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Modular automorphisms of the Drinfeld modular curves $X_0(\mathfrak{n})$

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Abstract

For $\mathfrak{n} \in \mathbb{F}_q[T]$, we determine the group of modular automorphisms of the Drinfeld modular curve $X_0(\mathfrak{n})$ or equivalently, the normalizer of the Hecke congruence subgroup $\Gamma_0(\mathfrak{n})$ in $GL_2(\mathbb{F}_q((T^{-1})))$. Some applications to the strong Weil uniformization of elliptic curves over $\mathbb{F}_q(T)$ are given.

Let $\mathbb{F}_q$ be the finite field with q elements, $A = \mathbb{F}_q[T]$ the polynomial ring, $K = \mathbb{F}_q(T)$ the rational function field, and K_∞ the completion of K at the place $\infty = \frac{1}{T}$. These are the characteristic p analogues of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. As an analogue of the complex numbers $\mathbb{C}$ we take C, the completion of the algebraic closure of K_∞. Throughout this paper, $\mathfrak{n}$ will denote a monic element of A and $\mathfrak{p}$ and $\mathfrak{p}_i$ will be primes (i.e., monic irreducible elements of A).

The group $GL_2(K_\infty)$ acts by fractional linear transformations on the Drinfeld upper halfplane $\Omega := C - K_\infty$. The quotient space $\Gamma_0(\mathfrak{n})\backslash\Omega$ by the Hecke congruence subgroup

$$\Gamma_0(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A) \ : \ \mathfrak{n}|c \right\}$$

is a rigid analytic space that can be compactified by adding the finite set of cusps $\Gamma_0(\mathfrak{n})\backslash\mathbb{P}^1(K)$. As in the classical situation, we thus obtain the Drinfeld modular curve

$$X_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n})\backslash\Omega \,\dot\cup\, \Gamma_0(\mathfrak{n})\backslash\mathbb{P}^1(K)\,,$$

which as a curve is defined over K. Without further explanation we mention that $X_0(\mathfrak{n})$ is a coarse moduli scheme for rank 2 Drinfeld A-modules with a fixed cyclic $\mathfrak{n}$-isogeny. For all this and more information on $X_0(\mathfrak{n})$, see [2].

We also need the Bruhat-Tits tree $\mathcal{T}$ of $GL_2(K_\infty)$. This is a $(q+1)$-valent tree, whose vertices are the cosets $GL_2(K_\infty)/K_\infty^\times \cdot GL_2(\mathcal{O}_\infty)$, where $\mathcal{O}_\infty$ is the valuation ring of K_∞. Its oriented edges are the cosets $GL_2(K_\infty)/K_\infty^\times \cdot \mathcal{J}$, where $\mathcal{J}$ is the group $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathcal{O}_\infty) \ : \ v_\infty(c) > 0\}$, and the canonical reduction maps an oriented edge to its terminal vertex. Inversion of an edge is given by multiplication from the right with $\left(\begin{smallmatrix} 0 & 1 \\ T^{-1} & 0 \end{smallmatrix}\right)$. Thus $GL_2(K_\infty)$ acts in an obvious way on $\mathcal{T}$.

There is a $GL_2(K_\infty)$-invariant mapping from Ω to $\mathcal{T}$ (see [4] or [3] for an exact treatment), which makes it possible to reduce some questions concerning Drinfeld modular curves to graph-theoretic problems.

The quotient graph $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$ may be considered as a rough picture of $X_0(\mathfrak{n})$. It is a finite graph with a finite number of half-lines (i.e., graphs of the form •—•—•— . . .) attached to it. These are in one-to-one correspondence with the cusps of $X_0(\mathfrak{n})$ and hence are also called cusps.

