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Height preserving linear transformations on semisimple K-algebras

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Abstract

Let A be a semisimple, commutative, finite K-algebra, K a number field. In this paper we study a family of height functions on A with special regard toward the characterization of height preserving K-linear transformations. The height functions that we examine are defined as a product over $\mathcal{M}_K$ (the set of places of K) of v-adic norms on the various completions $A_v = A\otimes_K K_v$.

Introduction

Let A be a semisimple, commutative, finite K-algebra, K a number field. An $\mathcal{M}_K$-family of norms on A is a collection $\mathcal{F} = \big\{\|\cdot\|_v\big\}_{v\in\mathcal{M}_K}$, where $\|\cdot\|_v$ is a K_v-norm on A_v. An $\mathcal{M}_K$-family $\mathcal{F}$ is called *admissible* if $\|a\|_v \neq 1$ only for finitely many $v\in\mathcal{M}_K$ for all non-zero $a\in A$. To any admissible $\mathcal{F}$ one associates a height function $H_{\mathcal{F}}$, defined by setting

$$H_{\mathcal{F}}(a) = \prod_{v\in\mathcal{M}_K} \|a_v\|_v^{d_v}$$

where $a \mapsto a_v$ denotes the canonical injection of A into $A_v = A\otimes_K K_v$ and $d_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$. We will construct, for each $1 \le q \le \infty$, a family $\mathcal{F}_q$, and hence a height function $H_q := H_{\mathcal{F}_q}$ which depends only on q and on the algebra structure of A. Our definition agrees with the classical Northcott-Weil ℓ^q-height in the case

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$A = K^n$ (for the Northcott-Weil heights the most frequently used values of q are $1, 2$ and ∞). Among our height functions there is a special one: H_∞. The peculiarity of H_∞ lies in the fact that it can be considered as the canonical height (in a sense analogous to that of [3]) for H_q and the homomorphism $\psi_k : A \to A$, $a \mapsto a^k$, see the remark after proposition 2.1 for a more complete discussion. We also obtain a description of the points of minimal height (proposition 2.6.), which for H_∞ is the analogue of corollary 1.1.1 of [3].

A useful tool in approaching the problem of characterizing the height preserving linear transformations of A is the ℓ^q-*operator height* on $\mathrm{GL}_K\big([A]\big)$, which is defined as

$$\begin{aligned} H_q^{op} : \mathrm{GL}_K\big([A]\big) &\longrightarrow \mathbb{R} \\ T \longmapsto H_q^{op}(T) &= \sup_{a \in A - \{0\}} \frac{H_q\big(T(a)\big)}{H_q(a)}. \end{aligned}$$

The notion of operator height certainly deserves a deeper study which we began in [7] and intend to pursue in a future paper. For the time being we will use it merely as a tool. The decomposition of H_q^{op} as a product of local norms that we obtain (theorem 3.2) reveals itself as the main ingredient to prove our first result about height preserving transformations. Before stating it we need the following definition: An element $a \in A$ is called $K-$*periodic* if the set $\big\{[a^n] \in \mathbb{P}\big([A]\big)\big\}$ is finite, $\mathbb{P}\big([A]\big)$ being the projective space associated to the K-vector space underlying A.

Theorem

Let A be an isotypical semisimple K-algebra. Given $a \in A$ let L_a be the "multiplication by a"map. Let T be an invertible K-linear transformation of A. Then T preserves H_q if and only if there exists $a \in A$ invertible and K-periodic such that $(L_a T)_v$ is an isometry for the v-adic norm of $\mathcal{F}_q$ for all $v \in \mathcal{M}_K$.

The above result combined with some results about isometries for the local norms yields.

Theorem

Let A be an isotypical semisimple K-algebra. Suppose that either A splits over K or $q = 1$ or $q = \infty$. Then $T \in \mathrm{GL}_K\big([A]\big)$ preserves H_q if and only if there exists $a \in A$ invertible and K-periodic such that $L_a T$ is a K-algebra automorphism.

The paper is organized as follows. In section 1 we give the definition of the local norms that will be used to define our height functions. We also prove some results about isometries for the archimedean case, that will be needed in section 3. Homogeneous heights are defined in section 2 where some of their properties, including the appropriate version of Northcott's finiteness theorem, are proved. Section 3 is devoted to the proof of our results about height preserving transformations.

Conventions and Notations. By a k-algebra we will always mean a finite commutative algebra with a unit, (where finite means that it is finite dimensional as a k-vector space). If A is a k-algebra we denote by $(X_A, \mathcal{O}_{X_A})$ the associated affine k-scheme, and by $a \mapsto \widehat{a}$ the canonical isomorphism $A \simeq \Gamma(X_A, \mathcal{O}_{X_A})$. From the structure theorem for semisimple k-algebras one sees immediately that $A_\mathfrak{p}$, the localization of A at any prime ideal $\mathfrak{p}$, is a field. Therefore the stalk of $\mathcal{O}_{X_A}$ at $x \in X$ coincides with $k(x)$ the residue field at x and the structure theorem can be seen as saying that $A \simeq \prod_{x\in X} k(x)$.

If K is a number field, we denote by $\mathcal{M}_K$ the set of equivalence classes of absolute values of K. Moreover $\mathcal{M}_K^0$ (respectively $\mathcal{M}_K^\infty$) is the subset of $\mathcal{M}_K$ formed by the equivalence classes of non-archimedean (resp. archimedean) absolute values. For $v \in \mathcal{M}_K$, $|\cdot|_v$ is the representative of the class v, normalized by requiring that $|\cdot|_v$ restricted to $\mathbb{Q}$ is either the standard p-adic absolute value or the standard archimedean absolute value. With K_v we denote the completion of K with respect to $|\cdot|_v$. With this normalization the product formula reads $\prod_{v\in\mathcal{M}_K} |\lambda|_v^{n_v} = 1$, where $n_v = [K_v : \mathbb{Q}_v]$. Finally we set $d_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$.

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§1. Local norms

In this section we will employ the following notations

F	a field complete with respect to the absolute value $\lvert\cdot\rvert$
A	a semisimple F-algebra
$(X, \mathcal{O}_X)$	the affine F-scheme associated to A
$\lvert\cdot\rvert_x$	the unique extension of $\lvert\cdot\rvert$ to $F(x)$, for $x \in X$.

Let us start with the non-archimedean case since it is the shortest of the two. Thus we assume that $|\cdot|$ is a non-archimedean absolute value. The ℓ^∞-norm on A is

$$\begin{aligned} \| \ \|_{A,\infty} : A &\longrightarrow \mathbb{R} \\ a &\longmapsto \sup_{x\in X} |\widehat{a}(x)|_x. \end{aligned}$$

A endowed with $\| \ \|_{A,\infty}$ becomes a non-archimedean Banach algebra.

