,dt\Big)^{1/2} \leq \Big(\int_0^\tau ||S_tx||^2\,dt\Big)^{1/2} + \Big(\int_0^\tau ||T_tA_0||\,dt\Big)\Big(\int_0^\tau ||S_tx||^2\,dt\Big)^{1/2}\,.$$

Let $m_1 = \int_0^\tau ||T_tA_0||\,dt$, then for $x \in \mathcal{D}(A)$,

$$\sqrt{m_0}||x|| \leq (1+m_1)\Big(\int_0^\tau ||S_tx||^2\,dt\Big)^{1/2}$$

$$\int_0^\tau ||S_tx||^2\,dt \geq m_0(1+m_1)^{-2}||x||^2$$

also, by the density of $\mathcal{D}(A)$ in $\mathcal{H}$, the last inequality holds for all $x \in \mathcal{H}$, i.e., $\{S_t\}$ still satisfies Assumption 6. □

Proof of Theorem 8. *Sufficiency.* Suppose that there exist constant $\varepsilon, \mu > 0$, and $P \in L(\mathcal{H}), P \geq 0$ such that Riccati inequality (2.1) holds. Consider the feedback law (2.2) and define the closed-loop system state operator

$$A_c(t) := A + DF(t)E_1 - \big[B_2 + DF(t)E_2\big]R_\varepsilon^{-1}\Big[B_2^*P + \frac{1}{\varepsilon}E_2^*E_1\Big]$$

then for all $x \in \mathcal{D}(A)$,

$$
\begin{aligned}
\langle Px, A_c(t)x\rangle + \langle A_c(t)x, Px\rangle = {} & \langle Ax, Px\rangle + \langle Px, Ax\rangle \\
& - 2\langle PB_2R_\varepsilon^{-1}B_2^*Px, x\rangle - \frac{1}{\varepsilon}\langle E_1^*E_2R_\varepsilon^{-1}B_2^*Px, x\rangle \\
& - \frac{1}{\varepsilon}\langle PB_2R_\varepsilon^{-1}E_2^*E_1x, x\rangle + \langle Y(t)x, x\rangle \qquad (3.1)
\end{aligned}
$$

here

$$
\begin{aligned}
X &:= E_1 - \frac{1}{\varepsilon}E_2R_\varepsilon^{-1}E_2^*E_1 - E_2R_\varepsilon^{-1}B_2^*P \\
Y(t) &:= PDF(t)X + X^*F^*(t)D^*P\,.
\end{aligned}
$$

Note that $||F(t)|| \le 1$, it is easy to see that

$$
Y(t) \le \varepsilon PDD^*P + \frac{1}{\varepsilon}X^*X\,. \qquad (3.2)
$$

By combining (3.1) with (2.2) and the fact

$$
\frac{1}{\varepsilon}R_\varepsilon^{-1}E_2^*E_2R_\varepsilon^{-1} = R_\varepsilon^{-1} - R_\varepsilon^{-2}
$$

we have

$$
\begin{aligned}
Y(t) + K_\varepsilon^*K_\varepsilon \le {} & \varepsilon PDD^*P + \frac{1}{\varepsilon}E_1^*E_1 \\
& - \frac{1}{\varepsilon^2}E_1^*E_2R_\varepsilon^{-1}E_2^*E_1 + PB_2R_\varepsilon^{-1}B_2^*P\,. \qquad (3.3)
\end{aligned}
$$

Now, via the application of (3.3) to (3.1) and Assumption 1, we have

$$
\begin{aligned}
& \big\langle A_c(t)x, Px\big\rangle + \big\langle Px, A_c(t)x\big\rangle \\
& \qquad + r^{-2}\big\langle PB_1B_1^*Px, x\big\rangle + \big\langle (C_1 + D_1K_\varepsilon)x, (C_1 + D_1K_\varepsilon)x\big\rangle \\
& \le \big\langle Ax, Px\big\rangle + \big\langle Px, Ax\big\rangle \\
& \quad + r^{-2}\big\langle PB_1B_1^*Px, x\big\rangle + \frac{1}{\varepsilon}\big\langle E_1x, E_1x\big\rangle + \varepsilon\big\langle PDD^*Px, x\big\rangle \\
& \quad + ||C_1x||^2 - \Big\langle R_\varepsilon^{-1}\big(B_2^*P + \frac{1}{\varepsilon}E_2^*E_1\big)x, \big(B_2^*P + \frac{1}{\varepsilon}E_2^*E_1\big)x\Big\rangle \\
& \le -\mu\big\langle x, x\big\rangle\,.
\end{aligned}
$$

Hence, system Σ_0 is quadratically stabilizable with the H_∞ norm bound r.

