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Counting exceptional units

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Abstract

The number of solutions of the "unit equation" $x + y = 1$ in units of (the ring of integers of) an algebraic number field of degree n and unit rank r is known to be bounded above by an exponential function of n and r, but the best known lower bounds are only polynomial in n, and the true counts have been computed only in a few cases.

We will present recently computed solution counts in number fields of unit rank $r \leq 5$, leading to a tentative formula for the largest number of solutions attained by at least one field of given signature. The formula agrees with the Stewart heuristic, predicting about $\exp(r^{2/3+o(1)})$ solutions. These counts are dominated by "small" solutions, whereas the smaller number of solutions which can be attained infinitely often by fields of a fixed signature hinges on the "large" ones.

1. Introduction

The unit equation $x + y = 1$, to be solved in invertible algebraic integers, first appears in C. L. Siegel's fundamental paper on diophantine approximation [26]. In spite of a 67-year history of research not very much is known in general about how many solutions $x, 1-x$ it can have in any given algebraic number field. Such x are called *exceptional units*, a term introduced by T. Nagell in 1969 [20]. Siegel already knew that their number in any given field is finite (an easy consequence of Satz 7, Zusatz 1 of his dissertation [25]), and it follows from A. Baker's theory of linear forms in logarithms that the solutions are effectively computable. This was first made explicit by K. GYŐRY in the 1970s [8]. J.-H. EVERTSE proved in 1983 [5] that the number of solutions is at most

$$3 \cdot 7^{n+2r+2},$$

where $n = [K : \mathbb{Q}]$ is the degree and r the rank of the group of units of the field K. (By DIRICHLET's theorem, $r = r_1 + r_2 - 1$ if the field has r_1 real and r_2 pairs of complex embeddings; since $r_1 + 2\,r_2 = n$, we have $n/2 \leq r + 1 \leq n$.) Although the numbers 3 and 7 can be reduced somewhat, Evertse's bound has not yet been improved in substance; all known upper bounds are of this form — exponential in n or in r or in some combination of n and r. Note that the method used by Evertse, H. P. SCHLICKEWEI and others (see the excellent survey [6]) is ineffective in the sense that it does not provide bounds on the solutions themselves.

There can be no positive lower bounds valid for *all* fields of given signature (n, r). E.g., whenever an ideal of norm 2 is present, there can be no exceptional units at all, since all the units must then lie in the nontrivial coset. The best one can strive for are lower bounds for some or for infinitely many fields of given signature. Nagell [19] noted that infinitely many non-isomorphic fields of degree n contain at least

$$12n - 30$$

exceptional units ([19], after Théorème 10*bis*, p. 125; note that Nagell counts *pairs* of solutions). This linear lower bound is the best known to date of its type. He also gave a linear lower bound $12n - 18$ for the number of exceptional units in at least one field of degree n ([19], Théorème 10*bis*) and observed that the largest numbers of solutions tend to occur in the fields of smallest absolute discriminant within each signature. Using cyclotomic units, I can construct examples with solution counts growing at least like n^3, but we will see that these are still untypically low.

The present author, with the benefit of access to an extensive synopsis of number fields of moderately large degrees (which is being made available in electronic form [21]), has been computing tables of exceptional units in many fields of small absolute discriminant up to degree 11 and rank 5. Some rank 6 computations are in progress at the time of writing. We will present a table of results below and attempt to predict the true rate of growth for the number of solutions attainable at least once per signature. Rather less is known about the (smaller) number of solutions which can be attained infinitely often. We will give lower bounds for a few signatures at the end of this note; more details can be found in an earlier version of it [23].