Proposition 1.1

Let A and B be semisimple F-algebras.
(a) If $\phi : A \rightarrow B$ is an isomorphism of F-algebras, then $\| a \|_{A,\infty} = \| \phi(a) \|_{B,\infty}$.
(b) $\| a^k \|_{A,\infty} = \|a\|_{A,\infty}^k$.
(c) Suppose $A = \prod_{i=1}^r A_i$ and let $\pi : A \rightarrow A_i$ denote the projection onto the i^{th} factor. Then

$$\| a \|_{A,\infty} = \sup_{1\le i\le r} \| \pi_i(a) \|_{A_i,\infty}.$$

Proof. (a) and (b) follow directly from the definition. To prove (c) let X_i be the affine scheme associated to A_i and denote by $\eta_i : X_i \rightarrow X$ the injection induced by π_i. Then

$$\| \pi_i(a) \|_{A_i,\infty} = \sup_{x\in\eta_i(X_i)} |\widehat{a}(x)|_x$$

and since $X = \coprod_{i=1}^n \eta_i(X_i)$, (c) follows. □

That is all we need in the non-archimedean case. From now on we assume that $|\cdot|$ is an archimedean absolute value. Let $1 \le q \le \infty$. We define the ℓ^q-norm on A, $\| \cdot \|_{A,q} : A \rightarrow \mathbb{R}$, by setting

$$\| a \|_{A,q} = \begin{cases} \left(\sum_{x\in X} \dim_F F(x)\, |\widehat{a}(x)|_x^q \right)^{1/q} & \text{if} \quad 1 \le q < \infty \\ \sup_{x\in X} |\widehat{a}(x)|_x & \text{if} \quad q = \infty. \end{cases}$$

A endowed with any of the above norms becomes a real or complex Banach algebra (depending on whether $F = \mathbb{R}$ or $\mathbb{C}$). Note that if A splits over F then $\| \cdot \|_{A,\infty}$ is nothing else than the standard ℓ^q-norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Proposition 1.2

Let A and B be semisimple F-algebras.

(a) If $\phi : A \to B$ is an isomorphism of F-algebras, then $\| a \|_{A,q} = \| \phi(a) \|_{B,q}$.
(b) $\| a^k \|_{A,\infty} = \|a\|_{A,\infty}^k$
(c) Suppose $A = \prod_{i=1}^r A_i$ and let $\pi_i : A \to A_i$ denote the projection onto the i^{th} factor. Then

$$\| a \|_{A,q} = \begin{cases} \left(\sum_{i=1}^{r} \|\pi_i(a)\|_{A_i,q}^q \right)^{1/q} & \text{if} \quad 1 \leq q < \infty \\ \max_{1 \leq i \leq r} \| \pi_i(a) \|_{A_i,q} & \text{if} \quad q{=}\infty. \end{cases}$$

(d) $\lim_{k\to\infty} \|a^k\|_{A,q}^{\frac{1}{k}} = \|a\|_{A,\infty}$.

Proof. (a), (b) and (c) are proved as in lemma 1.1. (d) follows either from a general result about real and complex Banach algebras, see e.g. [2, I.5.8 and I.13.7], or by a direct computation which is left to the reader. □

Let $\mathrm{GL}_F\big([A]\big)$ be the group of invertible F-linear transformations of A. We denote by $\mathbf{O}_q(A)$ the subgroup of $\mathrm{GL}_F\big([A]\big)$ formed by the isometries for the ℓ^q-norm. Our next goal is to prove a characterization for the elements of $\mathbf{O}_q(A)$. If A splits over F this sort of results are well known:

Proposition 1.3

Suppose that $A = F^n$. Let $\mathfrak{S}_n(\mathcal{U}) \subset \mathrm{GL}(n, F)$ be the subgroup of monomial matrices with entries in $\mathcal{U} = \big\{a{\in}F \;\big|\; |a| = 1\big\}$. Then

$$\mathbf{O}_q(F^n) = \begin{cases} \mathfrak{S}_n(\mathcal{U}) & \text{if} \quad q \neq 2 \\ O(n) & \text{if} \quad q = 2 \text{ and } F = \mathbb{R} \\ U(n) & \text{if} \quad q = 2 \text{ and } F = \mathbb{C} \end{cases}$$

where $O(n)$ $\big($ respectively $U(n)\big)$ denotes the subgroup of orthogonal (resp. unitary) matrices.

Proof. These results can be viewed as special cases of their infinite dimensional version, see [1]. □

It remains to deal with the case of a non-split real algebra. Thus from now on we assume that A is a real semisimple algebra. The characterization that we will be able to obtain is a corollary of the following generalization of the Banach-Stone theorem due to M. Grzesiak.

Theorem 1.4

Let Z be a compact Hausdorff space and $\tau : Z \longrightarrow Z$ be an involution. Set $C(Z,\tau) = \Big\{ f \in C(Z,\mathbb{C}) \;\Big|\; f\big(\tau(z)\big) = \overline{f(z)} \;\forall z \in Z \Big\}$. We always consider $C(Z,\tau)$ endowed with the sup-norm (which makes $C(Z,\tau)$ a real Banach algebra). A map $T : C(Z,\tau) \longrightarrow C(Z,\tau)$ is a surjective linear isometry if and only if there exists a homeomorphism $\alpha : Z \longrightarrow Z$ satisfying $\tau \circ \alpha = \alpha \circ \tau$ and an invertible function $g \in C(Z,\tau)$ satisfying $|g(z)| = 1 \;\forall z \in Z$ such that

$$(Tf)(z) = g(z) f\big(\alpha(z)\big)$$

for every $f \in C(Z,\tau)$ and $z \in Z$.

Proof. See [5]. $\square$

We have to reformulate this general result in our setting. The set $X(\mathbb{C})$ of $\mathbb{C}$-valued points of X is a compact hausdorff space. Recall that $X(\mathbb{C})$ can be interpreted as the set of $\mathbb{R}$-linear homomorphisms of A to $\mathbb{C}$. We define an involution τ on $X(\mathbb{C})$ by setting $\psi^\tau(a) = \overline{\psi(a)}$. Note that the assignment $a \mapsto a^g \in$, where $a^g(\psi) = \psi(a)$ defines an injection $\jmath : A \hookrightarrow C(X(\mathbb{C}),\tau)$ which is isometric if we endowed A with the ℓ^∞-norm. It is straightforward to check that $\dim_{\mathbb{R}} C(X(\mathbb{C}),\tau) = \dim_{\mathbb{R}} A$ and so $\jmath$ is an isometric isomorphism.

Corollary 1.5

Suppose A is a semisimple $\mathbb{R}$-algebra and let T belong to $\mathrm{GL}_{\mathbb{R}}\big([A]\big)$. Then $T \in \mathbf{O}_\infty(A)$ if and only if the following two conditions are satisfied

(1) $T(1) = b$ belongs to $A_1 = \Big\{ a \in A \;\Big|\; |\widehat{a}(x)| = 1 \;\forall x \in X \Big\}$.

(2) $L_b^{-1} T$ is an algebra automorphism.