Necessity. From Definition 2, there exist a fixed static state feedback $u(t) = Kx(t)$ and $P \in L(\mathcal{H}), P \geq 0$ such that for any $x \in \mathcal{D}(A)$,

$$\langle A_g(t)x, Px\rangle + \langle Px, A_g(t)x\rangle + r^{-2}\langle PB_1B_1^*Px, x\rangle + \langle C_gx, C_gx\rangle \leq -\alpha\langle x, x\rangle \quad (3.4)$$

where α is a positive constant, $A_g(t) := A + B_2K + DF(t)(E_1 + E_2K), C_g := C_1 + D_1K$. Choose $F(t) \equiv 0$. Then

$$\big\langle(A + B_2K)x, Px\big\rangle + \big\langle Px, (A + B_2K)x\big\rangle \leq -\alpha\langle x, x\rangle\,. \quad (3.5)$$

For any $x \in \mathcal{D}(A)$, from [2, Lemma 1.4], the semigroup $\{S_t\}$ generated by $(A + B_2K)$ is exponentially stable. Also from (3.5),

$$\langle Px, x\rangle \geq \alpha \int_0^{+\infty} ||S_t||^2\, dt \geq \frac{\alpha m_0}{(1 + m_1)^2)}||x||^2 \quad (I)$$

for all $x \in \mathcal{D}(A)$, where the right-hand inequality is deduced from Assumption 6 and Lemma 7 with $A_0 := B_2K$. Hence P is invertible in $\mathcal{H}$ by the density of $\mathcal{D}(A)$ in $\mathcal{H}$.

Without loss of generality, we assume that $E_1 + E_2K \neq 0$. Otherwise, we can make a sufficiently small perturbation K' to K such that $E_1 + E_2K \neq 0$ and inequality (3.4) is still valid with some modification on the positive constant α. In the following, we shall work under the condition that $E_1 + E_2K \neq 0$.

From (3.4), for any $x \in \mathcal{D}(A)$ with $x \neq 0$,

$$\begin{aligned}\big\langle(A + B_2K)x, Px\big\rangle + \big\langle Px, (A + B_2K)x\big\rangle + \big\langle C_1^*C_1x, x\big\rangle \\ + r^{-2}\big\langle PB_1B_1^*Px, x\big\rangle + ||Kx||^2 \\ < -2Re\big\langle F(t)(E_1 + E_2K)x, D^*Px\big\rangle\end{aligned}$$

for any admissible $F(t) \in L(\mathcal{H}_1, \mathcal{H}_2)$ with $||F(t)|| \leq 1$. So

$$\begin{aligned}\big\langle(A + B_2K)x, Px\big\rangle + \big\langle Px, (A + B_2K)x\big\rangle + r^{-2}\big\langle PB_1B_1^*x, x\big\rangle \\ + \big\langle C_1^*C_1x, x\big\rangle + \big\langle K^*Kx, x\big\rangle \\ \leq -2Sup\Big\{Re\big\langle F(t)(E_1 + E_2K)x, D^*Px\big\rangle : ||F(t)|| \leq 1\Big\} \quad (3.6)\end{aligned}$$

Choose $l_0 > 0$ such that $R(l_0) := (l_0I - A - B_2K)^{-1} \in L(\mathcal{H})$, i.e., $l_0 \in \rho(A + B_2K)$, and let $A_1 = l_0(A + B_2K)R(l_0)$. Then $A_1 \in L(\mathcal{H})$. Now, let $y = \frac{1}{l_0}(l_0I - A - B_2K)x$, then $A_1y = (A + B_2K)x$ and $x = l_0R(l_0)y$. From (3.6),

$$\begin{aligned}\big\langle A_1y, l_0PR(l_0)y\big\rangle + \big\langle l_0PR(l_0)y, A_1y\big\rangle \\ + r^{-2}\big\langle PB_1B_1^*Pl_0R(l_0)y, l_0R(l_0)y\big\rangle \\ + l_0^2\big\langle C_1^*C_1R(l_0)y, R(l_0)y\big\rangle + l_0^2\big\langle K^*KR(l_0)y, R(l_0)y\big\rangle \\ < -2l_0^2Sup\Big\{Re\big\langle F(t)(E_1 + E_2K)R(l_0)y, D^*PR(l_0)y\big\rangle; ||F(t)|| \leq 1\Big\}\end{aligned}$$