In order to guide our expectations, let us look at the more general case of the S-unit equation $x+y=1$, where the unknowns x and y now range over the subgroup of elements of $K^\times$ which are integral with integral inverse at all places except those in the finite set S. The rank of this group is $r+s$ where s is the number of finite places in S. Evertse's upper bound remains valid for this situation with the exponent $n+2r+2$ replaced accordingly by $n+2(r+s+1)$. We can now consider a fixed field K and let the set S vary. Congruence obstructions of the type mentioned above can be removed by making the offending prime ideals invertible, and one may hope for an improved lower bound in terms of $r+s$ which holds for suitably chosen sets S containing s finite places. A bound of this type, subexponential in s, was found by P. Erdős, C. L. Stewart and R. Tijdeman [4]. (A simplified proof was given by D. B. Zagier in [32], p. 425f.) It says that by choosing S carefully, one can produce at least a constant times

$$\exp\bigl((4+o(1))(s/\log s)^{1/2}\bigr)$$

exceptional S-units in $\mathbb{Q}$ (implying a similar lower bound for all number fields). Soon afterwards, Stewart suggested that an optimal choice of S ought to yield

$$\exp(s^{2/3+o(1)})$$

solutions ([6], p. 120).

Now experimental evidence suggests that in fields whose absolute discriminants are small for their signatures, the rings of integers behave "nicely" in many ways (e.g., they are often euclidean for the norm [11, 13, 12, 14]), whereas in a general number field such nice behavior is only obtained after passing to S-integers and S-units for large enough sets S. This observation has even been used in the opposite direction for finding fields with small absolute discriminants, see [13, 12, 21]. Thus it is not unreasonable to expect that Stewart's heuristic might carry over to the case of ordinary units in fields of small absolute discriminants. We will see that this expectation is indeed borne out by examples.

The remainder of this note is organized as follows. After recalling some basic facts and notation, we will draw a distinction between "small" and "large" solutions in section 3. The former kind will account for most of the exceptional units seen in fields of small absolute discriminant; they cannot contribute to infinitely many fields of fixed signature except insofar as they come from proper subfields. We will focus on these small solutions in section 4, presenting our conjectural formula for their true number and the numerical evidence supporting this formula. The last section is devoted to the large solutions.

2. Basic facts

We recall some well-known simple properties of exceptional units (see [14] and the references cited there for more details). Let R be a commutative ring with an identity element 1 for multiplication; write $R^\times$ for its group of units and $E(R) = R^\times \cap (1 - R^\times)$ for the set of exceptional units of R. Homomorphisms of such rings are understood to preserve identity elements, hence they map units to units and exceptional units to exceptional units. Nagell noted that exceptional units (e.u.s from now on) usually come in orbits of six:

Lemma 2.1

If $x \in E(R)$ is an exceptional unit, then so are

$$x^i := 1/x\,,\quad x^j := 1-x\,,\quad x^k := x/(x-1)\,,\quad x^{ij} = 1-1/x\,,\quad x^{ji} = 1/(1-x)\,.$$

These are distinct unless either $x^2 - x + 1 = 0$, in which case we have $x = x^{ij} = x^{ji}$ and $x^i = x^j = x^k$, or $1{+}1 \in R^\times$ and $x \in \{-1, 1{+}1, (1{+}1)^{-1}\}$, when $x = x^i$ or x^k or x^j, respectively. We write $H = \langle i, j, k\,;\ i^2 = j^2 = k^2 = (ij)^3 = ijik = 1\rangle$ for this nonabelian group of order 6 of homographic transformations. □

Now let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial, x a root of f (in some fixed algebraic closure of $\mathbb{Q}$) and $\mathbb{Z}[x]$ the subring of the ring of integers of the number field $K = \mathbb{Q}(x)$ generated by x. The intersection of $\mathbb{Z}[x]$ with the group of units of K is a subgroup of finite index and hence of full rank. In particular, whenever an element of $\mathbb{Z}[x]$ is invertible in the full ring of integers of K, then it is already a unit of $\mathbb{Z}[x]$.