The same characterization holds for the ℓ^1-norm of A as we shall now show. Recall that on any semisimple real algebra there is a unique involution $*$ which is positive with respect to the trace i.e. $\mathrm{tr}(aa^*) > 0$ for all non-zero $a \in A$. Then

$$\begin{aligned} <\,,\,> \;\; &: A \times A \longrightarrow \mathbb{R} \\ &(a,\, b) \longmapsto \mathrm{tr}(ab^*) \end{aligned}$$

is a positive definite bilinear form on A. Let us identify A with its dual (as real vector spaces) by means of $<\,,\,>$. Under this identification the dual norm of $\|\cdot\|_{A,q}$, denoted by $\|\cdot\|^{\vee}_{A,q}$, becomes a norm on A

$$\|\, a \,\|^{\vee}_{A,q} = \sup_{b \in A - \{0\}} \frac{|<b,a>|}{\|b\|_{A,q}}\,.$$

As in the split case, one checks immediately that $\|\cdot\|^{\vee}_{A,q} = \|\cdot\|_{A,q'}$, where q' is the conjugate exponent of q. By means of $< , >$ we can define an involution, that by an abuse of notation we denote by $*$ on $\mathrm{GL}_{\mathbb{R}}([A])$, by requiring that

$$< T(a), b > = < a, T^*(b) > \tag{1}$$

for all $a, b \in A$. Let $\mathrm{Aut}_{F-alg}(A) \subset \mathrm{GL}_{\mathbb{R}}([A])$ denote the group of automorphism of A as an F-algebra. Note that $T \in \mathrm{GL}_{\mathbb{R}}([A])$ is in $\mathrm{Aut}_{F-alg}(A)$ if and only if T^* is.

Corollary 1.6

The characterization of the isometries for the norm $\|\cdot\|_{A,\infty}$ obtained in corollary 1.5 holds also for $\|\cdot\|_{A,1}$.

Proof. Suppose $T \in \mathbf{O}_1(A)$, then also T^{-1} belongs to $\mathbf{O}_1(A)$. It follows at once from (1) that $(T^{-1})^*$ belongs to $\mathbf{O}_\infty(A)$. Let $c = (T^{-1})^*(1)$ then, by corollary 1.5, $L_c^{-1}(T^{-1})^*$ is an algebra automorphism. But then

$$T^* L_c = \left(L_c{-}1(T^{-1})^*\right)^{-1} \in \mathrm{Aut}_{F-alg}(A)$$

and so $L_c^* T \in \mathrm{Aut}_{F-alg}(A)$. Therefore $c^* = T(1)^{-1}$ and since in general $L_d^* = L_{d^*}$ and $L_d^{-1} = L_{d^{-1}}$ we have $L_{T(1)}^{-1} T \in \mathrm{Aut}_{F-alg}(A)$. Finally, it is immediate to verify that $T(1) = (c^*)^{-1}$ satisfies (1) of corollary 1.5 since c does. □

§ 2. Homogeneous heights

In this section we will employ the following notations:

K	a number field
A	a semisimple K-algebra
$(X, \mathcal{O}_X)$	the affine K-scheme associated to A
$(X_v, \mathcal{O}_{X_v})$	the affine K_v-scheme associated to $A_v = A \otimes_K K_v$
$i_v : A \to A_v, a \mapsto a_v$	the canonical injection
$\pi_v : X_v \to X$	the surjection induced by i_v
$\lvert\cdot\rvert_y$	the unique extension of $\lvert\cdot\rvert_v$ to $K_v(y)$, $y \in X_v$.

As pointed out in the introduction in order to define a height function on A we need only to exhibit an admissible $\mathcal{M}_K$-family. Given $1 \le q \le \infty$ consider the $\mathcal{M}_K$-family $\mathcal{F}_q = \left\{\|\cdot\|_{A_v,\infty}\right\}_{v \in \mathcal{M}_K^0} \bigcup \left\{\|\cdot\|_{A_v,q}\right\}_{v \in \mathcal{M}_K^\infty}$ where the local norms are the ones defined in the previous section. First of all we have to check that $\mathcal{F}_q$ is admissible.

Lemma 2.1

The $\mathcal{M}_K$-family $\mathcal{F}_q$ is admissible.

Proof. By propositions 1.1 and 1.2 we can reduce to the case of a simple K-algebra. Thus $A = E$ is a field extension of K. Then given $a \in E$ we have $\| a_v \|_{E_v,\infty} = \sup_{u\in\mathcal{M}_E^v} |a|_u$ where $\mathcal{M}_E^v = \{u\in\mathcal{M}_E \mid |\cdot|_u\big|_K = |\cdot|_v\}$. So the lemma follows from the standard fact that given $a\in E$ there are only finitely many $u\in\mathcal{M}_E$ such that $|a|_u \neq 1$. $\square$

When no confusion arises we will write $\|\cdot\|_{v,q}$ for $\|\cdot\|_{A_v,q}$. The *absolute homogeneous* ℓ^q-height on A, $H_q :\longrightarrow \mathbb{R}$, is the height associated to $\mathcal{F}_q$. More explicitly let $n_y = \dim_{K_v} K_v(y)$, and let $\sigma(\hat{a}_v) = \{y \in X_v \mid \hat{a}_v(y) \neq 0\}$, then

$$H_q(a) = \begin{cases} \displaystyle\prod_{v\in\mathcal{M}_K} \sup_{y\in\sigma(\hat{a}_v)} |\widehat{a}_v(y)|_y^{d_v} & \text{if } \ q = \infty \\[2ex] \displaystyle\prod_{v\in\mathcal{M}_K^0} \sup_{y\in\sigma(\hat{a}_v)} |\widehat{a}_v(y)|_y^{d_v} \prod_{v\in\mathcal{M}_K^\infty} \Bigg(\sum_{y\in\sigma(\hat{a}_v)} n_y |\widehat{a}_v(y)|_y^q\Bigg)^{d_v} & \text{if } \ 1 \le q < \infty\,. \end{cases}$$

We collect the first properties of H_q in the next proposition.

Proposition 2.2

Let A and B be semisimple K-algebras. Then

(a)	$H_q(\lambda a) = H_q$ *for $a\in A$ and $\lambda\in K^\times$.*	*(scalar invariance)*
(b)	$H_q(aa') \le H_q(a)\cdot H_q(a')$.	*(submultiplicativity)*
(c)	$H_\infty(a^k) = \big(H_\infty(a)\big)^k$.	*(power-multiplicativity)*
(d)	$\lim_{k\to\infty}\big(H_q(a^k)\big)^{\frac{1}{k}} = H_\infty(a)$.	*(Gelfand-Beurling formula)*
(e)	*If $\varphi : A \longrightarrow B$ is K-isomorphism, then $H_q(a) = H_q\big(\varphi(a)\big)$ for all $a\in A$.*	

Proof. (a) follows from the product formula. The remaining ones follow directly from the corresponding properties of the local norms of $\mathcal{F}_q$. $\square$

Remark. Note that (d) can also be proved (in its logarithmic version) by Tate's averaging procedure. In fact denote by ϕ_n the homomorphism $a \mapsto a^n$ and set $h_q = \log H_q$. Since $\mathcal{M}_K^\infty$ is finite we have that $nh_q - h_q\circ\phi_n$ is a bounded function on A. Then Tate's lemma, as described in [6, Lemma 3.1], yields the existence of a unique function $\hat{h}$ such that $h\circ\phi_n = nh$ and h is in the same class of h_q modulo bounded functions. But h_∞ has both these properties and so $\hat{h} = h_\infty$.