Assume

$$\begin{aligned}\tilde{Y} &= l_0 R^*(l_0) P A_1 + l_0 A_1^* P R(l_0) + r^{-2} l_0^2 R^*(l_0) P B_1 B_1^* P R(l_0) \\ &\quad + l_0^2 R^*(l_0) C_1^* C_1 R(l_0) + l_0^2 R^*(l_0) K^* K R(l_0) \\ \tilde{X} &= l_0^2 R^*(l_0) P D D^* P R(l_0) \\ \tilde{Z} &= l_0^2 R^*(l_0)(E_1 + E_2 K)^*(E_1 + E_2 K) R(l_0)\end{aligned}$$

then $\tilde{X} \geq 0, \tilde{Z} \geq 0, \tilde{Y} \leq 0$, and

$$\langle \tilde{Y} y, y\rangle^2 \,-\, 4\langle \tilde{X} y, y\rangle \langle \tilde{Z} y, y\rangle > 0$$

for any $y \in \mathcal{H}$ with $y \neq 0$. Hence, for any $\varepsilon > 0$, the triple $(\tilde{X}, \tilde{Y} - \varepsilon I, \tilde{Z})$ has property P. So, from Lemma 5, there is a $\mu(\varepsilon) > 0$ such that

$$\begin{aligned}&\mu(\varepsilon)^2 \tilde{X} + \mu(\varepsilon)(\tilde{Y} - \varepsilon I) + \tilde{Z} \leq 0, \quad \text{i.e.,} \\ &\mu(\varepsilon)^2 \tilde{X} + \,\mu(\varepsilon)\tilde{Y} \,+\, \tilde{Z} \leq \mu(\varepsilon)\varepsilon I\,. \end{aligned} \tag{3.7}$$

Alternatively, let $x = l_0 R(l_0) y$, where $y \in \mathcal{H}$, and $S_\varepsilon := \mu(\varepsilon) P$. we have $x \in \mathcal{D}(A)$ and

$$\begin{aligned}\langle (A + B_2 K)x, S_\varepsilon x\rangle &+ \langle S_\varepsilon x, (A + B_2 K)x\rangle + \mu(\varepsilon)\big(||C_1 x||^2 + ||Kx||^2\big) \\ &+ r^{-2}\mu^{-1}(\varepsilon)\langle S_\varepsilon B_1 B_1^* S_\varepsilon x, x\rangle + \langle S_\varepsilon D D^* S_\varepsilon x, x\rangle \\ &+ ||(E_1 + E_2 K)x||^2 \\ &\leq \mu(\varepsilon)\varepsilon \frac{1}{l_0^2} ||(l_0 I - A - B_2 K)x||^2\,.\end{aligned} \tag{3.8}$$

Now, we obtain bounds for $\mu(\varepsilon)$. From (3.7),

$$0 < \mu(\varepsilon) \leq \frac{-\langle \tilde{Y} y, y\rangle + \varepsilon}{\langle \tilde{X} y, y\rangle} \leq \frac{||\tilde{Y}|| + \varepsilon}{\langle \tilde{X} y, y\rangle}$$

for any $y \in \mathcal{H}$ such that $\tilde{X} y \neq 0$, and $||y|| = 1$. Hence

$$0 < \mu(\varepsilon) \leq \frac{||\tilde{Y}|| + \varepsilon}{||\tilde{X}||}\,. \tag{3.9}$$

We claim that

$$\inf_{1 \geq \varepsilon > 0} \mu(\varepsilon) > 0\,.$$

Otherwise there is a sequence of numbers $\varepsilon_n \in (0,\, 1](n = 1, 2, \cdots)$ such that

$$\lim_{n\to\infty} \mu(\varepsilon_n) \,=\, 0$$

and then (3.8) with (3.9) would imply that $E_1 + E_2K = 0$, contradicting our assumption that $E_1 + E_2K \neq 0$. Hence we can choose $\varepsilon_n \in (0,\, 1](n = 1, 2, \cdots)$ such that

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ and } \lim_{n\to\infty} \mu(\varepsilon_n) = \beta > 0\,.$$

Again via the use of (3.7) − (3.8) and let $Q_\beta := \beta P$, it follows

$$\begin{aligned}\langle (A + B_2K)x, Q_\beta x\rangle + \langle Q_\beta x, (A + B_2K)x\rangle + \beta(||C_1x||^2 + ||Kx||^2)\\ + r^{-2}\,\beta^{-1}\langle Q_\beta B_1B_1^* Q_\beta x, x\rangle + \langle Q_\beta DD^* Q_\beta x, x\rangle\\ + ||(E_1 + E_2K)x||^2 \le 0\,.\end{aligned}$$

Divided by β, we have

$$\begin{aligned}\langle (A + B_2K)x, Px\rangle + \langle Px, (A + B_2K)x\rangle\\ + \langle C_1^*C_1x, x\rangle + \langle K^*Kx, x\rangle + r^{-2}\langle PB_1B_1^*Px, x\rangle\\ + \beta\langle PDD^*Px, x\rangle + \frac{1}{\beta}||(E_1 + E_2K)x||^2 \le 0\end{aligned}$$

or yet

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle + \beta\langle PDD^*Px, x\rangle + \frac{1}{\beta}||E_1X||^2 + ||C_1x||^2 + \langle J(\beta)x, x\rangle \le 0$$

where

$$J(\beta) = K^*\Big(I + \frac{1}{\beta}E_2^*E_2\Big)K + K^*\Big(B_2^*P + \frac{1}{\beta}E_2^*E_1\Big) + \Big(PB_2 + \frac{1}{\beta}E_1^*E_2\Big)K\,.$$