Lemma 2.2

a) If $g \in \mathbb{Z}[X]$ is such that $g(x) \in \mathbb{Z}[x]^\times$, then the canonical homomorphism from $\mathbb{Z}[X]$ onto $\mathbb{Z}[x]$ which sends X to x extends to a unique homomorphism of the ring of Laurent polynomials $\mathbb{Z}[X][g^{-1}]$ onto $\mathbb{Z}[x]$. The same is true, mutatis mutandis, when the single polynomial g is replaced with a subset of nonconstant elements of $\mathbb{Z}[X]$.

b) ([11], Lemma (2.5)) *If $g \in \mathbb{Z}[X]$ is also monic and irreducible, and if y denotes a root of g, then $g(x)$ is a unit in $\mathbb{Z}[x]$ if and only if $f(y)$ is a unit in $\mathbb{Z}[y]$.* □

Combining a) and b), we see that the monic irreducible polynomial $f \in \mathbb{Z}[X]$ is the minimal polynomial of an exceptional unit if and only if $f(0), f(1) \in \{\pm 1\}$. The following proposition is an immediate consequence.

Proposition 2.3 ([17], Théorème 1)

The only exceptional units in (the rings of integers of) quadratic number fields are the roots of the four polynomials

$$X^2 - X + 1\,, \quad X^2 - X - 1\,, \quad X^2 + X - 1\,, \quad X^2 - 3\,X + 1\,;$$

i.e., the sixth roots of unity $\zeta_6^{\pm 1}$ *and the* H*-orbit of the golden ratio* $\vartheta = \frac{1}{2}(1+\sqrt{5})$. $\square$

In [14] it is shown how one can compute the e.u.s of rings of Laurent polynomials (finitely generated quotient rings of $\mathbb{Z}[X]$). This can be used to construct rings containing at least a prescribed number of e.u.s (see [14, 23]), but finding "good" large sets of generator polynomials for the units of the Laurent ring is a problem in its own right.

3. The large and the small solutions

Evertse's bound is the sum of two contributions estimated by different methods, covering respectively the solutions of small and of large height. There are various notions of height (or size, or measure) for algebraic numbers ([15], Ch. 4), each of which can be bounded in terms of (the degree and) any of the others, and all of which have the fundamental property that any set of algebraic numbers of bounded degree and height is finite and effectively computable, at least in principle. (Any choice of upper bounds for degree and height constitutes or entails bounds for the coefficients of the minimal polynomials.)

For the purpose of the following discussion, think of some fixed notion of height and assume that a positive bound $B(n, r)$ has been chosen. Among all exceptional units in fields of degree n and unit rank r (always working in a fixed algebraic closure of $\mathbb{Q}$), call those of height not exceeding $B(n, r)$ the "small" ones and the others the "large" ones. By the fundamental property of heights, the small e.u.s in all fields of this signature are the roots of a finite set of polynomials. Therefore they generate only finitely many distinct fields, and we deduce Theorem 3.1.

Theorem 3.1

Fix a signature (n, r) and a notion of smallness as explained above. Then small exceptional units which are themselves of degree n, i.e., which are primitive elements of the fields under consideration, can exist only in finitely many distinct number fields of the prescribed signature, and these fields can be effectively determined. If any other field of the given signature contains a small exceptional unit x, then x along with its H-orbit must lie in a proper subfield. $\square$

Thus it is appropriate to subdivide the problem of counting e.u.s in number fields into two subproblems as follows. For given n and r, let $C_1(n, r)$ denote the maximal number $\#E(R)$ of exceptional units where R ranges over the rings of integers of all fields of degree n and unit rank r. Furthermore, let $C_2(n, r)$ be the largest integer N such that infinitely many distinct fields of this signature possess N e.u.s *of degree n, i.e. not counting e.u.s contained in proper subfields.* The existence of these maxima follows at once from Evertse's bound; clearly we have $C_2(n, r) \leq C_1(n, r) \leq 3 \cdot 7^{n+2r+2}$.