Example 1: If $A = K^n$ then H_q coincides with the (absolute) Northcott-Weil ℓ^q-height, i.e.

$$H_q(a) = \begin{cases} \prod_{v\in\mathcal{M}_K} \sup_{1\le i\le n} |a_i|_v^{d_v} & \text{if } q=\infty \\ \prod_{v\in\mathcal{M}_K^0} \sup_{1\le i\le n} |a_i|_v^{d_v} \prod_{v\in\mathcal{M}_K^\infty} \left(\sum_{i=1}^n |a_i|_v^q\right)^{d_v/q} & \text{if } 1\le q<\infty \end{cases}$$

where $a = (a_1, \ldots, a_n) \in K^n$.

2. Let $A = E$ be a field extension of K. Then

$$H_q(a) = \begin{cases} \prod_{v\in\mathcal{M}_K} \sup_{u\in\mathcal{M}_E^v} |a|_u^{d_u} & \text{if } q=\infty \\ \prod_{v\in\mathcal{M}_K^0} \sup_{u\in\mathcal{M}_E^v} |a|_u^{d_u} \prod_{v\in\mathcal{M}_K^\infty} \left(\frac{1}{n_v} \sum_{u\in\mathcal{M}_E^v} n_u |a|_u^q\right)^{d_v/q} & \text{if } 1\le q<\infty. \end{cases}$$

Let L be a finite extension of K. We denote by $\iota_L : A \longrightarrow A_L$ the canonical injection of A into $A_L = A\otimes_K L$. We say that L is a splitting field for A if L is a Galois extension of K and A_L is isomorphic, as L-algebra to L^n $(n = \dim_K A)$. The next proposition gives a useful method for computing H_q.

Proposition 2.3

Let A be a semisimple K-algebra and $1 \le q \le \infty$. Suppose that L is a splitting field of A. Then $H_q\big(\iota_L(a)\big) = H_q(a)$ for all $a\in A$.

Proof. Since H_q is invariant under isomorphisms it is enough to show that $H_q\big((\psi\circ\iota_L)(a)\big) = H_q(a)$, where $\psi : A_L \to L^n$ is any L-isomorphism. Note that the invariance of H_q under K-isomorphisms does not prove the proposition since A and A_L are considered as algebras over different fields. Let $\mathcal{G}_q = \big\{\| \ \|_{w,\infty}\big\}_{w\in\mathcal{M}_L^0} \bigcup \big\{\| \ \|_{w,q}\big\}_{v\in\mathcal{M}_L^\infty}$ be the $\mathcal{M}_L$-family defining H_q on L^n. Since $\mathcal{M}_L = \coprod_{v\in\mathcal{M}_K} \mathcal{M}_L^v$ and $\sum_{w\in\mathcal{M}_L^v} d_w = d_v$ it suffices to prove that for all $a\in A$

$$\| (\psi\circ\iota_L)(a) \|_{w,q} = \| a \|_{v,q} \quad \text{for all } v\in\mathcal{M}_K^\infty \text{ and all } v\in\mathcal{M}_K^0 \text{ (but only } q=\infty). \qquad (*)$$

By propositions 1.1 and 1.2 we need to prove $(*)$ only for simple algebras. Thus we assume that $A = E$ is a field extension of K. Since L is Galois over K there exist $n = [E : K]$ distinct embeddings of E into L over K, say $\phi_1, \ldots, \phi_n$. The map

$$\begin{aligned} \phi : E\otimes_K L &\longrightarrow L^n \\ a\otimes\lambda &\longmapsto \lambda\big(\phi_1(a), \ldots, \phi_n(a)\big) \end{aligned}$$

is an isomorphism of L-algebras and we shall prove that $(*)$ holds for $\phi\circ\iota_L$. Since L is Galois over K, the sets $\{|\cdot|_u\}_{u\in\mathcal{M}_E^v}$ and $\{|\cdot|_w\circ\phi_i\}_{i=1}^n$ contain the same distinct absolute values, yielding $(*)$ for $q=\infty$. Moreover the only difference between the two sets is that in $\{|\cdot|_w\circ\phi_i\}_{i=1}^n$ the same absolute value can appear more than once. The number of times that $|\cdot|_u$ appears in $\{|\cdot|_w\circ\phi_i\}_{i=1}^n$ is $\frac{n_u}{n_v}$ (cf. [4, III.1.20]). Therefore

$$\|\phi(a)\|_{w,q}=\left(\sum_{i=1}^n|\phi_i(a)|_w^q\right)^{1/q}=\left(\sum_{u\in\mathcal{M}_E^v}\frac{n_u}{n_v}|a|_u^q\right)^{1/q}=\|a\|_{v,q}.\ \square$$

Corollary 2.4

Let A and B be semisimple K-algebras. Then

(a) $H_q(a)\geq 1$ *for* $a\neq 0$. *(positivity)*

(b) $H_q(a\otimes b)=H_q(a)H_q(b)$. *(Segre invariance)*

(c) *Let L be any extension of K. Then* $H_q(a)=H_q(\iota_L(a))$ *for all* $a\in A$.

Proposition 2.2 enables us to prove Northcott's Finiteness Theorem for H_q on $\mathbb{P}([A])$.

Corollary 2.5 (Northcott's Finiteness Theorem)

Let A be a semisimple K-algebra. Then for any constant C the set

$$\mathcal{N}_q(\mathbb{P}([A]),C)=\{P\in\mathbb{P}([A])\mid H_q(P)\leq C\}$$

is finite.

Proof. Let L be a splitting field of A and denote by $\varphi: A\longrightarrow L^n$ the composition of ι_L with an isomorphism of A_L into L^n. By Northcott's Finiteness Theorem for projective spaces we know that $\mathcal{N}_q(\mathbb{P}^{n-1}(L),B)$ is finite. Thus the corollary follows from proposition 2.2 and the fact that the map $\widetilde{\varphi}:\mathbb{P}([A])\to\mathbb{P}^{n-1}(L)$ induced by φ is injective. $\square$

Given $f\in\Gamma(X,\mathcal{O}_X)$, the set $\sigma(f)=\{x\in X\mid f(x)\neq 0\}$ is called the *support* of f. An element a of A is called K-*periodic* if there exist $\lambda\in K^\times$ and a positive integer r such that $\widehat{a}^r(x)=\lambda$ for all $x\in\sigma(\widehat{a})$, (or equivalently if the set $\{[a^n]\in\mathbb{P}([A])\}$ is finite). Note that if A is simple, then $a\in A$ is K-periodic if and only if a is a root of a polynomial in $K[X]$ of the form $X^r - \lambda$. The set of K-periodic elements of A is denoted by $\mathrm{Per}_K\big(A\big)$. Finally for $a \in A$ we set $\delta(a) = \sum_{x\in\sigma(\hat{a})} \dim_K K(x)$.