Denote by $\mathcal{N}_{GL_2(K_\infty)}(\Gamma_0(\mathfrak{n}))$ the normalizer of $\Gamma_0(\mathfrak{n})$ in $GL_2(K_\infty)$. It is not too difficult to show that the operation of $GL_2(K_\infty)$ on Ω resp. $\mathcal{T}$ induces an injective mapping from

$$\mathcal{M}(\mathfrak{n}) := \mathcal{N}_{GL_2(K_\infty)}(\Gamma_0(\mathfrak{n}))/(K_\infty^\times \cdot \Gamma_0(\mathfrak{n}))$$

into $Aut(X_0(\mathfrak{n}))$ resp. $Aut(\Gamma_0(\mathfrak{n})\backslash\mathcal{T})$. Its image is called the subgroup of *modular automorphisms* of $X_0(\mathfrak{n})$ resp. $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$.

For example, fix a monic $\mathfrak{m} \in A$ with $\mathfrak{m}|\mathfrak{n}$ and $(\mathfrak{m}, \frac{\mathfrak{n}}{\mathfrak{m}}) = 1$. Then all the matrices $\left(\begin{smallmatrix} \mathfrak{m}a & b \\ \mathfrak{n}c & \mathfrak{m}d \end{smallmatrix}\right)$ with determinant $\varepsilon\mathfrak{m}$ ($a, b, c, d \in A$ and $\varepsilon \in \mathbb{F}_q^\times$) are in $\mathcal{N}_{GL_2(K)}(\Gamma_0(\mathfrak{n}))$. They are even all in the same coset modulo $\Gamma_0(\mathfrak{n})$, so they all induce the same modular automorphism of $X_0(\mathfrak{n})$ or $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$, the so-called (partial) Atkin-Lehner involution $W_\mathfrak{m}$.

Clearly, $W_\mathfrak{m}^2 = id$, and for divisors $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{n}$ as above, we have $W_{\mathfrak{m}_1}W_{\mathfrak{m}_2} = W_{\mathfrak{m}_3}$ with $\mathfrak{m}_3 = \frac{\mathfrak{m}_1\mathfrak{m}_2}{(\mathfrak{m}_1,\mathfrak{m}_2)^2}$. Hence the Atkin-Lehner involutions form a 2-elementary abelian subgroup $\mathcal{W}(\mathfrak{n})$ of $\mathcal{M}(\mathfrak{n})$ of cardinality 2^s, where s is the number of different prime divisors of $\mathfrak{n}$. As automorphisms of $X_0(\mathfrak{n})$ the Atkin-Lehner involutions are rational over K. For their interpretation on the moduli problem "Drinfeld modules plus $\mathfrak{n}$-isogeny" see [8].

By $\underline{H}_!(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{n})}$ we denote the space of $\mathbb{R}$-valued, alternating, harmonic, $\Gamma_0(\mathfrak{n})$-invariant functions on the oriented edges of $\mathcal{T}$, having finite support modulo $\Gamma_0(\mathfrak{n})$. Its dimension is $g(X_0(\mathfrak{n}))$, the genus of $X_0(\mathfrak{n})$. There exists a Petersson scalar product $(\cdot, \cdot)$ on $\underline{H}_!(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{n})}$, taking integral values on $\underline{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$.

More visible is the homology of the graph $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$, denoted by