The theorem implies that the small solutions do not affect $C_2(n, r)$ at all. When we are interested in counts *including* the e.u.s in subfields, the best approach is probably to consider extensions of one fixed subfield at a time. E.g., Nagell [18] proved $C_2(n, 1) = 0$ for all three cases of $n \in \{2, 3, 4\}$ (clearly $C_2(2, 0) = C_2(2, 1) = 0$ follows from proposition 2.3) and pointed out that there are infinitely many totally complex quadratic extensions of the real quadratic field $\mathbb{Q}(\vartheta)$; every one of these fields contains the six e.u.s of the subfield, but except for the cyclotomic field $\mathbb{Q}(\zeta_5)$ and the composite $\mathbb{Q}(\zeta_6, \vartheta)$ they contain no further e.u.s. A similar statement holds for quadratic extensions of $\mathbb{Q}(\zeta_6)$. Along the way, Nagell determined all e.u.s which generate fields of unit rank one, establishing $C_1(2, 1) = 6$ (proposition 2.3 again), $C_1(3, 1) = 12$ attained by the field of minimal discriminant -23 and $C_1(4, 1) = 20$ attained by the complex quartic field of smallest discriminant 117, an extension of $\mathbb{Q}(\zeta_6)$. All this was possible using elementary estimates. Linear forms in logarithms only enter the scene when $r > 1$.

4. Large numbers of small solutions

In unit ranks $r \geq 2$, exact numbers of e.u.s are known only for a few fields. These include Nagell's results $\#E(R) = 42$ and 18 for the rings of integers of the cyclic cubic fields of discriminants 7^2, smallest among all real cubic fields, and 9^2 [20]. Work in progress [22, 28] will provide many more examples in degrees 3 and 4.

B. M. M. de Weger [pers. comm.] computed $\#E(R) = 570$ for the ring of integers $R = \mathbb{Z}[\zeta_{11} + \zeta_{11}^{-1}]$ of the real subfield of the 11th cyclotomic field, which has the smallest discriminant 14641 among all totally real quintic fields. Beyond these examples, CM extensions of totally real fields already treated can be handled using an idea of Győry ([7], Lemme 12); this covers e.g. the seventh, ninth and eleventh cyclotomic fields with 72, 38 and 660 e.u.s respectively.

Recent improved estimates for linear forms in logarithms [1, 9, 31] have considerably cut down on the amount of computation required for solving the unit equation completely with Baker's method, but complete solutions are still expensive to obtain. Our experiments were therefore restricted to finding *most* solutions in each of a large number of fields, thus establishing lower bounds on $C_1(n,r)$ which we hope are quite close to the truth. We investigated the fields with $r \leq 5$ and minimal discriminants for those signatures for which they are known, $n \leq 7$ and $(n,r) = (8,3)$, and the fields with smallest known absolute discriminants for the remaining signatures which appear below, with defining polynomials taken from [13] and [12]; see also [21]. For many signatures, further fields with discriminants close to the minima were inspected. For each field, a suitably reduced set of fundamental units $\eta_1, \ldots, \eta_r$ and, if necessary, a root of unity ζ of maximal order were computed using PARI/GP. Then for all units $\varepsilon = \zeta^{a_0} \prod_1^r \eta_j^{a_j}$ with individually bounded exponents a_j, we checked whether the norm of $\varepsilon - 1$ was ± 1, using exact arithmetic. The largest exponents actually seen to occur in solutions were recorded, and the exponent bounds were extended until the actual maxima stayed strictly within the bounds, and until the total number of solutions (other than sixth roots of unity) was divisible by 6. (With additional bookkeeping, it would have been possible to detect missing members of individual H-orbits, but we were more concerned with speed for the time being. Indeed, we exploited the symmetry with respect to taking inverses for halving the range of one of the exponents.) The examples where full lists of e.u.s are known suggest that this approach will rarely miss any (large) solutions.

We found not a single example where the largest number of e.u.s was not attained by the field of minimal discriminant, although in a few cases the same count was attained by the field of second smallest absolute discriminant. The counts we obtained are listed in the following table.