Proposition 2.6

Let A be a semisimple K-algebra and $a \in A$ be non-zero. Then

(a) $H_\infty(a) = 1$ if and only if a is a K-root.

(b) If $1 \le q < \infty$, then

$$H_q(a) \geq \delta(a)^{1/q}$$

and the equality holds if and only if a is K-periodic.

Proof. (a) Suppose first that $a \in \mathrm{Per}_K\big(A\big)$. Then there exists $\lambda \in K^\times$ such that $\lambda \widehat{a}^r(x) = 1$ for all $x \in \sigma(\hat{a})$. Thus $H_\infty(a)^r = H_\infty(a^r) = 1$, which yields $H_\infty(a) = 1$. Suppose instead that $H_\infty(a) = 1$. Then, by proposition 2.1.(d), $H_\infty(a^r) = 1$ for all integers $r \geq 1$. Thus $\big\{[a^n] \in \mathbb{P}\big([A]\big), n \geq 1\big\} \subset \mathcal{N}_q\big(\mathbb{P}\big([A]\big), 1\big)$, but the latter set is finite by Northcott's Finiteness Theorem, hence a is K-periodic.

(b) Let $a \in A$ be non-zero. Since H_q is invariant under multiplication by scalars we can assume $\| a \|_{v,\infty}^{d_v} \geq 1$ for all $v \in \mathcal{M}_K^0$, so

$$\Lambda(a) = \prod_{v\in\mathcal{M}_K^0} \| a \|_{v,\infty}^{d_v} \geq 1 \,.$$

For $x \in X$ set $d_x = \dim_K K(x)$ and $d_y = \dim_{K_v} K_v(y)$, for $y \in X_v$. Then $\sum_{y\in\pi_v^{-1}(x)} d_y = d_x$ which yields $\delta(a_v) = \delta(a)$ for all $v \in \mathcal{M}_K$. Moreover with our notation the product formula $\big($for the number field $K(x)\big)$ reads

$$\prod_{v\in\mathcal{M}_K^0} \prod_{y\in\pi_v^{-1}(x)} |\hat{a}_v(y)|_y^{d_y d_v} \prod_{v\in\mathcal{M}_K^\infty} \prod_{y\in\pi_v^{-1}(x)} |\hat{a}_v(y)|_y^{d_y d_v} = 1 \,.$$

Hence

$$\Lambda(a)^{d_x} \prod_{v\in\mathcal{M}_K^\infty} \prod_{y\in\pi_v^{-1}(x)} |\hat{a}_v(y)|_y^{d_y d_v} \geq 1. \qquad (*)$$

for every $x \in X$. Finally, given $v \in \mathcal{M}_K^\infty$ from the inequality between the arithmetic and the geometric mean we get

$$\sum_{y\in\sigma(\hat{a}_v)} d_y |\hat{a}_v(y)|_y^q \geq \delta(a_v) \left(\prod_{y\in\sigma(\hat{a}_v)} |\hat{a}_v(y)|_y^{q d_y} \right)^{1/\delta(a)} . \qquad (**)$$

Now we have all we need to obtain the lower bound for H_q:

$$\begin{aligned}
H_q(a)^q &= \Lambda(a)^q \prod_{v\in\mathcal{M}_K^\infty} \left(\sum_{y\in\sigma(\hat{a}_v)} d_y |\hat{a}_v(y)|_y^q \right)^{d_v} \\
&\geq \Lambda(a)^q \prod_{v\in\mathcal{M}_K^\infty} \delta(a_v) \left(\prod_{y\in\sigma(\hat{a}_v)} |\hat{a}_v(y)|_y^{qd_y} \right)^{d_v/\delta(a)} && \text{(by (**))} \\
&= \Lambda(a)^q \delta(a) \prod_{v\in\mathcal{M}_K^\infty} \prod_{x\in\sigma(\hat{a})} \left(\prod_{y\in\pi_v^{-1}(x)} |\hat{a}_v(y)|_y^{qd_y} \right)^{d_v/\delta(a)} \\
&= \delta(a) \prod_{x\in\sigma(\hat{a})} \Lambda(a)^{\frac{qd_x}{\delta(a)}} \prod_{v\in\mathcal{M}_K^\infty} \left(\prod_{y\in\pi_v^{-1}(x)} |\hat{a}_v(y)|_y^{d_v d_y} \right)^{q/\delta(a)} \\
&\geq \delta(a). && \text{(by (*))}
\end{aligned}$$

It remains to show that $H_q(a) = \delta(a)^{\frac{1}{q}}$ if and only if a belongs to $\mathrm{Per}_K\big(A\big)$. Suppose a is K-periodic. Then there exists $\lambda\in K^\times$ such that $|\hat{a}_v(y)|_v = |\lambda|_v^{\frac{1}{r}}$ for all $y\in X_v$. Thus

$$H_q(a) = \prod_{v\in\mathcal{M}_K^0} |\lambda|_y^{\frac{d_v}{r}} \prod_{v\in\mathcal{M}_K^\infty} \left(\sum_{y\in\sigma(\hat{a}_v)} d_y |\lambda|_y^{\frac{q}{r}} \right)^{d_v/q} = \prod_{v\in\mathcal{M}_K} |\lambda|_y^{\frac{d_v}{r}} \prod_{v\in\mathcal{M}_K^\infty} \delta(a)^{\frac{d_v}{q}} = \delta(a)^{\frac{1}{q}}\,.$$

Suppose now that $H_q(a) = \delta(a)^{\frac{1}{q}}$. Then in both $(*)$ and $(**)$ the equality holds. For $(*)$ this implies that the equality holds also for a^n (for all $n \geq 1$). In $(**)$ the equality holds if and only if $|\hat{a}_v(y)|$ is independent of y for every $v\in\mathcal{M}_K^\infty$. Thus also in $(**)$ the equality holds for all a^n's . Hence

$$H_q(a^n) = \delta(a^n)^{\frac{1}{q}} = \delta(a)^{\frac{1}{q}}$$

and so Northcott's Finiteness Theorem yields the K-periodicity of a. □

Corollary 2.7

Let A be a semisimple K-algebra and $1 \leq q < \infty$. If $a\in A$ is non-zero, then

$$H_q(a) \geq \left(\min_{x\in X} \dim_K K(x) \right)^{1/q} .$$

and the equality holds iff a is K-periodic, $\sigma(\hat{a}) = \{x_0\}$ and $\dim_K K(x_0) = \min_{x\in X} \dim_K K(x)$.