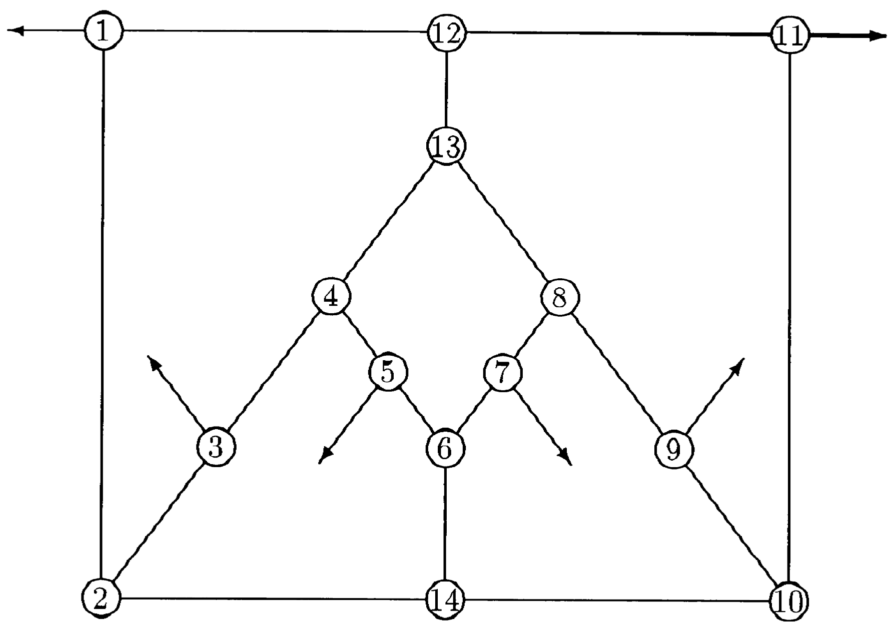
$$H_1(\Gamma_0(\mathfrak{n})\backslash\mathcal{T}, \mathbb{R}).$$

The modules $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ and $\mathrm{H}_1(\Gamma_0(\mathfrak{n})\backslash\mathcal{T},\mathbb{Z})$ are isomorphic. If $q=2$, the isomorphism is induced by the canonical mapping from $\mathcal{T}$ to $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$ and the scalar product of φ and ψ in $\mathrm{H}_1(\Gamma_0(\mathfrak{n})\backslash\mathcal{T},\mathbb{R})$ is just $\frac{1}{2}\sum_e \varphi(e)\psi(e)$, the sum being taken over the oriented edges of $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$. If $q>2$, one has to introduce weight factors (see [4] or [3] for more details).

In any case the modular automorphisms operate from the right on $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{R})^{\Gamma_0(\mathfrak{n})}$ and on $\mathrm{H}_1(\Gamma_0(\mathfrak{n})\backslash\mathcal{T},\mathbb{R})$ by acting from the left on the edges. Further, for $G\leq\mathcal{M}(\mathfrak{n})$ the dimension of the subspaces of G-invariants equals the genus of $G\backslash X_0(\mathfrak{n})$.

Example: For $q=2$ and $\mathfrak{n}=T^2(T^2+T+1)$ the graph $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$ is given in the picture below. The cusps are abbreviated by arrows.

We see that the curve $X_0(\mathfrak{n})$ has 6 cusps and that its genus is 5. The full Atkin-Lehner involution $W_\mathfrak{n}$ is the reflection at the middle axis, so the genus of the curve $W_\mathfrak{n}\backslash X_0(\mathfrak{n})$ is 2.



Theorem 1

a) $\mathcal{N}_{GL_2(K_\infty)}(\Gamma_0(\mathfrak{n})) = K_\infty^\times \cdot \mathcal{N}_{GL_2(K)}(\Gamma_0(\mathfrak{n}))$.
b) If $q > 2$ *then* $\mathcal{M}(\mathfrak{n}) = \mathcal{W}(\mathfrak{n})$*, that is, the partial Atkin-Lehner involutions are the only modular automorphisms.*
c) If $q = 2$ *and* $\mathfrak{n} = \prod \mathfrak{p}_i^{e_i}$*, we define* $U_1 = \left(\begin{smallmatrix} 1 & 0 \\ \frac{\mathfrak{n}}{T} & 1 \end{smallmatrix}\right)$ *and* $U_2 = \left(\begin{smallmatrix} 1 & 0 \\ \frac{\mathfrak{n}}{T^2} & 1 \end{smallmatrix}\right)$ *and*