Table 4.1: $C_1(n,r) \geq \dots$

$n =$	2	3	4	5	6	7	8	9	10	11
$r = 0$	2									
1	6	12	20							
2		42	54	78	110					
3			162	228	288	366	438			
4				570	750	954	1110	1344	1584	
5					2070	2310	2958	3342	3840	4482

Although 5 is not a very large number, one is tempted to try and fit a smooth function through these data points. Writing $T_1(n,r)$ for the tabulated counts, the best fit I found was

$$\log T_1(n,r) \approx r^{2/3}(\log r)^{1/3} + r^{1/3}\log n + 1\,. \tag{$*$}$$

For $2 \leq r \leq 5$, the difference between the lefthand and the righthand side of $(*)$ does not exceed 0.163 in magnitude, with the largest deviation occurring for $(n,r) = (4,2)$. The relative error is less than 4.1% in this case and less than 2% for all other signatures with $2 \leq r \leq 5$.

Whereas the dominant $r^{2/3}$ in $(*)$ is well constrained by the tabulated values, the exponent attached to the $\log r$ and the second and third summands are not. That exponent could be a real number close to 1/3, there may well be a numerical coefficient and/or an exponent close to 1 missing from the $\log n$ term, *etc.* More importantly, there ought to be a (negative) contribution from (the logarithm of) the absolute discriminant and from the regulator. This will probably show up for $(n,r) = (7,6)$ where the smallest discriminant 20 134 393 [24] is significantly larger than the Odlyzko bound, and such a contribution might also describe the decrease of the counts as one walks away from the minimal discriminant, but it is too early for guessing the shape of suitable terms.

We therefore propose the more modest Conjecture 4.2.

Conjecture 4.2 *As $n, r \to \infty$, we have* $\log C_1(n,r) = r^{2/3+o(1)}$.

In the signatures with larger ranks the numbers of solutions were seen to vary almost monotonically with the discriminant, apparently unaffected by the presence or absence of subfields and of nontrivial roots of unity. Therefore we expect $C_1(n,r)$ to be independent of such accidental properties of the field or fields with minimal discriminants.

Let us confront the conjecture with the contribution from small solutions to Evertse's upper bound, where the precise meaning of "small" is explained in [5]. This bound is given by the last displayed formula (70) of [5] which reads, in our notation,

$$5 \cdot A^{r+1} \cdot \left(2e^{24/A}\right)^n$$

where A is an arbitrary positive integer parameter. Evertse proceeds to set $A = 49$ in order to obtain a simple formula for the combined upper bound, but clearly the optimal values for our application lie between $A = 24$ for totally real fields ($n = r+1$) and $A = 48$ for totally complex ones ($n = 2r + 2$). They lead to $5 \cdot (48e)^{r+1}$ and $5 \cdot (192e)^{r+1}$, respectively, as upper bounds for the number of small e.u.s, expressions which are purely exponential in r. Evertse's proof is based on the product formula for absolute values; a number-geometric argument invoking the discriminant and/or the regulator might perhaps yield a sharper result.

5. A few large solutions, infinitely often

Evertse's upper bound for the number of large solutions (for some precise meaning of "large") and *a fortiori* for $C_2(n, r)$ is ([5], Lemma 9)

$$2 \cdot 7^{2r+2} .$$

One must bear in mind that this covers more general kinds of unit equations than the ones we are considering. At any rate, the known lower bounds for $C_2(n, r)$ are much smaller.

As has been explained in much more detail in [23], such lower bounds are readily obtained by using Lemma 2.2 to construct suitable infinite families of polynomials of fixed degree n. With some extra care, one can ensure that up to finitely many exceptions, these polynomials will have n real roots, or $n-2$ real roots and a pair of complex roots; larger numbers of non-real places are more difficult to force. Within such a family, one has good control over the locations of the real and complex roots, and the author suggested in [23] that the methods pioneered by E. Thomas [30] and M. Mignotte [16] and further developed by A. Pethő *et al.* could be used to show that equation orders arising from a well-constructed family contain no other e.u.s than the forced ones. These methods combine diophantine approximation techniques (a gap principle) with linear forms in logarithms to obtain exact solution counts for entire families of exponential diophantine equations. Some such examples are now being worked out in full detail by N. P. Smart and the author [28, 22]. It is another matter, but still possible in some cases, to show that there are no spurious solutions outside the equation order (which *a priori* might be and sometimes is a proper subring of the full ring of integers of the field).