§3. Height preserving linear transformations

Let $\mathrm{GL}_K\big([A]\big)$ denote the group of invertible linear transformations of $[A]$. Our first necessity is a way to measure how far a linear transformation is from being height preserving. This role can be interpreted by

$$H_q^{op} : \mathrm{GL}_K\big([A]\big) \longrightarrow \mathbb{R}$$
$$T \longmapsto H_q^{op}(T) = \sup_{a\in A-\{0\}} \frac{H_q\big(T(a)\big)}{H_q(a)} .$$

which we call the *operator ℓ^q-height* on $\mathrm{GL}_K\big([A]\big)$. The following properties of H_q^{op} are immediate from the definition.

Proposition 3.1

Let A be a semisimple K-algebra, $T,S\in\mathrm{GL}_K\big([A]\big)$, $T,S\neq 0$ and $\lambda\in K^\times$. Then
(a) $H_q^{op}(T)\geq 1$.
(b) $H_q^{op}(\lambda T) = H_q^{op}(T)$.
(c) $H_q^{op}(ST)\leq H_q^{op}(S)\,H_q^{op}(T)$.

For $v\in\mathcal{M}_K$, we denote by $T\mapsto T_v$ the canonical injection of $\mathrm{GL}_K\big([A]\big)$ into $\mathrm{GL}_{K_v}\big([A_v]\big)$. Our next goal is to have a decomposition of H_q^{op} as product of local norms. The local norms that we intend to use are, in view of the definition of the operator ℓ^q-height, the operator norms on $\mathrm{GL}_{K_v}\big([A_v]\big)$ associated to the norms of $\mathcal{F}_q$. By an abuse of notation we denote by $\|\cdot\|_{v,q}$ the operator norm on $\mathrm{GL}_{K_v}\big([A_v]\big)$ associated to $\|\cdot\|_{v,q}$.

Theorem 3.2

Let A be a semisimple K-algebra. Then

$$H_q^{op}(T) = \prod_{v\in\mathcal{M}_K^0} \|\, T_v \,\|_{v,\infty}^{d_v} \prod_{v\in\mathcal{M}_K^\infty} \|\, T_v \,\|_{v,q}^{d_v}$$

for all $T\in\mathrm{GL}_K\big([A]\big)$.

Before proving theorem 3.2 we need some preparatory work. The subgroup of $\mathrm{GL}_{K_v}\big([A_v]\big)$ formed by the isometries for the norm $\|\cdot\|_{v,q}$ ($q=\infty$ only if v is non archimedean) is denoted by $\mathbf{O}_q(A_v)$.

Lemma 3.3

Let A be a semisimple K-algebra. If $T\in\mathrm{GL}_K\big([A]\big)$, then the set

$$\mathcal{S}_T = \Big\{ v\in\mathcal{M}_K^0 \,|\, T_v \notin \mathbf{O}_\infty(A_v) \Big\}$$

is finite.

Proof. Let L be a splitting field of A, and $\phi_L : A_L \longrightarrow L^n$ be an isomorphism. Let S be the L-linear transformation of L^n defined by $S = (\phi_L \circ T_L \circ \phi_L^{-1})$, where T_L is obtained by extending T by L-linearity to A_L. Suppose that $S_w \in \mathbf{O}_\infty(L_w^n)$ and let $v \in \mathcal{M}_K^0$ be such that w belongs to $\mathcal{M}_L^v$. From the proof of proposition 2.3 (in particular from $(*)$) it follows that $v \in \mathcal{S}_T$ if and only if $\mathcal{M}_L^v \subset \mathcal{S}_S$. Thus it suffices to prove the proposition in the case $A = K^n$. Then we can identify $GL_K\big([K^n]\big)$ with $GL_n(K)$, the group of invertible $n \times n$ matrices with entries in K, and $GL_{K_v}\big([K_v^n]\big)$ with $GL_n(K_v)$. Under these identifications $\mathbf{O}_\infty(K_v^n) \cap GL_K\big([K^n]\big)$ corresponds to $GL_n(\mathcal{O}_v)$, where $\mathcal{O}_v = \big\{\lambda \in K \;\big|\; |\lambda|_v \leq 1\big\}$, and so the lemma follows. □

Given a finite subset $\mathcal{S}$ of $\mathcal{M}_K$ we set $A_{\mathcal{S}} = \prod_{v \in \mathcal{S}} A_v$ and we consider A as embedded diagonally into $A_{\mathcal{S}}$. Set $\mathcal{S}^0 = \mathcal{S} \cap \mathcal{M}_K^0$ and $\mathcal{S}^\infty = \mathcal{S} \cap \mathcal{M}_K^\infty$. We define a metric on A by setting

$$\begin{aligned} d_q : A_{\mathcal{S}} \times A_{\mathcal{S}} &\longrightarrow \mathbb{R} \\ (\overline{\alpha}\,, \overline{\beta}) &\longmapsto d_q(\overline{\alpha}, \overline{\beta}) = \max\left\{ \sup_{v \in \mathcal{S}^0} \| \alpha_v - \beta_v \|_{v,\infty}, \sup_{v \in \mathcal{S}^\infty} \| \alpha_v - \beta_v \|_{v,q} \right\} \end{aligned}$$

where $\overline{\alpha} = \{\alpha_v\}_{v \in \mathcal{S}}$, and $\overline{\beta} = \{\beta_v\}_{v \in \mathcal{S}}$.

Proposition 3.4

Let A be a semisimple K-algebra, $\mathcal{S}$ a finite subset of $\mathcal{M}_K$ and $1 \leq q \leq \infty$. Then A is dense in $A_{\mathcal{S}}$ with respect to the metric d_q.

Proof. If A is simple the proposition follows from the weak approximation theorem. The general case is reduced to the case of A simple by means of propositions 1.1 and 1.2. □

Corollary 3.5

Let A be a semisimple K-algebra, $\mathcal{S}$ a finite subset of $\mathcal{M}_K$, $T \in \mathrm{GL}_K\big([A]\big)$ and $1 \leq q \leq \infty$. Then for every $\varepsilon > 0$ there exists $a \in A$ such that

$$\| T_v \|_{v,q} < \frac{\| T(a) \|_{v,q}}{\| a \|_{v,q}^{d_v}} + \varepsilon \quad \forall\, v \in \mathcal{S}^\infty \quad \textit{and} \quad \| T_v \|_{v,\infty} < \frac{\| T(a) \|_{v,\infty}}{\| a \|_{v,\infty}^{d_v}} + \varepsilon \quad \forall\, v \in \mathcal{S}^0\,.$$

We can now proceed to the proof of theorem 3.2.