$$\mathcal{M}_T(\mathfrak{n}) = \begin{cases} \langle id \rangle & \text{if } ord_T(\mathfrak{n}) = 0 \\ \langle W_T \rangle \cong C_2 & \text{if } ord_T(\mathfrak{n}) = 1 \\ \langle W_{T^2}, U_1 \rangle \cong S_3 & \text{if } ord_T(\mathfrak{n}) = 2 \\ \langle W_{T^3}, U_1 \rangle \cong D_4 & \text{if } ord_T(\mathfrak{n}) = 3 \\ \langle W_{T^4}, U_1, U_2 \rangle \cong S_4 & \text{if } ord_T(\mathfrak{n}) = 4 \\ \langle W_{T^5}, U_1, U_2 \rangle \cong D_8 \rtimes C_2 & \text{if } ord_T(\mathfrak{n}) = 5 \\ \langle W_{T^\nu}, U_1, U_2 \rangle & \text{if } ord_T(\mathfrak{n}) = \nu > 5\,. \end{cases}$$

For $ord_T(\mathfrak{n}) > 5$ *the group* $\mathcal{M}_T(\mathfrak{n})$ *is non-abelian of order 32. As an automorphism of* $X_0(\mathfrak{n})$ *the involution* U_1 *is rational over* K*, whereas the involution* U_2 *is rational only over* $K(\alpha)$ *with* $\alpha^2 + \alpha = T^{-1}$.

$\mathcal{M}_{T+1}(\mathfrak{n})$ *is similarly defined with* $V_i = \left(\begin{smallmatrix} 1 & 0 \\ \frac{\mathfrak{n}}{(T+1)^i} & 1 \end{smallmatrix}\right)$.
The involutions U_1, U_2, V_1, *and* V_2 *commute with each other. For every* $W_\mathfrak{m}$ *with* $T \nmid \mathfrak{m}$ *we have* $W_\mathfrak{m} U_1 = U_1 W_\mathfrak{m}$ *and*

$$W_\mathfrak{m} U_2 = \begin{cases} U_2 W_\mathfrak{m} & \text{if } \mathfrak{m} \equiv 1 \text{ mod } T^2, \\ U_1 U_2 W_\mathfrak{m} & \text{if } \mathfrak{m} \equiv T+1 \text{ mod } T^2\,. \end{cases}$$

Analogously for V_1 *and* V_2.

There exists a semi-direct product decomposition

$$\mathcal{M}(\mathfrak{n}) = \langle \mathcal{M}_T(\mathfrak{n}), \mathcal{M}_{T+1}(\mathfrak{n}) \rangle \rtimes \langle W_{\mathfrak{p}_i^{e_i}} : \mathfrak{p}_i \neq T, T+1 \rangle$$

with operation given by the relations above. Moreover,

$$\langle \mathcal{M}_T(\mathfrak{n}), \mathcal{M}_{T+1}(\mathfrak{n}) \rangle = \mathcal{M}_T(\mathfrak{n}) \mathcal{M}_{T+1}(\mathfrak{n}),$$

which means that every $M \in \langle \mathcal{M}_T(\mathfrak{n}), \mathcal{M}_{T+1}(\mathfrak{n}) \rangle$ *may be written as* $M = M_T M_{T+1}$ *with uniquely determined* $M_T \in \mathcal{M}_T(\mathfrak{n})$ *and* $M_{T+1} \in \mathcal{M}_{T+1}(\mathfrak{n})$.

One sees that $\mathcal{M}(\mathfrak{n})$ shows a similar feature as in the classical situation (compare [1] Theorem 8), where the existence of modular automorphisms that are no Atkin-Lehner involutions depends on divisibility of $\mathfrak{n}$ by 4 or 9. As in [6] p. 289, the modular automorphisms U_1 and U_2 can be given a modular interpretation on Drinfeld modules and $\mathfrak{n}$-isogenies.

We only give the idea of the proof of b) to show why the situation is different for $q = 2$.