It is easy to see that

$$J(\beta) \ge -\Big(PB_2 + \frac{1}{\beta}E_1^*E_2\Big)\Big(I + \frac{1}{\beta}E_2^*E_2\Big)^{-1}\Big(B_2^*P + \frac{1}{\beta}E_2^*E_1\Big)\,.$$

Hence, for all $x \in \mathcal{D}(A)$

$$\begin{aligned}\langle Ax, Px\rangle + \langle Px, Ax\rangle + \beta\langle PDD^*Px, x\rangle\\ + \frac{1}{\beta}||E_1x||^2 + ||C_1x||^2\\ - \Big\langle R_\beta^{-1}\big(B_2^*P + \frac{1}{\beta}E_2^*E_1\big)x, \big(B_2^*P + \frac{1}{\beta}E_2^*E_1\big)x\Big\rangle \le 0\,.\end{aligned}$$

On the other hand, from Lemma 3, there exists a $\delta_0 > 0$ such that $\Sigma(\delta_0)$ is still quadratically stabilizable with H_∞ norm bound $r > 0$. Note that the difference of $\Sigma(\delta_0)$ with Σ_0 is just in the state operator. For $\Sigma(\delta_0)$ we also have a $\bar{\varepsilon} > 0$ such that following inequality holds for all $x \in \mathcal{D}(A)$

$$\begin{aligned}\langle (A+\delta_0 I)x, Px\rangle + \langle Px, (A+\delta_0 I)\rangle + \bar{\varepsilon}\langle PDD^*Px, x\rangle \\ + \frac{1}{\bar{\varepsilon}}||E_1x||^2 + ||C_1x||^2 \\ - \Big\langle R_{\bar{\varepsilon}}^{-1}\big(B_2^*P + \frac{1}{\bar{\varepsilon}}E_2^*E_1\big)x, \big(B_2^*P + \frac{1}{\bar{\varepsilon}}E_2^*E_1\big)x\Big\rangle \le 0\end{aligned}$$

i.e.,

$$\begin{aligned}&\langle Ax, Px\rangle + \langle Px, Ax\rangle + \bar{\varepsilon}\langle PDD^*Px, x\rangle + \frac{1}{\bar{\varepsilon}}||E_1x||^2 \\ &\quad + ||C_1x||^2 - \Big\langle R_{\bar{\varepsilon}}^{-1}\big(B_2^*P + \frac{1}{\bar{\varepsilon}}E_2^*E_1\big)x, \big(B_2^*P + \frac{1}{\bar{\varepsilon}}E_2^*E_1\big)x\Big\rangle \\ &\le -2\delta_0\langle Px, x\rangle \le -2\,\frac{\delta_0\alpha m_0}{(1+m_1)^2}\langle x, x\rangle\end{aligned}$$

where the last inequality is deduced from previous inequality (I). □

Corollary 9 is a natural implication of Theorem 8, while the proof of Corollary 10 can be finished by combining Theorem 8 with some similar argument from [1, 2].

4. Conclusions

This paper has presented a state feedback law for uncertain distributed parameter systems with time-varying norm-bounded perturbations. Based on the solvability of some Riccati inequalities, a necessary and sufficient condition is given for these uncertain plants to be quadratically stabilizable with an H_∞ norm constraint. Moreover, we also point out some other interesting results.

References

1. Wanyi Chen and Fengsheng Tu, Quadratic stability of uncertain distributed parameter systems, *IMA J. Math. Control Inform.* **11** (1994), 201–207.
2. R. F. Curtain, The strict bounded real lemma in infinite dimensions, *Systems Control Lett.* **20** (1993), 113–116.

3. J. C. Doyle, K. Glover, P. P. Khargonekar and B. A. Francis, State-space solutions to standard H_2 and H_∞ control problems, *IEEE Trans. Automat. Control* **34** (1989), 831–847.
4. B. A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag, 1987.
5. I. R. Petersen, A stabilization algorithm for a class of uncertain systems, *Systems Control Lett.* **8** (1987), 351–357.
6. S. E. Rebiai and A. S. I. Zinober, Stabilization of uncertain distributed parameter systems, *Internat. J. Control* **57** (1993), 1167–1175.
7. W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.
8. Lihua Xie and Carl E. De Souza, Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty, *IEEE Trans. Automat. Control* **37** (1992), 1188–1191.