Proof of theorem 3.2. The inequality

$$H_q^{op}(T) \leq \prod_{v \in \mathcal{M}_K^0} \| T_v \|_{v,\infty}^{d_v} \prod_{v \in \mathcal{M}_K^\infty} \| T_v \|_{v,q}^{d_v}$$

is clear. Thus it suffices to show that for every $\varepsilon > 0$ there exists $a \in A$ such that

$$\prod_{v\in\mathcal{S}^0} \| T_v \|_{v,\infty}^{d_v} \prod_{v\in\mathcal{S}^\infty} \| T_v \|_{v,q}^{d_v} < \frac{H_q(T(a))}{H_q(a)} + \varepsilon$$

where $\mathcal{S} = \mathcal{S}^0 \bigcup \mathcal{S}^\infty$, $\mathcal{S}^0 = \{v \in \mathcal{M}_K^0 \mid T \notin \mathbf{O}_\infty(A_v)\}$ and $\mathcal{S}^\infty = \{v \in \mathcal{M}_K^\infty \mid T \notin \mathbf{O}_q(A_v)\}$. Fix $\varepsilon > 0$. By lemma 3.3 $\mathcal{S}$ is finite and so we can find $\delta > 0$ such that

$$\left(\prod_{v\in\mathcal{S}^0} \| T_v \|_{v,\infty}^{d_v} \prod_{v\in\mathcal{S}^\infty} \| T_v \|_{v,q}^{d_v}\right) - \varepsilon < \prod_{v\in\mathcal{S}^0} \left(\| T_v \|_{v,\infty}^{d_v} - \delta\right) \prod_{v\in\mathcal{S}^\infty} \left(\| T_v \|_{v,q}^{d_v} - \delta\right).$$

By corollary 3.5 there exists $a \in A$ such that

$$\| T_v \|_{v,\infty}^{d_v} - \delta < \frac{\| T(a) \|_{v,\infty}^{d_v}}{\| a \|_{v,\infty}^{d_v}} \qquad \forall v \in \mathcal{S}^0$$

and

$$\| T_v \|_{v,q}^{d_v} - \delta < \frac{\| T(y) \|_{v,q}^{d_v}}{\| \vec{y} \|_{v,q}^{d_v}} \qquad \forall v \in \mathcal{S}^\infty.$$

Taking the product over $v \in S$ we have

$$\begin{aligned}\left(\prod_{v\in\mathcal{S}^0} \| T_v \|_{v,\infty}^{d_v} \prod_{v\in\mathcal{S}^\infty} \| T_v \|_{v,q}^{d_v}\right) - \varepsilon &< \prod_{v\in\mathcal{S}^0} \left(\| T \|_{v,\infty}^{d_v} - \delta\right) \prod_{v\in\mathcal{S}^\infty} \left(\| T \|_{v,q}^{d_v} - \delta\right) \\ &< \prod_{v\in\mathcal{S}^0} \frac{\| T(a) \|_{v,\infty}^{d_v}}{\| a \|_{v,\infty}^{d_v}} \prod_{v\in\mathcal{S}^\infty} \frac{\| T(a) \|_{v,q}^{d_v}}{\| a \|_{v,q}^{d_v}} \\ &= \frac{H_q(T(a))}{H_q(a)}. \ \square\end{aligned}$$

As we said in the introduction our main interest is to give an explicit description of the linear transformations that preserve the ℓ^q-height on a semisimple K-algebra. Set

$$\mathcal{H}_q(A) = \left\{T \in \mathrm{GL}_K\big([A]\big) \mid H_q\big(T(a)\big) = H_q(a) \ \forall a \in A\right\}.$$

Thus $\mathcal{H}_q(A) \subset \mathrm{GL}_K\big([A]\big)$ is the subgroup of linear transformations that preserve the ℓ^q-height on A. Note that $\mathrm{Aut}_{K-alg}\big(A\big) \subset \mathcal{H}_q(A)$. If $a \in A$ is invertible $L_a \in \mathrm{GL}_K\big([A]\big)$ denotes the invertible linear transformation given by "multiplication by a".

Lemma 3.6

Let A be a semisimple K-algebra. If $a \in A$ is invertible, then $L_a \in \mathcal{H}_q(A)$ if and only if a is K-periodic.

Proof. If $L_a \in \mathcal{H}_q(A)$, then

$$H_q(a) = H_q\bigl(L_a(1)\bigr) = \begin{cases} 1 & \text{if } q = \infty \\ (\dim_K A)^{\frac{1}{q}} & \text{if } 1 \le q < \infty. \end{cases}$$

Thus, by proposition 2.5 $a \in \mathrm{Per}_K\bigl(A\bigr)$. Viceversa suppose a is K-periodic and invertible. Then there exists $\lambda \in K^\times$ such that $|\widehat{a}_v(y)|_v = |\lambda|_v^{\frac{1}{r}}$ for all $y \in X_v$. Hence

$$\begin{aligned} H_q\bigl(L_a(b)\bigr) &= \prod_{v\in\mathcal{M}_K^0} \sup_{y\in\sigma(\hat{b}_v)} |\widehat{a}_v(y)\widehat{b}_v(y)|_v^{d_v} \prod_{v\in\mathcal{M}_K^\infty} \left(\sum_{y\in\sigma(\hat{b}_v)} n_y |\widehat{a}_v(y)\widehat{b}_v(y)|_v^q \right)^{d_v} \\ &= \prod_{v\in\mathcal{M}_K} |\lambda|_v^{\frac{d_v}{r}} \prod_{v\in\mathcal{M}_K^0} \sup_{y\in\sigma(\hat{b}_v)} |\widehat{b}_v(y)|_v^{d_v} \prod_{v\in\mathcal{M}_K^\infty} \left(\sum_{y\in\sigma(\hat{a}_v)} n_y |\widehat{b}_v(y)|_v^q \right)^{d_v} \\ &= H_q(b). \end{aligned}$$

The analogous computation holds for $q = \infty$. $\square$

Let $\mathrm{Per}_K^\times\bigl(A\bigr) \subset \mathrm{Per}_K\bigl(A\bigr)$ be formed by the K-periodic elements of A that are invertible. Note that $\mathrm{Per}_K^\times\bigl(A\bigr)$ is a subgroup of $A^\times$. A K-algebra A is called isotypical if all its simple components are isomorphic or equivalently if $K(x) \simeq K(y)$ for all $x, y \in X$.

Theorem 3.7

Suppose A is an isotypical K-algebra and $1 \le q \le \infty$. If $T \in \mathrm{GL}_K\bigl([A]\bigr)$, then T belongs to $\mathcal{H}_q(A)$ if and only if there exists $a \in \mathrm{Per}_K^\times\bigl(A\bigr)$, such that $(L_aT)_v \in \mathbf{O}_\infty(A_v)$ for all $v \in \mathcal{M}_K^0$ and $(L_aT)_v \in \mathbf{O}_q(A_v)$ for all $v \in \mathcal{M}_K^\infty$.