If $M \in \mathcal{M}(\mathfrak{n})$ then by a) we may suppose $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in A$ and $gcd(a, b, c, d) = 1$. With $D = det(M)$, from $M\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)M^{-1} \in \Gamma_0(\mathfrak{n})$ and $M\left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{n} & 1 \end{smallmatrix}\right)M^{-1} \in \Gamma_0(\mathfrak{n})$ one obtains $D|\mathfrak{n}$, $D|a^2$, $(D, b) = 1$, $D\mathfrak{n}|c^2$ and $D|d^2$.

If $q > 2$ then $\mathbb{F}_q^\times$ contains an $\varepsilon \neq 1$ and from $M\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)M^{-1} \in \Gamma_0(\mathfrak{n})$ one can calculate that M is an Atkin-Lehner involution.

For $q = 2$ one may give necessary and sufficient conditions for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to be in $\mathcal{N}_{GL_2(K)}(\Gamma_0(\mathfrak{n}))$ but several more pages are needed to derive statement c). A complete proof will be included in [7].

In $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{n})}$ there exists the subspace of newforms $\underline{\mathrm{H}}_!^{new}(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{n})}$, the orthogonal complement of certain embeddings of $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{m})}$ for proper divisors $\mathfrak{m}$ of $\mathfrak{n}$. And for every $\mathfrak{p} \nmid \mathfrak{n}$ there exists a Hecke operator $\mathcal{H}_\mathfrak{p}$ on $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{n})}$ (compare [4] or [3] for exact definitions). These Hecke operators are simultaneously diagonalizable on $\underline{\mathrm{H}}_!^{new}(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{n})}$.

Now in our situation we dispose of the following deep theorem, analogous to the Shimura-Taniyama-Weil conjecture in the classical context.

Theorem 2 ([4], [3])

a) The $\mathbb{F}_q(T)$-isogeny classes of elliptic curves over $\mathbb{F}_q(T)$ with conductor $\infty \cdot \mathfrak{n}$ and split multiplicative reduction at ∞ are in one-to-one correspondence with the 1-dimensional simultaneous eigenspaces of $\underline{\mathrm{H}}_!^{new}(\mathcal{T}, \mathbb{R})^{\Gamma_0(\mathfrak{n})}$ with rational eigenvalues for the Hecke operators $\mathcal{H}_\mathfrak{p}$. Moreover, for $\mathfrak{p} \nmid \mathfrak{n}$ the number of $A/\mathfrak{p}$-rational points of the reduction mod $\mathfrak{p}$ of any such curve is $q^{deg(\mathfrak{p})} + 1 - c_\mathfrak{p}$, where $c_\mathfrak{p}$ is the $\mathcal{H}_\mathfrak{p}$-eigenvalue of the corresponding simultaneous eigenspace.
b) Every such eigenspace contains a (up to sign unique) primitive $\varphi \in \underline{\mathrm{H}}_!^{new}(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$. The degree $-v_\infty(j(E))$ of the j-invariant of the strong Weil curve E in the corresponding isogeny class is the minimal positive scalar product of φ with elements of $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$. The degree of the strong Weil uniformization $X_0(\mathfrak{n}) \to E$ is $\frac{(\varphi,\varphi)}{-v_\infty(j(E))}$.

EXAMPLE (continued): Take again $q = 2$ and $\mathfrak{n} = T^2(T^2 + T + 1)$. Then the space of newforms is 1-dimensional; more precisely $\mathrm{H}_1^{new}(\Gamma_0(\mathfrak{n})\backslash\mathcal{T}, \mathbb{R}) = \mathbb{R}\cdot\varphi$, where φ takes the value 1 on the edges that form the closed path connecting the vertices 1, 2, 3,. . .,12, 1. So there exists exactly one $\mathbb{F}_2(T)$-isogeny class of elliptic curves over $\mathbb{F}_2(T)$ with conductor $\infty\cdot\mathfrak{n}$ and split multiplicative reduction at ∞. The minimal positive scalar product of φ with elements of $\mathrm{H}_1(\Gamma_0(\mathfrak{n})\backslash\mathcal{T}, \mathbb{Z})$ is 4. Hence the degree of the j-invariant of the strong Weil curve E in this class is 4. Since $q = 2$ and the curves are Tate curves at ∞, every curve in this isogeny class is determined by the degree of its j-invariant. Thus one can verify that