Conjecture 5.1 *For each of $n = 3, 4, 5, 6$, we should have $C_2(n, 2) = 6$.*

We have already seen in section 3 that $C_2(n, 1) = 0$ for all three possible values of n. The case $(n, r) = (3, 2)$ of real cubic fields has been the subject of numerous studies (see e.g. [19, 3, 10] and the references cited there). By Lemma 2.2, the minimal polynomials of the possible e.u.s, up to the action of H, form two infinite one-parameter families, one of which produces only cyclic fields (and some of them more than once), whereas the other avoids the cyclic fields with finitely many exceptions, and never seems to hit a non-cyclic real cubic fields more than once. Nagell conjectured that the non-cyclic family produces pairwise distinct fields.

For quartic and for quintic rank two fields, [23] gives families which show $C_2 \geq 6$; in the totally complex degree 6 case one can easily obtain the same result using suitable families of cubic extensions of fixed imaginary quadratic fields. The point of the conjecture is that C_2 should not be larger than 6.

A proof of the full conjecture probably cannot be based on such families alone, as there are too many free parameters when $n > 3$. I have a heuristic argument which ought to furnish a proof at least for $n = 4$ and $n = 5$, but the details remain to be worked out.

Proceeding to higher ranks, we have $C_2(4, 3) \geq 18$ since infinitely many such fields arise from minimal polynomials f satisfying $\{f(0), f(1), f(-1)\} \subseteq \{\pm 1\}$ and producing what J.-D. THÉROND calls *unités vraiment exceptionnelles* [29]. Similarly, the conditions

$$\{f(0), f(1)\} \subseteq \{\pm 1\} \quad \text{and} \quad f(\zeta_6) \in \{\pm 1, \pm\,\zeta_6^{\pm 1}\}$$

lead to families establishing $C_2(5, 3) \geq 24$, one of which is the main example discussed in [23], and $C_2(5, 4) \geq 48$ follows from considering

$$\{f(0), f(1)\} \subseteq \{\pm 1\} \quad \text{and} \quad f(\vartheta) \in \pm\,\vartheta^{\mathbb{Z}}\,.$$

I believe that these lower bounds are already the correct values but do not know at the moment how one could go about proving this. An argument *à la* Thomas [30] ought to show that with at most finitely many exceptions, the equation orders $\mathbb{Z}[x]$ defined by the family polynomials contain no extraneous solutions, but one would also need to deal with possible solutions outside $\mathbb{Z}[x]$.

At present, we possess insufficient evidence to predict the behaviour of $C_2(n,r)$ for larger degrees and ranks, except that it should stay rather smaller than C_1. It may turn out that factors other than the signature are relevant here, such as the Galois group of the normal closure or the presence of nontrivial roots of unity in proper subfields. One would then have to consider $C_2(n,r)$ as the maximum of several similarly defined quantities $C_2'(n,r,\ldots)$ which need to be studied separately.

Acknowledgements. The question, "How many exceptional units. . . ?", was first posed to the author by K. Győry in 1989, and since then repeatedly by several others, in conversation and in correspondence. To A. LEUTBECHER I owe my introduction into this fascinating field as well as long lists of polynomials. Direct and electronic conversations with Győry, Mignotte, Pethő, Schlickewei, Smart, de Weger and Zagier, as well as the referees' comments, have contributed in many ways to the exposition. Any remaining obscurities are of course my own responsibility.

The computational results mentioned above were obtained with the aid of Maple V.3 running on a Sun SPARCstation ELC and, mainly, PARI/GP [2] on a Sun SPARCclassic and on an Intel P100 Linux workstation.

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