Proof. The "if "part follows directly from lemma 3.6. Suppose now that T belongs to $\mathcal{H}_q(A)$. Choose $z \in X$ such that $\dim_K K(z) = \min_{x\in X} \dim_K K(x)$ and let $b \in A$ be such that $\widehat{b}(y) = 0$ if $y \ne x$ and $\widehat{b}(z) = 1$. Then $H_q(b) = 1$ and since $T \in \mathcal{H}_q(A)$ corollary 2.7 yields that $T(b)$ is K-periodic. Since A is isotypical there exists $a \in \mathrm{Per}_K^\times\bigl(A\bigr)$, such that

$$\widehat{a}(x)\widehat{T(b)}(x) = 1 \qquad \text{for all } x \in \sigma\left(\widehat{a}\right). \tag{$*$}$$

Then $\| (L_aT)_v \|_{v,\infty} \ge 1 \ \ \forall v \in \mathcal{M}_K^0$ and $\| (L_aT)_v \|_{v,q} \ge 1 \ \forall v \in \mathcal{M}_K^\infty$. By lemma 3.6. L_a belongs to $\mathcal{H}_q(A)$ and so does L_aT. Then, by theorem 3.4, we have

$$1 = H_q^{op}(L_aT) = \prod_{v\in\mathcal{M}_K^0} \| (L_aT)_v \|_{v,\infty}^{d_v} \prod_{v\in\mathcal{M}_K^\infty} \| (L_aT)_v \|_{v,q}^{d_v}$$

which combined with $(*)$ yields

$$\| (L_aT)_v \|_{v,\infty} = 1 \qquad \forall v \in \mathcal{M}_K^0 \qquad \text{and} \qquad \| (L_aT)_v \|_{v,q} = 1 \qquad \forall v \in \mathcal{M}_K^\infty . \quad (**)$$

Suppose there exists $u \in \mathcal{M}_K^0$ such that $(L_aT)_u \notin \mathbf{O}_\infty(A_u)$. Hence we can find $a \in A$ such that $\| L_aT(a) \|_{u,\infty} \neq \| a \|_{u,\infty}$. By $(**)$ we must have $\| L_aT(a) \|_{u,\infty} < \| a \|_{u,\infty}$. But then

$$\begin{aligned} H_q(a) &= \prod_{v \in \mathcal{M}_K^0} \| L_aT(a) \|_{v,\infty}^{d_v} \prod_{v \in \mathcal{M}_K^\infty} \| L_aT(a) \|_{v,q}^{d_v} \\ &< \prod_{v \in \mathcal{M}_K^0} \| a \|_{v,\infty}^{d_v} \prod_{v \in \mathcal{M}_K^\infty} \| a \|_{v,q}^{d_v} = H_q(a) \end{aligned}$$

which is a contradiction since $L_aT \in \mathcal{H}_q(a)$. The same computation shows the impossibility of the existence of $u \in \mathcal{M}_K^\infty$ such that $(L_aT)_u \notin \mathbf{O}_q(A_u)$. $\square$

We would like to have a more explicit characterization of the height preserving transformations. As we already remarked $\mathcal{H}_q(a)$ contains both $\mathrm{Aut}_{K-alg}\big(A\big)$ and $\big\{ L_a \;\big|\; a \in \mathrm{Per}_K^\times\big(A\big)\big\}$ and thus the subgroup that they generate, which is isomorphic to the semidirect product of the two subgroups. The next theorem shows that for a large class of algebras that is all.

Theorem 3.8

Let A be an isotypical semisimple K-algebra and $1 \le q \le \infty$. Suppose that one of the following conditions is satisfied:

(1) either $q = 1$ or $q = \infty$

(2) A splits over K,

then T belongs to $\mathcal{H}_q(a)$ if and only if there exists $a \in \mathrm{Per}_K^\times\big(A\big)$ such that L_aT is a K-algebra automorphism.

Proof. Suppose first that either $q = 1$ or $q = \infty$. By theorem 3.7 there exists $c \in \mathrm{Per}_K^\times\big(A\big)$, such that $S = (L_aT)_v \in \mathbf{O}_q(A_v)$. Since $S_v \in \mathbf{O}_q(A_v)$ theorem 1.6, implies that $(L_{b^{-1}})_v S_v$ is an algebra automorphism, with $b = S(1)$. But then $L_b^{-1} = (L_b^{-1}S)S^{-1} \in \mathcal{H}_q(A)$, so by lemma 3.6 $b \in \mathrm{Per}_K^\times\big(A\big)$. Set $a = b^{-1}c$, then L_aT is a K-algebra automorphism of A.

Suppose now that A splits over K so that we can assume $A = K^n$. Let us identify $\mathrm{GL}_K\big([K^n]\big)$ with $\mathrm{GL}(n, K)$ the group of invertible $n \times n$ matrices with coefficient in K. Let $\mathfrak{S}_n(\Gamma) \subset \mathrm{GL}(n, K)$ denote the subgroup of monomial matrices with entries in Γ, where $\Gamma \subset K^\times$ is a subgroup. Since $a = (a_1, \ldots, a_n) \in K^n$ is invertible and K-periodic if and only if there exists $\lambda \in K^\times$ such that $\lambda a_i \in \mu_k$ for all $i = 1, \ldots, n$, theorem 3.7 implies that it is enough to show that

$$\bigcap_{v \in \mathcal{M}_K^0} \mathrm{O}_{v,\infty}(K^n) \bigcap_{v \in \mathcal{M}_K^\infty} \mathrm{O}_{v,q}(K^n) = \mathfrak{S}_n(\mu_k).$$

where $\mathrm{O}_{v,q}(K^n) = \mathbf{O}_q(K_v^n) \cap \mathrm{GL}(n, K)$. Let $\mathcal{O}_v = \{\lambda \in K \mid |\lambda|_v \leq 1\}$. If $v \in \mathcal{M}_K^0$, then $\mathrm{O}_{v,\infty}(K^n) = \mathrm{GL}(n, \mathcal{O}_v)$, so that

$$\bigcap_{v \in \mathcal{M}_K^0} \mathrm{O}_{v,\infty}(K^n) = \bigcap_{v \in \mathcal{M}_K^0} \mathrm{GL}(n, \mathcal{O}_v) = \mathrm{GL}(n, \mathcal{O}_K).$$

where $\mathcal{O}_K$ is the ring of integers of K. Thus all that is left to prove is the following assertion: if $S = (s_{ij}) \in \mathrm{GL}(n, \mathcal{O}_K)$ is such that $S \in \mathrm{O}_{v,q}(K^n)$ for all $v \in \mathcal{M}_K^\infty$ then $S \in \mathfrak{S}_n(\mu_k)$. If $q \neq 2$, then, by proposition 1.5, $\mathrm{O}_{v,q}(K^n) = \mathfrak{S}_n(\mathcal{U}_v)$ where $\mathcal{U}_v = \{\lambda \in K \mid |\lambda|_v = 1\}$. By Kronecker's theorem every non-zero entry of S must be a root of unity. If $q = 2$, let $\{e_1, \ldots e_n\}$ denote the canonical basis of K^n. Then

$$1 = \| e_i \|_{v,2} = \| S(e_i) \|_{v,2} = \left(\sum_{j=0}^{n} |s_{ij}|_v \right)^{1/2}. \qquad (*)$$

It follows that $|s_{ij}|_v \leq 1$ for all $v \in \mathcal{M}_K^\infty$, and since we already know that the s_{ij}'s are algebraic integers, Kronecker's theorem implies again that all the non-zero s_{ij}'s are roots of unity. Then, looking back at $(*)$, we see that the only possibility is that $S \in \mathfrak{S}_n(\mu_k)$. □

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