$$E:\ Y^2 + TXY + TY = X^3 + T^2$$

is the strong Weil curve in this class. The degree of the strong Weil uniformization $X_0(\mathfrak{n}) \to E$ is $(\varphi, \varphi)/4 = 3$.

Now the modular automorphism $U_1W_{T^2}$ maps the edge from vertex 14 to vertex 2, that is the edge whose double-coset representative in $GL_2(K_\infty)$ is $\binom{1\ \ 0}{T+1\ \ 1}$, to the edge with representative $\binom{0\ \ 1}{1\ \ T^3}$, that is the edge from vertex 14 to vertex 6. This determines $U_1W_{T^2}$ as an automorphism of the graph; namely: $U_1W_{T^2}$ fixes the vertices 13 and 14 and acts as permutation $(2, 6, 10)(4, 8, 12)$ on their neighbours. One easily sees that the subspace of $U_1W_{T^2}$-invariant elements of $\mathrm{H}_1(\Gamma_0(\mathfrak{n})\backslash\mathcal{T}, \mathbb{R})$ is just $\mathbb{R}\cdot\varphi$, so

$$E = \langle U_1W_{T^2}\rangle\backslash X_0(\mathfrak{n}).$$

In the same way (i.e., calculation of the graph $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$ by computer program as described in [5] and calculation of some Hecke operators and modular automorphisms by hand) one obtains.

Proposition 1

There are exactly 24 different $\mathbb{F}_2(T)$-isogeny classes of elliptic curves over $\mathbb{F}_2(T)$ with conductor $\infty\cdot\mathfrak{n}$ and split multiplicative reduction at ∞, where $\mathfrak{n}\in\mathbb{F}_2[T]$ and $deg(\mathfrak{n}) \le 4$. The table below shows twelve of these, and replacing T by $T + 1$ gives the other twelve.

Here ∂ denotes the degree of the strong Weil uniformization. G is a subgroup of $\mathcal{M}(\mathfrak{n})$ such that $E = G\backslash X_0(\mathfrak{n})$, and $-$ means that such a subgroup doesn't exist.

$\mathfrak{n}$	*equation of strong Weil curve E*	∂	G
T^3	$Y^2+TXY=X^3+T^2$	1	$\langle id\rangle$
$T^2(T+1)$	$Y^2+TXY+TY=X^3$	1	$\langle id\rangle$
$T(T^2+T+1)$	$Y^2+(T+1)XY+Y=X^3+T(T^2+T+1)$ $Y^2+(T+1)XY+Y=X^3+X^2+T+1$	2 2	$\langle W_T\rangle$ $\langle W_{T^2+T+1}\rangle$
T^4+T^3+1	$Y^2+TXY+Y=X^3+X^2$	2	$\langle W_{T^4+T^3+1}\rangle$
T^4	$Y^2+TXY=X^3+TX^2+T^2$	2	$\langle (W_{T^4}U_1)^2\rangle$
$T^3(T+1)$	$Y^2+TXY=X^3+(T+1)^2X$	4	$\langle W_{T^3}, U_1W_{T^3}U_1\rangle$
$T^2(T^2+T+1)$	$Y^2+TXY+TY=X^3+T^2$	3	$\langle U_1W_{T^2}\rangle$
$T(T^3+T+1)$	$Y^2+(T+1)XY+TY=X^3+X^2$ $Y^2+(T+1)XY+TY=X^3+T^3$	4 4	$\langle W_T, W_{T^3+T+1}\rangle$ $-$
$T(T^3+T^2+1)$	$Y^2+(T+1)XY+TY=X^3$ $Y^2+(T+1)XY+TY=$ $X^3+(T+1)X^2+T^3X+T^2$	2 14	$\langle W_{T^3+T^2+1}\rangle$ $-$

Finding the equations of the strong Weil curves involves some trial and error, but a posteriori they can be proved to be correct.

The first five curves in the table and the corresponding graphs are already in [3]. Some further elliptic curves of the form $G\backslash X_0(\mathfrak{n})$, even some with conductor $\infty\cdot\mathfrak{m}$ for a proper divisor $\mathfrak{m}$ of $\mathfrak{n}$, are listed in [8]. However, all in all there exist only finitely many ones.

Proposition 2

Let $\mathfrak{n}=\prod_{i=1}^{s}\mathfrak{p}_i^{e_i}$ *be such that* $G\backslash X_0(\mathfrak{n})$ *is elliptic for a subgroup* G *of* $\mathcal{M}(\mathfrak{n})$. *Then (with* $d=deg(\mathfrak{n})$*) one of the following assertions must hold:*

a) $q=2$, $d\le 15$, $s\le 4$,
b) $q=3$, $d\le 7$, $s\le 4$,
c) $q=4$, $d\le 5$, $s\le 4$,
d) $q=5$, $d\le 5$, $s\le 5$,
e) $q=7$, $d\in\{3,4\}$, $s\in\{3,4\}$,
f) $q\in\{8,9\}$, $d=4$, $s\in\{3,4\}$,
g) $q\in\{11,13\}$, $d=4$, $s=4$.

Proof. Except for the bound $d\le 15$, $s\le 4$ in case $q=2$ this is the statement of Proposition 17 in [8], where the case $G\le\mathcal{W}(\mathfrak{n})$ is treated. In case $q=2$, where $\mathcal{M}(\mathfrak{n})$ can be larger than $\mathcal{W}(\mathfrak{n})$, arguments and calculations similar to those developed in section 3 of [8] yield the bound given above. □

Without proof (compare again [7]) we also state.

Proposition 3

Let $q = 2$.

a) If $T^3|\mathfrak{n}$ *then* U_1 *acts as* -1 *on* $\underline{\mathrm{H}}_!^{new}(\mathcal{T},\mathbb{R})^{\Gamma_0(\mathfrak{n})}$.

b) If $T^5|\mathfrak{n}$ *then* U_2 *acts on* $\underline{\mathrm{H}}_!^{new}(\mathcal{T},\mathbb{R})^{\Gamma_0(\mathfrak{n})}$ *as twist by* T^{-1}*, that is: If* $\varphi \in \underline{\mathrm{H}}_!^{new}(\mathcal{T},\mathbb{R})^{\Gamma_0(\mathfrak{n})}$ *is a simultaneous eigenform for the Hecke operators with eigenvalues* $c_\mathfrak{p}$ *then* $\varphi \circ U_2$ *has Hecke eigenvalues* $\chi(\mathfrak{p})c_\mathfrak{p}$*, where* χ *is the character of the field extension of* K *generated by* $X^2 + X = T^{-1}$.

Similar statements hold for V_1 *and* V_2.

Statement b) implies: If $T^5|\mathfrak{n}$ and E belongs to $\varphi \in \underline{\mathrm{H}}_!^{new}(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ then $\varphi \circ U_2$ belongs to the T^{-1}-twist of E.

This holds also for $ord_T(\mathfrak{n}) = 4$, but then $\varphi \circ U_2$ is not necessarily a newform, that is, the conductor of the T^{-1}-twist of E might be smaller. For example in the table in Proposition 2 one sees that the T^{-1}-twist of the curve with conductor $\infty \cdot T^4$ is the curve with conductor $\infty \cdot T^3$.

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